The Helmholtz decomposition in weighted L^p spaces in cones

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Abstract

In this paper we prove the Helmholtz decomposition in conical domains of $\mathbb{R}^3$ in weighted L^p_β spaces under a spectral condition on β, p. The basic elements are the transformation of the original problem into a problem set in cylindrical domains and the combination of a priori bounds from [5] with the vector-valued multiplier theorem [29].

1 Introduction

The Helmholtz decomposition, namely the decomposition of vector fields into a solenoidal field and a gradient part, is an indispensable tool in the analysis of incompressible fluid flows. While in the L^2-setting such a decomposition is easily obtained for any domains of $\mathbb{R}^d, d \geq 2$ using elementary Hilbert space properties [20, Lemma 2.5.1], the situation becomes more delicate in the L^p-setting. Nevertheless, there are numerous results for bounded domains [3, 10, 12, 20] or exterior domains (complement of a bounded domains) [17, 18, 28]; see also [6, 7, 23, 22] for such results in weighted spaces. A common feature of such domains is that they have a compact boundary. On the other hand there exist domains with non compact boundary (sector-like) for which the L^p-Helmholtz decomposition fails [1, 2, 16]. Nevertheless it is still valid for basic domains with a smooth boundary like half-spaces or perturbations of them [11, 13, 27], aperture domains [4, 17, 6], cylinders or layers [17, 21, 5, 19, 26, 25], or domains whose boundary is the graph of smooth functions [2, 11]. But to the best of our knowledge, the question of

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the validity of the L^p-Helmholtz decomposition for domains with a non compact and non-smooth boundary remains open. Nevertheless let us mention the paper [15] that obtains a Helmholtz decomposition in anisotropic L^p spaces for domains whose boundary is the graph of a globally Lipschitz functions (hence conical domains are admitted).

The goal of this paper is therefore to prove such a Helmholtz decomposition in conical domains of $\mathbb{R}^3$ in weighted L^p_β spaces (L^p-spaces being a particular one when $\beta = 0$) under a condition between α, p and the eigenvalues of the Laplace-Beltrami operator with Neumann boundary conditions on the intersection of the cone with the unit sphere. The basic tools are: First to transform the problem into a problem set in cylindrical domains using the Euler change of variables, second applying a formal Fourier transformation. In this way, we find a problem that is a lower order perturbation of a problem considered in [5]. Finally, combining a priori bounds from this paper with the vector-valued multiplier theorem [29], we conclude the result. Note that this result can be derived via a duality argument, we refer to [24] in this respect.

The paper is organized as follows. The second section deals with the weak Neumann problem in the Fourier space, namely we study the resolvent of the operator obtained after Euler change of variables and Fourier transform and show its $\mathcal{R}$-boundedness. In section 3, we show the well-posedness of the weak Neumann problem in a cone and conclude with the Helmholtz decomposition.

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2 The weak Neumann problem in the Fourier space

Let us fix an open subset S of the unit sphere of $\mathbb{R}^3$ with a $C^{1,1}$ boundary. On such a domain, we want to consider a Neumann type problem with complex parameters ξ on $L^p(S)$ spaces. Hence as in [5], in order to have uniform estimates in ξ and to use the vector-valued multiplier theorem [29], we enlarge the setting to weighted L^p spaces.

Before let us fix some notations. Let us denote by $(\mu_k)_{k\geq 0}$, the sequence of the eigenvalues (repeated according to their multiplicities) of the (non negative) Laplace-Beltrami operator L_{LB}^{Neu} on S with Neumann boundary conditions (obviously $\mu_0 = 0$ and $\mu_1 > 0$).

