Sporadic finite simple groups and block designs*

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Abstract

The purpose of this paper is to classify all pairs $(\mathcal{D}, G)$, where $\mathcal{D}$ is a non-trivial 2-(v, k, λ) design with $\lambda \leq 10$, and $G \leq \mathrm{Aut}(\mathcal{D})$ acts transitively on the set of blocks of $\mathcal{D}$ and primitively on the set of points of $\mathcal{D}$ with sporadic socle. We prove that there are exactly 15 such pairs $(\mathcal{D}, G)$.

1 Introduction

A 2-(v, k, λ) *design* is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ consisting of a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks. Every block is incident with k points, every point is incident with r blocks, and any two distinct points are incident with λ blocks. The number of blocks is conventionally denoted b and the parameters of such a design $\mathcal{D}$ are v, b, r, k, λ, but, since b and r may easily be determined from v, k and λ it is conventional to speak of a 2-(v, k, λ) *design*.

In particular, when $\mathcal{B}$ is the set of all k-subsets of $\mathcal{P}$, then $\mathcal{D}$ is a 2-$(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2})$ design, which is called *complete*. $\mathcal{D}$ is *non-trivial* if $b > 1$ and $2 < k < v$, and *symmetric* if $v = b$. A symmetric design is called a *projective plane*, a *biplane* and a *triplane* if $\lambda = 1, 2, 3$ respectively.

An *automorphism* of $\mathcal{D}$ is a permutation of the points of $\mathcal{D}$ that also permutes the blocks of $\mathcal{D}$. The *full automorphism group* of $\mathcal{D}$ is the group of all automorphisms of $\mathcal{D}$ and is denoted by $\mathrm{Aut}(\mathcal{D})$. If $G \leq \mathrm{Aut}(\mathcal{D})$, then G is called an *automorphism group* of $\mathcal{D}$. We say that G is *block-transitive* (resp. *point-transitive*)

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if G acts transitively on $\mathcal{B}$ (resp. $\mathcal{P}$) and *block-primitive* (resp. *point-primitive*) if G acts primitively on $\mathcal{B}$ (resp. $\mathcal{P}$).

By a *flag* (resp. *antiflag*) of $\mathcal{D}$ we mean an incident (resp. non-incident) pair (α, B), where α is a point of $\mathcal{D}$ and B is a block of $\mathcal{D}$. If $G \leq \mathrm{Aut}(\mathcal{D})$, then G is *flag-transitive* (resp. *antiflag-transitive*) if it is transitive on the set of flags (resp. antiflags) of $\mathcal{D}$.

Flag-transitive 2-designs with small λ have been widely studied. For the flag-transitive projective planes, Kantor [10] proved that either $\mathcal{D}$ is Desarguesian and $\mathrm{PSL}(3,n) \lhd G$, or G is a Frobenius group of odd order $(n^2+n+1)(n+1)$ and n^2+n+1 is prime. In [1], Buekenhout *et al.* classified flag-transitive 2-$(v,k,1)$ designs. The classification of flag-transitive symmetric 2-$(v,k,2)$ designs has almost been completed by Regueiro in a sequence of four papers, see [12, 13, 14, 15], but for the non-symmetric case, there is relatively less research. If G acts flag-transitively and point-primitively on a 2-(v,k,λ) symmetric design with $\lambda \leq 4$, then G is of affine or almost simple type [8, 12], and in particular, if G is almost simple, then $\mathrm{Soc}(G)$ cannot be a sporadic simple group [7, 8]. In 2005, Tian and Zhou completely classified flag-transitive point-primitive symmetric designs with sporadic socle [16].