Let us further fix an atlas (U_ℓ, Φ_ℓ), $\ell = 1, \cdots, L$ of S in the sense that for each ℓ, U_ℓ is a bounded open set of $\mathbb{R}^2$ and

$$\Phi_\ell : U_\ell \to S$$

is an injective and C^1 mapping such that

$$\cup_{\ell=1}^L \Phi_\ell(U_\ell) = S.$$

Now a non-negative function $\omega \in L^1(S)$ is called an A_p-weight if and only if $\omega \circ \Phi_\ell$ is the restriction to U_ℓ of an A_p-weight of $\mathbb{R}^2$. Finally we introduce the space $L^p_\omega(S)$ defined by

$$L^p_\omega(S) := \{u \in L^1(S) : \|u\|_{p,\omega} = \left(\int_S |u(\vartheta)|^p \omega(\vartheta)\, d\sigma(\vartheta)\right)^{\frac{1}{p}} < \infty\}.$$

In the same manner, we define

$$\mathbf{L}^p_{T,\omega}(S) := \{\mathbf{u} \in L^p_\omega(S)^3 : \mathbf{u}(\vartheta)\cdot\vartheta = 0 \text{ a.e. in } S\},$$

as well as

$$H^1_{p,\omega}(S) := \{u \in L^p_\omega(S) : \nabla_T u \in \mathbf{L}^p_{T,\omega}(S)\},$$

equipped with their natural norm. Here and below, $\nabla_T u$ means the tangential gradient of u on S.

The main goal of this section is to study the next problem: given two real parameters η, δ, and functions $\mathbf{g} \in \mathbf{L}^p_{T,\omega}(S)$ and $g \in L^p_\omega(S)$, we look for a solution $\psi \in H^1_{p,\omega}(S)$ to

$$\begin{aligned}\int_S \Big(\nabla_T\psi\cdot\nabla_T\varphi \quad &- \quad (i\xi-\eta)(i\xi+\delta)\psi\bar\varphi\Big)\, d\sigma \\ &= \quad \int_S (\mathbf{g}(\vartheta)\cdot\nabla_T\bar\varphi - g(\vartheta)(i\xi+\delta)\bar\varphi)\, d\sigma, \forall\varphi \in H^1_{p',\omega'}(S),\end{aligned} \tag{2.1}$$

where p' is the conjugate of p and $\omega' = \omega^{-\frac{1}{p-1}}$.

The case when $\delta = -\eta$ with $|\eta| < \mu_1$ is treated in [5, Theorem 3.6] (since Theorem 3.2 of [5] is valid by replacing Δ' by the operator

$$\sum_{i,j} a_{ij}\partial_i\partial_j,$$

with constant coefficients $a_{ij} = a_{ji}$ that is elliptic in the (usual) sense that the matrix $(a_{ij})_{1\le i,j\le 2}$ is positive definite). We here extend this result under one condition on η, δ. Namely we have the next result.

Theorem 2.1. *Assume that*

$$\eta\delta \neq -\mu_k, \forall k \in \mathbb{N}. \tag{2.2}$$

Then for all $\mathbf{g} \in \mathbf{L}^p_{T,\omega}(S)$ and $g \in L^p_\omega(S)$, there exists a unique solution $\psi \in H^1_{p,\omega}(S)$ of (2.1) *with the estimate*

$$\|\nabla\psi\|_{p,\omega} + (1+|\xi|)\|\psi\|_{p,\omega} \le C(\|\mathbf{g}\|_{p,\omega} + \|g\|_{p,\omega}), \tag{2.3}$$

with a positive constant C that may depend on the A_p-constant $\mathcal{A}_p(\omega)$ but is independent of ξ.

Proof. We first prove the existence and uniqueness of the solution. For that purpose, we introduce the family of operators (compare with [5, p. 375])

$$T_{p,\omega}(\eta,\delta,\xi) : H^{1,0}_{p,\omega}(S) \to (H^{1,0}_{p',\omega'}(S))^* : \psi \to T_{p,\omega}(\eta,\delta,\xi)\psi,$$

where

$$H^{1,0}_{p,\omega}(S) := \{\psi \in H^{1,0}_{p,\omega}(S) : \int_S \psi\, d\sigma = 0\},$$

and

$$\langle T_{p,\omega}(\eta,\delta,\xi)\psi,\varphi\rangle = \int_S \Big(\nabla_T\psi\cdot\nabla_T\varphi - (i\xi-\eta)(i\xi+\delta)\psi\bar{\varphi}\Big)\,d\sigma, \forall\varphi\in H^{1,0}_{p',\omega'}(S).$$

The proof of Theorem 3.6 of [5] shows that $T_{p,\omega}(0,0,\xi)$ is an isomorphism. Now as

$$(T_{p,\omega}(0,0,\xi) - T_{p,\omega}(\eta,\delta,\xi))\psi = (\xi^2 + (i\xi-\eta)(i\xi+\delta))\psi,$$

and since by Proposition 2.5 of [5], $H^1_{p,\omega}(S)$ is compactly embedded into $L^p_\omega(S)$, we deduce that $T_{p,\omega}(0,0,\xi) - T_{p,\omega}(\eta,\delta,\xi)$ is a compact operator. Therefore $T_{p,\omega}(\eta,\delta,\xi)$ is a Fredholm operator of index 0 and it will be an isomorphism if and only if it is injective. So let $\psi\in\ker T_{p,\omega}(\eta,\delta,\xi)$ or equivalently $\psi\in H^{1,0}_{p,\omega}(S)$ satisfies

$$\int_S \Big(\nabla_T\psi\cdot\nabla_T\bar{\varphi} - (i\xi-\eta)(i\xi+\delta)\psi\bar{\varphi}\Big)\,d\sigma = 0, \forall\varphi\in H^{1,0}_{p',\omega'}(S). \tag{2.4}$$

But as ψ has zero mean value, we have

$$\int_S \Big(\nabla_T\psi\cdot\nabla_T 1 - (i\xi-\eta)(i\xi+\delta)\psi 1\Big)\,d\sigma = -(i\xi-\eta)(i\xi+\delta)\int_S \psi\,d\sigma = 0,$$

and therefore (2.4) extends to the whole $H^1_{p',\omega'}(S)$, namely

$$\int_S \Big(\nabla_T\psi\cdot\nabla_T\bar{\varphi} - (i\xi-\eta)(i\xi+\delta)\psi\bar{\varphi}\Big)\,d\sigma = 0, \forall\varphi\in H^{1}_{p',\omega'}(S). \tag{2.5}$$

But Lemma 2.2 of [8] guarantees that $H^1_{p,\omega}(S)$ is continuously embedded into $W^{1,s}(S)$ for some $s>1$ and by the Sobolev embedding theorem, $W^{1,s}(S)$ being continuously embedded into $L^2(S)$, we deduce that $\psi\in L^2(S)$.

Now for any $h\in L^2(S)$, as by our assumption $(-i\xi-\eta)(-i\xi+\delta)$ is never an eigenvalue of L^{Neu}_{LB}, there exists a unique solution $\varphi\in H^2(S)$ solution of

$$\Big(L^{\mathrm{Neu}}_{LB} - (-i\xi-\eta)(-i\xi+\delta)\Big)\varphi = h.$$

Since $H^2(S)$ is continuously embedded into $W^{1,q}(S)$ for all $q>1$, by Lemma 2.4 of [5], $H^2(S)$ is continuously embedded into $H^1_{p',\omega'}(S)$. Consequently the above identity implies that

$$\int_S \Big(\nabla_T\psi\cdot\nabla_T\bar{\varphi} - (i\xi-\eta)(i\xi+\delta)\psi\bar{\varphi}\Big)\,d\sigma = \int_S \psi\bar{h}\,d\sigma.$$

Comparing with (2.5), we see that

$$\int_S \psi\bar{h}\,d\sigma = 0,$$

for any $h\in L^2(S)$ and we conclude that $\psi=0$.