Camina and Spiezia have proved in [5] that if G is an almost simple group which acts block-transitively on a 2-$(v,k,1)$ design, then $\mathrm{Soc}(G)$ cannot be a sporadic simple group. However, for block-transitive 2-(v,k,λ) designs with $\lambda > 1$, there are only a few known results. Recently the authors generalized the result of Liang and Zhou [11] by weakening the assumption of flag-transitivity. They have shown in [18] that the only block-transitive point-primitive 2-$(v,k,2)$ designs having an automorphism group with sporadic socle is the unique 2-$(176,8,2)$ design that admits the Higman–Sims simple group HS as an automorphism group.

In this paper we continue the study of block-transitive 2-(v,k,λ) designs with $\lambda \leq 10$ that admit an automorphism group with sporadic socle. Our main result is the following:

Theorem 1. *Let $\mathcal{D}$ be a non-trivial 2-(v,k,λ) design with $\lambda \leq 10$. If $\mathcal{D}$ has an automorphism group G that is block-transitive and point-primitive with $\mathrm{Soc}(G)$ a sporadic simple group, then one of the following holds:*

- (i) *$\mathcal{D}$ has parameters $(11,3,9)$, $(12,6,5)$, $(12,3,10)$, $(55,3,4)$, $(55,3,8)$, $(55,4,8)$, $(55,9,8)$, or $(55,6,10)$ and $G = \mathrm{M}_{11}$.*
- (ii) *$\mathcal{D}$ has parameters $(12,3,10)$ and $G = \mathrm{M}_{12}$.*
- (iii) *$\mathcal{D}$ has parameters $(22,6,5)$, $(176,5,4)$, or $(176,16,9)$ and $G = \mathrm{M}_{22}$.*
- (iv) *$\mathcal{D}$ has parameters $(22,6,5)$ and $G = \mathrm{M}_{22}\colon 2$.*
- (v) *$\mathcal{D}$ has parameters $(176,8,2)$ and $G = \mathrm{HS}$.*

Remark 1. (1) There are up to isomorphism two 2-$(176,16,9)$ designs admitting M_{22} as a block-transitive point-primitive automorphism group and A_7 as a point stabilizer. All the other designs mentioned above are unique up to isomorphism.

(2) The 2-$(11, 3, 9)$, 2-$(12, 6, 5)$, 2-$(12, 3, 10)$ and 2-$(55, 4, 8)$ designs with $G = \mathrm{M}_{11}$, 2-$(12, 3, 10)$ design with $G = \mathrm{M}_{12}$, 2-$(22, 6, 5)$ and 2-$(176, 16, 9)$ designs with $G = \mathrm{M}_{22}$, 2-$(22, 6, 5)$ design with $G = \mathrm{M}_{22} : 2$ are flag-transitive.

(3) The 2-$(11, 3, 9)$ and 2-$(12, 6, 5)$ designs with $G = \mathrm{M}_{11}$, 2-$(12, 3, 10)$ design with $G = \mathrm{M}_{12}$, 2-$(22, 6, 5)$ design with $G = \mathrm{M}_{22}$, 2-$(22, 6, 5)$ design with $G = \mathrm{M}_{22} : 2$ are antiflag-transitive.

(4) The 2-$(11, 3, 9)$ and 2-$(12, 3, 10)$ designs are complete.

(5) The two 2-$(22, 6, 5)$ designs are isomorphic.

(6) From Theorem 1, we deduce that there is no block-transitive point-primitive symmetric 2-(v, k, λ) design with $\lambda \leq 10$ admitting an automorphism group with sporadic socle. This generalizes the result in [6] to non-symmetric designs.

2 Preliminaries

In this section we collect some basic results that will be used throughout the proof of Theorem 1.

Lemma 2.1. [2] *Let $\mathcal{D}$ be a non-trivial 2-(v, k, λ) design. If $G \leq \mathrm{Aut}(\mathcal{D})$ acts block-transitively on $\mathcal{D}$, then G acts point-transitively on $\mathcal{D}$.*

Lemma 2.2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a non-trivial 2-(v, k, λ) design, $G \leq \mathrm{Aut}(\mathcal{D})$ and $B \in \mathcal{B}$. Then the block-length k can be written as a sum of orbit-lengths of G_B on $\mathcal{B}$.*

Proof. G_B leaves B invariant, and so it partitions the set of points of B into point-orbits. ∎

The following lemma is an immediate consequence of the definitions.