At this stage it remains to show the estimate (2.3). Let us then fix $\mathbf{g} \in \mathbf{L}^p_{T,\omega}(S)$ and $g \in L^p_\omega(S)$ and the unique solution $\psi \in H^1_{p,\omega}(S)$ of (2.1). Then ψ can be viewed as the solution of

$$\int_S \Big(\nabla_T\psi \cdot \nabla_T\varphi + (\xi + i\alpha)^2\psi\bar{\varphi}\Big)\, d\sigma = \int_S \Big((\mathbf{g}(\vartheta)\cdot \nabla_T\bar{\varphi} - g(\vartheta)(i\xi+\delta)\bar{\varphi}) \\ + (i\xi(\delta-\eta+\alpha) - \eta\kappa - \alpha^2)\psi\bar{\varphi}\Big)\, d\sigma, \forall \varphi \in H^1_{p',\omega'}(S),$$

with a fixed $\alpha \in (0, \sqrt{\mu_1})$, or equivalently

$$\int_S \Big(\nabla_T\psi \cdot \nabla_T\varphi + (\xi + i\alpha)^2\psi\bar{\varphi}\Big)\, d\sigma = \\ \int_S (\mathbf{g}(\vartheta)\cdot \nabla_T\bar{\varphi} - i(\xi+i\alpha)h(\vartheta)\bar{\varphi})\, d\sigma, \forall \varphi \in H^1_{p',\omega'}(S),$$

where h is given by

$$h(\vartheta) \quad = \quad -\frac{i\xi(\delta-\eta+\alpha) - \eta\kappa - \alpha^2}{i\xi - \alpha}\psi(\vartheta) + \frac{i\xi+\delta}{i\xi-\alpha}g(\vartheta).$$

Hence using Theorem 3.6 of [5] to this last problem, we find that

$$\|\nabla\psi\|_{p,\omega} + |\xi|\|\psi\|_{p,\omega} \leq C(\|\mathbf{g}\|_{p,\omega} + \|h\|_{p,\omega}),$$

with a positive constant C that may depend on the A_p-constant $\mathcal{A}_p(\omega)$ but is independent of ξ. From the form of h, we find that

$$\|\nabla\psi\|_{p,\omega} + |\xi|\|\psi\|_{p,\omega} \leq C(\|\mathbf{g}\|_{p,\omega} + \|g\|_{p,\omega} + \|\psi\|_{p,\omega}),$$

since the ratios $\frac{i\xi(\delta-\eta+\alpha)-\eta\kappa-\alpha^2}{i\xi-\alpha}$ and $\frac{i\xi+\delta}{i\xi-\alpha}$ are bounded for any $\xi \in \mathbb{R}$. This proves (2.3) for $|\xi|$ large. For $|\xi|$ small, the previous estimate (compare with the estimate (3.11) of [5]) allows to use the arguments of the proof of Theorem 3.6 of [5] to deduce that (2.3) is also valid (using our assumption (2.2)). ■

At this stage for a fixed pair of real numbers (δ, η) fulfilling (2.2), for any $\xi \in \mathbb{R}$ and any $(\mathbf{g}, g) \in X_{p,\omega} := \mathbf{L}^p_{T,\omega}(S) \times L^p_\omega(S)$, we set

$$\mathcal{M}_{\delta,\eta}(\xi)(\mathbf{g}, g) = (\nabla_T\psi, i\xi\psi), \tag{2.6}$$

where $\psi \in H^1_{p,\omega}(S)$ is the unique solution of (2.1). According to Theorem 2.1, the operator $\mathcal{M}_{\delta,\eta}(\xi)$ is bounded from $X_{p,\omega}$ into itself. For $T \in \mathcal{L}(X_{p,\omega})$ denote by $|||T|||$ its operator norm. Then the previous Theorem mainly allows to obtain the next result.

Corollary 2.2. *Given $p \in (1, \infty)$, fix a pair of real numbers (δ, η) fulfilling* (2.2). *Then $\mathcal{M}_{\delta,\eta}(\xi)$ is Fréchet differentiable with respect to $\xi \in \mathbb{R}$ and there exists an A_p-consistent constant c (depending on (δ, η)) such that*

$$|||\mathcal{M}_{\delta,\eta}(\xi)||| + |\xi||||\mathcal{M}'_{\delta,\eta}(\xi)||| \leq c, \forall \xi \in \mathbb{R}. \tag{2.7}$$

Proof. The estimate of $|||\mathcal{M}_{\delta,\eta}(\xi)|||$ is nothing else than (2.3). But it is easy to check that the Fréchet derivative $\frac{\partial}{\partial\xi}\mathcal{M}_{\delta,\eta}(\xi)(\mathbf{g},g) = (\nabla_T\psi', i\xi\psi') + (0, i\psi)$, where $\psi \in H^1_{p,\omega}(S)$ is solution of (2.1) and $\psi' \in H^1_{p,\omega}(S)$ is solution of (compare with (2.1))

$$\begin{aligned}\int_S \left(\nabla_T\psi' \cdot \nabla_T\varphi \right. &- \left.(i\xi - \eta)(i\xi + \delta)\psi'\bar{\varphi}\right) d\sigma \\ &= -i\int_S (g(\vartheta) + (2i\xi + \delta - \eta)\psi\bar{\varphi})\, d\sigma, \forall\varphi \in H^1_{p',\omega'}(S).\end{aligned}$$

Hence applying Theorem 2.1 to this problem, we find that

$$\|\nabla\psi'\|_{p,\omega} + |\xi|\|\psi'\|_{p,\omega} \leq C(\|\psi\|_{p,\omega} + \frac{1}{1+|\xi|}\|g\|_{p,\omega}),$$

with a positive constant C that may depend on the A_p-constant $\mathcal{A}_p(\omega)$ but is independent of ξ. By the estimate (2.3) satisfied by ψ, we find that

$$\|\nabla\psi'\|_{p,\omega} + |\xi|\|\psi'\|_{p,\omega} \leq \frac{C_1}{1+|\xi|}(\|\mathbf{g}\|_{p,\omega} + \|g\|_{p,\omega}),$$

with a positive constant C_1 that may depend on the A_p-constant $\mathcal{A}_p(\omega)$ but is independent of ξ, which yields the requested on $|||\mathcal{M}'_{\delta,\eta}(\xi)|||$. ■

Remark 2.3. Notice that the statement of Corollary 2.2 remains valid for the operator $\widetilde{\mathcal{M}}_{\delta,\eta}(\xi)$ defined by

$$\widetilde{\mathcal{M}}_{\delta,\eta}(\xi)(\mathbf{g},g) = (\nabla_T\psi, (1+i\xi)\psi),$$

for any $\xi \in \mathbb{R}$ and any $(\mathbf{g},g) \in X_{p,\omega}$.

To end up our preliminary results, we need an extrapolation property on $X_{p,\omega}$ and its consequence concerning $\mathcal{R}$-boundedness, see Theorem 4.3 of [9] in $L^p_\omega(\Omega)$ or Theorem 4.3 of [5] in $L^p_\omega(\Omega)^n$ for an open set of $\mathbb{R}^n$, $n \in \mathbb{N}^*$.

Theorem 2.4. *Let $1 < p, s < \infty$, $\omega \in A_p$. Furthermore, let $\mathcal{T}$ be a family of $\mathcal{L}(X_{p,\omega})$ that satisfies*

$$\|TF\|_{s,\nu} \leq C\|F\|_{s,\nu}, \forall T \in \mathcal{T}, F \in X_{p,\omega} \cap X_{s,\nu}, \tag{2.8}$$

for every $\nu \in A_s$ with a constant C that depends only on the A_s-constant $\mathcal{A}_s(\nu)$. Then $\mathcal{T}$ is $\mathcal{R}$-bounded on $\mathcal{L}(X_{p,\omega})$, in the sense that there exists $C > 0$ such that for any $N \in \mathbb{N}$, $T_j \in \mathcal{T}$ and $F_j \in X_{p,\omega}$, it holds

$$\| \left(\sum_{j=1}^N |T_jF_j|^2\right)^{\frac{1}{2}} \|_{p,\omega} \leq C\| \left(\sum_{j=1}^N |F_j|^2\right)^{\frac{1}{2}} \|_{p,\omega}.$$