Lemma 2.3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a non-trivial 2-(v, k, λ) design, $G \leq \mathrm{Aut}(\mathcal{D})$ and $B \in \mathcal{B}$. Then*

(1) *G is flag-transitive if and only if G is block-transitive and G_B is transitive on B;*

(2) *G is antiflag-transitive if and only if G is block-transitive and G_B is transitive on $\mathcal{P} \setminus B$.*

3 Proof of Theorem 1

Given a pair (v, λ) of integers where $v > 2$ and $3 \leq \lambda \leq 10$, there are only finitely many pairs (k, b) of integers such that there exists a 2-(v, k, λ) design with b blocks admitting a block-transitive and point-primitive automorphism group

G. The parameters of a non-trivial block-transitive 2-design have the following properties:

$$k-1 \mid \lambda(v-1); \tag{1}$$
$$2 < k < v \leq b; \tag{2}$$
$$b = \frac{\lambda v(v-1)}{k(k-1)} \in \mathbb{N}; \tag{3}$$
$$b \mid |G|. \tag{4}$$

Let $\mathcal{D}$ be a 2-(v,k,λ) design with parameter set (v,b,r,k,λ), and let $G \leq \mathrm{Aut}(\mathcal{D})$ be block-transitive and point-primitive, such that $\mathrm{Soc}(G)$ is a sporadic simple group. Since the cases in which $\lambda \leq 2$ are treated in [5] and [18], we assume that $3 \leq \lambda \leq 10$. We prove Theorem 1 in two steps: we first apply properties (1)-(4) and obtain 553 possible parameter sets (v,b,r,k,λ), then we analyze these potential parameter sets further and construct 15 pairs $(\mathcal{D},G)$ up to isomorphism.

3.1 Potential 2-(v,k,λ) designs

Let S be an arbitrary sporadic simple group. Then $|\mathrm{Out}(S)| = 1$ or 2 from 'The online ATLAS' [17], and so $G = S$ or $S:2$.

Since G is point-transitive (Lemma 2.1), G contains a subgroup G_α, the stabilizer of a point α, with index v. Since G is block-transitive, G contains a subgroup G_B, the stabilizer of a block B, with index b. As G is point-primitive, G_α must be maximal in G. For all sporadic simple groups, except the Monster M, the complete list of maximal subgroups is given in [17]. Therefore, for each such group, we can find the possible values for $|G_\alpha|$, and consequently for v. For a fixed pair (v,λ), if no k and b satisfy properties (1)-(4), we can exclude this pair. Otherwise, for potential values of k and b, we will construct a 2-(v,k,λ) design with b blocks or prove that there is no such design.

In this subsection, apart from the Monster M, we have examined all the other 25 sporadic simple groups with the aid of the computer algebra system GAP [9]. When G is one of the following groups

$$\mathrm{Co}_1,\ \mathrm{Suz},\ \mathrm{Suz}:2,\ \mathrm{He},\ \mathrm{He}:2,\ \mathrm{HN},\ \mathrm{HN}:2,$$
$$\mathrm{Th},\ \mathrm{Fi}_{22},\ \mathrm{Fi}_{22}:2,\ \mathrm{Fi}_{23},\ \mathrm{B},\ \mathrm{O'N}:2,\ \mathrm{J}_4,\ \mathrm{Ly}$$

there is no parameter set (v,b,r,k,λ) satisfying properties (1)-(4) and the other sporadic simple groups and small 'almost simple' extensions of sporadic simple groups yield 553 parameter sets (v,b,r,k,λ) altogether. In Table 1 we only list 4 such groups: M_{22}, Co_2, $\mathrm{J}_3:2$, and Ru, for which we get 57 potential parameter sets (v,b,r,k,λ) ($\mathrm{M}_{22}:2$ is listed in Table 2, and as the parameter sets of J_3 can be excluded like those of groups listed in Table 1, there is no need to list the group J_3). Among them 54 parameter sets are ruled out and up to isomorphism 4 designs are constructed. Except for the Monster M which will be dealt with separately in Lemma 3.7, the remaining automorphism groups and 496 parameter sets are discussed by using the same methods and 10 other designs are constructed. We omit the proof and list the 10 designs in Table 2 at the end of the paper.