Proof. First by using an atlas of S, we notice that the result is valid in $L^p_\omega(S)^4$, consequently it suffices to use an extension argument from $X_{p,\omega}$ to $L^p_\omega(S)^4$. For that purpose, for $T \in \mathcal{T}$, we introduce the operator $\tilde{T}$ as follows: for any $(\mathbf{g}, g) \in L^p_\omega(S)^4$, we denote by

$$\tilde{T}(\mathbf{g}, g) := T(\mathbf{g} - (\mathbf{g} \cdot \vartheta)\vartheta, g).$$

Note that $\tilde{T}$ is well-defined since $\mathbf{g} - (\mathbf{g} \cdot \vartheta)\vartheta$ belongs to $\mathbf{L}^p_{T,\omega}(S)$. Hence we find a family $\widetilde{\mathcal{T}} = \{\tilde{T} : T \in \mathcal{T}\}$ of elements of $\mathcal{L}(L^p_\omega(S)^4)$. Furthermore, by our assumption (2.8), one has

$$\|\tilde{T}(\mathbf{g}, g)\|_{s,\nu} \leq C\|(\mathbf{g} - (\mathbf{g} \cdot \vartheta)\vartheta, g)\|_{s,\nu},$$

with $C > 0$ from (2.8). As there exists a positive constant C_1 depending only on s and S (and that we may suppose to be ≥ 1) such that

$$\|\mathbf{g} - (\mathbf{g} \cdot \vartheta)\vartheta\|_{s,\nu} \leq C_1\|\mathbf{g}\|_{s,\nu},$$

we deduce that

$$\|\tilde{T}(\mathbf{g}, g)\|_{s,\nu} \leq CC_1\|(\mathbf{g}, g)\|_{s,\nu}.$$

Hence $\tilde{T}$ satisfies the statement of the Theorem in $L^p_\omega(S)^4$, hence it is $\mathcal{R}$-bounded.

Let us now show that this property implies a similar property for T. Indeed, taking any $N \in \mathbb{N}$, $T_j \in \mathcal{T}$ and $F_j \in X_{p,\omega}$, we notice that

$$T_j F_j = \tilde{T}_j F_j,$$

and therefore

$$\begin{aligned} \| \left(\sum_{j=1}^N |T_j F_j|^2 \right)^{\frac{1}{2}} \|_{p,\omega} &= \| \left(\sum_{j=1}^N |\tilde{T}_j F_j|^2 \right)^{\frac{1}{2}} \|_{p,\omega} \\ &\leq C \| \left(\sum_{j=1}^N |F_j|^2 \right)^{\frac{1}{2}} \|_{p,\omega}, \end{aligned}$$

for some $C > 0$ due to the $\mathcal{R}$-boundedness of the family $\widetilde{\mathcal{T}}$. ■

This Theorem and Corollary 2.2 directly imply the next result.

Theorem 2.5. *Given $p \in (1, \infty)$, fix a pair of real numbers (δ, η) fulfilling* (2.2). *Then the sets $\{\mathcal{M}_{\delta,\eta}(\xi) : \xi \in \mathbb{R}\}$ and $\{\xi\mathcal{M}'_{\delta,\eta}(\xi) : \xi \in \mathbb{R}\}$ are $\mathcal{R}$-bounded on $\mathcal{L}(X_{p,\omega})$.*

3 The weak Neumann problem in a cone

In the whole section let Γ be a three-dimensional cone with a section S with a $C^{1,1}$ boundary, in the sense that

$$\Gamma = \{x \in \mathbb{R}^3 : \vartheta = \frac{x}{|x|} \in S\}.$$

On such domain, we recall that the weighted Sobolev space of Kondrat'ev's type $V^{k,p}_\beta(\Omega)$ with $k \in \mathbb{N}$, $p \in (1, \infty)$ and $\beta \in \mathbb{R}$ is defined by

$$V^{k,p}_\beta(\Gamma) = \{u \in L^2_{\mathrm{loc}}(\Gamma) : \|u\|_{V^{k,p}_\beta(\Gamma)} < \infty\},$$

where

$$\|u\|^p_{V^{k,p}_\beta(\Gamma)} = \sum\nolimits_{|\alpha|\le k} \|r^{\beta+|\alpha|-k} D^\alpha u\|^p_{L^p(\Gamma)},$$