Table 1: Potential 2-designs with $G = \mathrm{M}_{22}$, Co_2, $\mathrm{J}_3 : 2$ and Ru

Case	G	G_α	(v, b, r, k, λ)	Lemma
1	M_{22}	$\mathrm{L}_3(4)$	$(22, 308, 42, 3, 4)$	3.1
2			$(22, 154, 28, 4, 4)$	3.1
3			$(22, 44, 14, 7, 4)$	3.1
4			$(22, 33, 12, 8, 4)$	3.1
5			$(22, 77, 21, 6, 5)$	$\mathcal{D}_1$
6			$(22, 462, 63, 3, 6)$	3.2
7			$(22, 231, 42, 4, 6)$	3.2
8			$(22, 66, 21, 7, 6)$	3.1
9			$(22, 616, 84, 3, 8)$	3.2
10			$(22, 308, 56, 4, 8)$	3.1
11			$(22, 88, 28, 7, 8)$	3.1
12			$(22, 66, 24, 8, 8)$	3.1
13			$(22, 770, 105, 3, 10)$	3.2
14			$(22, 385, 70, 4, 10)$	3.1
15			$(22, 154, 42, 6, 10)$	3.1
16			$(22, 110, 35, 7, 10)$	3.1
17			$(22, 42, 21, 11, 10)$	3.1
18			$(22, 22, 15, 15, 10)$	3.2
19		$2^4 : \mathrm{A}_6$	$(77, 77, 20, 20, 5)$	3.2
20			$(77, 154, 38, 19, 9)$	3.1
21			$(77, 154, 40, 20, 10)$	3.1
22		A_7	$(176, 3080, 105, 6, 3)$	3.2
23			$(176, 385, 35, 16, 3)$	3.1
24			$(176, 6160, 175, 5, 4)$	$\mathcal{D}_2$
25			$(176, 1120, 70, 11, 4)$	3.1
26			$(176, 6160, 210, 6, 6)$	3.2
27			$(176, 1680, 105, 11, 6)$	3.1
28			$(176, 880, 75, 15, 6)$	3.1
29			$(176, 770, 70, 16, 6)$	3.2
30			$(176, 12320, 350, 5, 8)$	3.3
31			$(176, 2240, 140, 11, 8)$	3.1
32			$(176, 9240, 315, 6, 9)$	3.4
33			$(176, 3080, 175, 10, 9)$	3.2
34			$(176, 1155, 105, 16, 9)$	$\mathcal{D}_3$, $\mathcal{D}_4$
35			$(176, 220, 45, 36, 9)$	3.1
36		$2^4 : \mathrm{S}_5$	$(231, 385, 40, 24, 4)$	3.1
37			$(231, 770, 80, 24, 8)$	3.2
38			$(231, 231, 46, 46, 9)$	3.2
39		$2^3 : \mathrm{L}_3(2)$	$(330, 385, 56, 48, 8)$	3.1
40		M_{10}	$(616, 660, 45, 42, 3)$	3.1
41			$(616, 880, 60, 42, 4)$	3.1
42			$(616, 1320, 90, 42, 6)$	3.1

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43			$(616, 1540, 105, 42, 7)$	3.2
44			$(616, 1848, 123, 41, 8)$	3.1
45			$(616, 1760, 120, 42, 8)$	3.1
46			$(616, 1980, 135, 42, 9)$	3.1
47	Co_2	$2^{10}\colon \mathrm{M}_{22}\colon 2$	$(46575, 512325, 1606, 146, 5)$	3.1
48			$(46575, 1024650, 3212, 146, 10)$	3.2
49	$\mathrm{J}_3\colon 2$	$3^{2+1+2}\colon 8.2$	$(25840, 261630, 891, 88, 3)$	3.5
50			$(25840, 348840, 1188, 88, 4)$	3.5
51			$(25840, 26163, 324, 320, 4)$	3.5
52			$(25840, 523260, 1782, 88, 6)$	3.5
53			$(25840, 87210, 783, 232, 7)$	3.2
54			$(25840, 697680, 2376, 88, 8)$	3.5
55			$(25840, 52326, 648, 320, 8)$	3.5
56			$(25840, 784890, 2673, 88, 9)$	3.5
57	Ru	${}^2\mathrm{F}_4(2)$	$(4060, 4872, 198, 165, 8)$	3.6

Remark 2. In each case, the last column of Table 1 indicates that we rule it out by the corresponding lemma in subsection 3.2, and the symbols $\mathcal{D}_i$, $i = 1, 2, 3, 4$ refer to the four designs which are constructed in Lemma 3.8.

3.2 Analyzing the parameters and the corresponding groups

Let B be a block. Then

(1) G has at least one subgroup of index b;

(2) The block-length k can be written as a sum of orbit-lengths of G_B on $\mathcal{P}$;

(3) $|B^G| = b$.

First, Lemmas 3.1-3.6 below exclude some cases (except $G = \mathrm{M}$): Lemmas 3.1-3.3 rule out the cases that contradict (1), (2), (3) respectively, while Lemmas 3.4, 3.5 rule out the cases that contradict the definition of a design, and the proof of Lemma 3.6 only relates to the order of maximal subgroups of Ru. Next, Lemma 3.7 deals with the Monster group M. Finally, Lemma 3.8 deals with the remaining three cases, and we construct three designs.

The commands mentioned in the proof below are performed by the computer algebra system MAGMA [3].

Lemma 3.1. *Cases 1-4, 8, 10-12, 14-17, 20, 21, 23, 25, 27, 28, 31, 35, 36, 39-42 and 44-47 cannot occur.*

Proof. As an example, we analyze Case 1, the other cases being similar. Given a degree v in the required range and a positive integer i, the command `PrimitiveGroup(v,i)` returns the i-th primitive group of degree v. Also returns a string (possibly empty) giving a description of the group and a string giving the type of the group in the O'Nan–Scott classification of finite primitive permutation groups. `G:=PrimitiveGroup(22, 1)` returns the primitive group M_{22} of almost simple type, acting on a set of cardinality 22. Since G is block-transitive, $|G : G_B| = b$. Applying the command `Subgroups(G:OrderEqual:=n)` which lists all subgroups of G of order n in this case, we find that such a subgroup G_B does not exist when $n = 443520/308 = 1440$. So Case 1 cannot occur. ∎

Lemma 3.2. *Cases 6, 7, 9, 13, 18, 19, 22, 26, 29, 33, 37, 38, 43, 48 and 53 cannot occur.*

Proof. In Case 26, $G = M_{22}$, $b = 6160$, so $|G_B| = 72$. `Subgroups(G:OrderEqual:=n)` where $n = 72$ shows that there are two conjugacy classes of subgroups with index 6160, whose representatives are denoted by L_1 and L_2. The commands `O:=Orbits(L)` where $L = L_1, L_2$, and `♯O[j]` ($j = 1, 2, 3, 4$) show that the 4 smallest orbit-lengths are 2, 12, 18^2 for L_1, 1, 3, 4 and 12 for L_2, where a^b means that the orbit-length a appears b times. Since $k = 6$ is not the sum of any orbit-lengths of L_1 or L_2, this case is ruled out by Lemma 2.2.