$r(x) = |x|$ being the distance from x to the origin, that is the corner point of Γ. In particular, we write $L^p_\beta(\Gamma) = V^{0,p}_\beta(\Gamma)$. Let us also recall that the singular functions of the Laplace operator with Neumann boundary conditions are in the form

$$r^{\lambda_{k\pm}} \varphi_k,$$

where φ_k is the eigenvector of L^{Neu}_{LB} of eigenvalue μ_k and

$$\lambda_{k\pm} = \frac{-1 \pm \sqrt{1 + 4\mu_k^2}}{2}. \tag{3.1}$$

In this section, we prove the next result.

Theorem 3.1. *Let $p \in (1, \infty)$ and $\beta \in \mathbb{R}$ be such that*

$$1 - \frac{3}{p} - \beta \neq \lambda_{k\pm}, \forall k \in \mathbb{N}. \tag{3.2}$$

Then for all $\mathbf{f} \in L^p_\beta(\Gamma)^3$ there exists a unique solution $u \in V^{1,p}_\beta(\Gamma)$ of

$$\int_\Gamma \nabla u \cdot \nabla \bar{v}\, dx = \int_\Gamma \mathbf{f} \cdot \nabla \bar{v}\, dx, \forall v \in V^{1,q}_{-\beta}(\Gamma), \tag{3.3}$$

where $q > 1$ is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore there exists $C > 0$ (independent of u and $\mathbf{f}$) such that

$$\|u\|_{V^{1,p}_\beta(\Gamma)} \le C \|\mathbf{f}\|_{L^p_\beta(\Gamma)^3}, \tag{3.4}$$

Proof. Assume that such a solution u exists. We perform the Euler change of variables $r = e^t$ that transform Γ into the strip $B = \mathbb{R} \times S$ and for any $v \in V^{1,q}_{-\beta}(\Gamma)$, set

$$U(t, \vartheta) = e^{\eta t} u(e^t, \vartheta), V(t, \vartheta) = e^{\delta t} v(e^t, \vartheta),$$

with $\eta = -(1-\frac{3}{p}-\beta)$ and $\delta = 1-\eta$. By the arguments of [14, p. 193] (easily extended to the L^p-setting), we deduce equivalently that U belongs to $W^{1,p}(B)$ and satisfies

$$\int_B \Big(\nabla_T U\cdot\nabla_T V \quad + \quad (\partial_t U-\eta U)(\partial_t\bar V-\delta\bar V)\Big)\,d\sigma \tag{3.5}$$
$$= \quad \int_B (\mathbf{h}_T(t,\vartheta)\cdot\nabla_T\bar V + h_r(t,\vartheta)(\partial_t\bar V-\delta\bar V))\,d\sigma, \forall V\in W^{1,q}(B),$$

where we have set

$$h_r(t,\vartheta) = e^{(2-\delta)t}\mathbf{f}(e^t,\vartheta)\cdot\vartheta, \quad \mathbf{h}_T(t,\vartheta) = e^{(2-\delta)t}\mathbf{f}(e^t,\vartheta) - h_r(t,\vartheta)\vartheta,$$

which, respectively, belong to $L^p(B)$ and $\mathbf{L}^p_T(B) := \{\mathbf{h}\in L^p(B)^3 : \mathbf{h}\cdot\vartheta = 0\}$ with

$$\|\mathbf{h}_T\|_{\mathbf{L}^p_T(B)} + \|h_r\|_{L^p(B)} \le C\|\mathbf{f}\|_{L^p_\beta(\Gamma)^3}, \tag{3.6}$$

for some positive constant C. Hence it suffices to show that (3.5) has a unique solution $U\in W^{1,p}(B)$ for $(\mathbf{h}_T,h_r)\in\mathbf{L}^p_T(B)\times L^p(B)$.