Sometimes v or $|G|$ is so big that the ensuing computation is likely to run out of memory. We deal with this situation by using some group-theoretic properties of G. Consider Case 48 for example: `M:=PermRepKeys(Co2)` gives the permutation representations of the group Co_2 of degrees 2300 and 4600 only. Since $v = 46575$, we will construct the permutation representation of Co_2 of degree 46575. Let G be its representation of degree 2300 (the other representation yields identical results). It is easy to find all maximal subgroups of G, and we only choose one, say H, with order $|G|/46575 = 908328960$. The command `F1,N:=CosetAction(G,H)`, which constructs the permutation representation of G acting on the set of right cosets of H in G, yields a group N which is the permutation representation of Co_2 of degree 46575. To avoid memory overflow, we have to get the subgroups of G with index $b = 1024650$ from its maximal subgroups or even from its 2-maximal subgroups. Using this method, we get two such subgroups of G, say H_1 and H_2. Since G and N are isomorphic and have two generators, the command `F:=hom<G->N|G.1->N.1,G.2->N.2>` gives a map from generators of G to generators of N. Commands `N1:=PermutationGroup<46575|F(H1)>` and `N2:= PermutationGroup<46575|F(H2)>` yield the subgroups of N with index $b = 1024650$, namely N_1 and N_2. The 3 smallest orbit-lengths of N_1 and N_2 are 15, 120, 840 and 1, 42, 420 respectively. But $k = 146$ contradicts Lemma 2.2.

The other cases can be ruled out similarly. ∎

Lemma 3.3. *Case 30 cannot occur.*

Proof. $G = M_{22}$ has 4 conjugacy classes of subgroups of index $b = 12320$, namely H_1, H_2, H_3 and H_4. The 4 smallest orbit-lengths are 2, 6^2 and 9 for H_1, 1^2, 6^2 for H_2, 2, 6^2, 18 for H_3, and 1, 3, 4, 12 for H_4. As $k = 5$, we only need to consider H_4. Denote the orbits of H_4 of lengths 1 and 4 by Δ_1 and Δ_2 respectively, and let $B = \Delta_1 \cup \Delta_2$. Then B is a potential block. However, the block orbit-length $|B^G| = 6160 \neq 12320 = b$, contradicting the fact that G is block-transitive. ∎

Lemma 3.4. *Case 32 cannot occur.*

Proof. M_{22} has two inequivalent but isomorphic transitive permutation representations of degree 176. Since these two representations always yield identical results, we only choose one to check. It is easy to get 9 conjugacy classes of subgroups of index $b = 9240$ for each permutation representation of $G = M_{22}$ acting on 176 points. After calculating the orbit lengths of the 9 subgroups, we first rule out 8 subgroups by Lemma 2.2. The 6 smallest orbit-lengths of the remaining subgroup are 2^2, 4, 6^2 and 12. Denote the first 5 orbits by Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5, respectively. Then $B_1 = \Delta_1 \cup \Delta_3$, $B_2 = \Delta_2 \cup \Delta_3$, $B_3 = \Delta_4$, $B_4 = \Delta_5$ can be regarded as blocks, and $|B_i^G| = 9240 = b$ for $i = 1, 2, 3, 4$. However, the command `Design<2,v|C>` shows that the structure is not a 2-design when $v = 176$, $C = B_i^G (i = 1, 2, 3, 4)$, a contradiction. ∎

Lemma 3.5. *Cases 49-56 cannot occur.*

Proof. Suppose that Case 56 holds, we first construct the action of $G = J_3 : 2$ on 25840 points and get 7 conjugacy classes of subgroups of index $b = 784890$ by using the methods of Case 48 in Lemma 3.2. The smallest orbit-lengths of the 7 conjugacy classes of subgroups are 16^2, 8, 16^4, respectively. As $|G_B|$ =128, all the orbit-lengths of G_B are divisors of 128. But $k = 88$ is not the sum of any such divisors bigger than 8. The only possible G_B is the third conjugacy class with the 69 smallest orbit-lengths being 8^2, 16^4, 32^{15}, 64^{47}, and 128. As an example, we let $B = O_2 \cup O_3 \cup O_{22}$, where O_i denotes the i-th orbit. It is easy to find two points which are not contained in $\lambda = 9$ blocks of B^G, contradicting the definition of a design. Actually, there is no need to examine all possible blocks one by one. Instead, we write a program in MAGMA to check these possible blocks simultaneously. ∎

Lemma 3.6. *Case 57 cannot occur.*

Proof. In this case, $G = \mathrm{Ru}$, $|G_B| = 29952000$. However, Ru has no maximal subgroup whose order is divisible by 29952000 (see [17, p.126]), a contradiction. ∎

Lemma 3.7. *G is not* M.

Proof. Assume that $G = \mathrm{M}$. The complete list of maximal subgroups is not yet available for the Monster group M. 43 maximal subgroups of M are given in [17], but none of them gives rise to a set of parameters satisfying properties (1)-(4). By [4], each maximal subgroup H of M which is not listed in [17] is almost simple with $\mathrm{Soc}(H)$ isomorphic to $L_2(13)$, $U_3(4)$, $U_3(8)$ or $\mathrm{Sz}(8)$, and these cases are easily ruled out by the methods in Subsection 3.1. ∎

Lemma 3.8. *Cases 5, 24 and 34 can occur.*

Proof. **Case 5**: $G = \mathrm{M}_{22}$ and $(v, b, r, k, \lambda) = (22, 77, 21, 6, 5)$. First we get the unique permutation representation of $G = \mathrm{M}_{22}$ acting on 22 points by `G:=PrimitiveGroup(22,1)`, and $G = \langle g_1, g_2, g_3 \rangle$ where

$$\begin{aligned}
g_1 &= (2,14,18,20,8)(3,7,12,13,19)(4,21,17,15,10)(5,11,16,6,9),\\
g_2 &= (1,22)(2,10)(3,14)(4,17)(8,15)(9,11)(13,20)(19,21),\\
g_3 &= (1,7,12,16,19,21,6)(2,8,13,17,20,5,11)(3,9,14,18,4,10,15).
\end{aligned}$$

There exists only one conjugacy class of subgroups with index 77, with representative H. The orbit-lengths of H on $\mathcal{P}$ are 6 and 16. If Δ denotes the orbit of length 6, then $|\Delta^G| = 77 = b$ and Δ can be regarded as a block and `Design<2,22|C>` where $C = \Delta^G$ returns a 2-(22, 6, 5) design with 77 blocks, denoted by $\mathcal{D}_1$. The basic block is

$$\Delta = \{6, 8, 10, 11, 16, 17\}.$$

Clearly, $\mathrm{M}_{22} \leq \mathrm{Aut}(\mathcal{D}_1)$ is block-transitive and point-primitive. Since H has exactly two orbits, which means that H acts transitively on both Δ and $\mathcal{P} \backslash \Delta$, we conclude that M_{22} is both flag-transitive and antiflag-transitive on $\mathcal{D}_1$ by Lemma 2.3.

Case 24: $G = \mathrm{M}_{22}$ and $(v, b, r, k, \lambda) = (176, 6160, 175, 5, 4)$. For one of the two transitive permutation representations of degree 176, $G = \langle g_1, g_2 \rangle$ where

$$\begin{aligned}
g_1 = &(1,2)(3,5)(4,7)(6,10)(8,13)(9,15)(11,17)(12,18)(14,21)(16,23)(19,27)(20,29)\\
&(22,32)(24,34)(25,36)(28,40)(30,42)(33,46)(35,49)(37,51)(38,53)(39,54)(41,57)\\
&(43,60)(44,61)(45,62)(47,65)(48,66)(52,71)(55,74)(56,76)(59,64)(63,83)(67,80)\\
&(68,88)(69,90)(70,92)(72,94)(73,81)(75,97)(77,99)(78,101)(79,87)(84,104)(85,105)\\
&(86,107)(93,102)(95,115)(96,117)(98,119)(103,111)(106,124)(108,126)(109,116)\\
&(110,129)(112,131)(113,132)(114,134)(118,135)(120,137)(121,139)(122,141)\\
&(123,143)(125,145)(127,130)(128,146)(133,140)(136,152)(138,149)(142,158)\\
&(147,160)(148,161)(150,163)(151,164)(153,159)(154,162)(156,167)(165,171)\\
&(169,170)(173,174),
\end{aligned}$$