Performing a formal Fourier transform $\mathcal{F}$ in t, we see that $\hat U(\xi,\vartheta) = (\mathcal{F}_{t\to\xi}U(\cdot,\vartheta))(\xi)$ satisfies

$$\int_S (\nabla_T\hat U\cdot\nabla_T\varphi + (i\xi-\eta)(-i\xi-\delta)\hat U\bar\varphi)\,d\sigma =$$
$$\int_S \Big(\hat{\mathbf{h}}_T(\xi,\vartheta)\cdot\nabla_T\bar\varphi - \hat h_r(\xi,\vartheta)(i\xi+\delta)\bar\varphi\Big)\,d\sigma,$$

for all suitable test-functions φ. Comparing with (2.1), we see that formally $(\nabla_T\hat U, i\xi\hat U) = \mathcal{M}_{\delta,\eta}(\xi)(\hat{\mathbf{h}}_T,\hat h_r)$. To be more correct, recalling the definition (2.6), as our assumption (3.2) guarantees that the pair (δ,η), defined above, satisfies (2.2), by Theorem 2.5, the sets $\{\mathcal{M}_{\delta,\eta}(\xi) : \xi\in\mathbb{R}\}$ and $\{\xi\mathcal{M}'_{\delta,\eta}(\xi) : \xi\in\mathbb{R}\}$ are $\mathcal{R}$-bounded on $L^p(S)$. Consequently by the vector-valued multiplier Theorem (see [29, Theorem 3.4]), we deduce that $\mathcal{F}^{-1}\mathcal{M}_{\delta,\eta}(\cdot)\mathcal{F}$ defines a bounded linear operator on $L^p(\mathbb{R};X_{p,1})$ into itself. Consequently the vector-valued function $(\mathbf{u},v) = (\mathcal{F}^{-1}\mathcal{M}_{\delta,\eta}\mathcal{F})(\mathbf{h}_T,h_r)$ belongs to $L^p(\mathbb{R};L^p_T(S)\times L^p(S))$. By taking a sequence of functions $\varphi_n\in\mathcal{D}(\mathbb{R},X_{p,1})$ that converges to $(\mathbf{h}_T,h_r)$, using Corollary 2.2, Remark 2.3 and passing to the limit the next results can be obtained. There exists $U\in W^{1,p}(B)$ such that

$$\mathbf{u} = \nabla_T U, v = i\partial_t U,$$

with

$$\|U\|_{W^{1,p}(B)} \le C(\|\mathbf{h}_T\|_{\mathbf{L}^p_T(B)} + \|h_r\|_{L^p(B)}),$$

for some $C>0$. Furthermore it will satisfy (3.5) since $\mathcal{D}(\mathbb{R};W^{1,q}(S))$ is dense in $W^{1,q}(B)$. Finally as

$$\|u\|_{V^{1,p}_\beta(\Gamma)} \le C\|U\|_{W^{1,p}(B)},$$

for some $C>0$, with (3.6) we conclude that (3.4) holds. ∎

Corollary 3.2. *Let $p \in (1,\infty)$ and $\beta \in \mathbb{R}$ satisfy* (3.2). *Then any $\mathbf{f} \in L^p_\beta(\Gamma)^3$ admits the Helmholtz decomposition*

$$\mathbf{f} = \mathbf{f}_0 + \nabla u,$$

with $\mathbf{f}_0 \in L^p_\beta(\Gamma)^3$ divergence free and $u \in V^{1,p}_\beta(\Gamma)$ such that

$$\|u\|_{V^{1,p}_\beta(\Gamma)} + \|\mathbf{f}_0\|_{L^p_\beta(\Gamma)^3} \leq C\|\mathbf{f}\|_{L^p_\beta(\Gamma)^3}, \tag{3.7}$$

for some positive constant C independent of $\mathbf{f}$.

Proof. Given $\mathbf{f} \in L^p_\beta(\Gamma)^3$, the previous Theorem furnishes $u \in V^{1,p}_\beta(\Gamma)$ solution of (3.3) and we conclude by setting

$$\mathbf{f}_0 = \mathbf{f} - \nabla u,$$

that is obviously divergence free by taking test-functions $v \in \mathcal{D}(\Gamma)$ in (3.3) and satisfies (3.7) owing to (3.4). ∎

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