$$\begin{aligned}
g_2 = &(1,3,6,11)(2,4,8,14)(5,9)(7,12,19,28)(10,16,24,35)(13,20,30,43)(15,22,33,47)\\
&(17,25,37,52)(18,26,38,23)(21,31,44,60)(27,39,55,75)(29,41,58,79)(32,45,63,84)\\
&(34,48,67,87)(36,50,69,91)(40,56,77,100)(42,59,80,83)(46,64,85,106)(49,68,\\
&89,110)(51,70)(53,72,95,116)(54,73,96,99)(57,78,102,66)(61,81,103,115)(62,82)\\
&(65,86,108,127)(71,93,113,133)(74,90,111,130)(76,98,120,138)(88,109,128,147)\\
&(92,112)(94,114)(97,118,136,153)(101,121,140,156)(104,122,142,143)(105,123)\\
&(107,125,132,150)(117,134,139,155)(119,124,144,126)(129,148,162,146)(131,149,\\
&141,157)(135,151,165,172)(137,154,166,173)(145,159,168,171)(152,161,167,174)\\
&(158,164)(160,169)(163,170,175,176).
\end{aligned}$$

There are two conjugacy classes of subgroups with index 6160, whose representatives are denoted by S_1 and S_2. The 4 smallest orbit-lengths are 2, 12, 18^2 for S_1, and 1, 3, 4, 12 for S_2. Denote the two orbits of lengths 1 and 4 of S_2 by Δ_1 and Δ_2, and let $B = \Delta_1 \cup \Delta_2$. The command `Design<2,176|C>` where $C = B^G$ returns a 2-$(176, 5, 4)$ design with 6160 blocks, denoted by $\mathcal{D}_2$, with basic block

$$B = \{1, 33, 39, 92, 167\}.$$

For the other transitive permutation representation of degree 176, we also get a design, isomorphic to $\mathcal{D}_2$.

It is obvious that $\mathrm{M}_{22} \leq \mathrm{Aut}(\mathcal{D}_2)$ is block-transitive and point-primitive. However, as S_2 is not transitive on B or $\mathcal{P}\backslash B$, M_{22} is neither flag-transitive nor antiflag-transitive on $\mathcal{D}_2$.

Case 34: $G = \mathrm{M}_{22}$ and $(v, b, r, k, \lambda) = (176, 1155, 105, 16, 9)$. For one of the two transitive permutation representations of degree 176, we get two conjugacy classes of subgroups with index 1155, whose representatives are denoted by H_1 and H_2. The 3 smallest orbit-lengths are $16^2, 32$ for H_1, and $16^2, 48$ for H_2. Denote the two orbits of length 16 of H_1 by Δ_1, Δ_2, and those of length 16 of H_2 by Δ_3 and Δ_4, respectively. We have $|\Delta_1^G| = |\Delta_3^G| = |\Delta_4^G| = 1155 = b$, $|\Delta_2^G| = 231$, so Δ_1, Δ_3 and Δ_4 can be regarded as blocks and the command `Design<2,176|C>` where $C = \Delta_1^G$ shows that the structure is not a 2-design, but when $C = \Delta_3^G$ and Δ_4^G we get two non-isomorphic 2-$(176, 16, 9)$ designs with 1155 blocks, denoted by $\mathcal{D}_3$ and $\mathcal{D}_4$. Furthermore, the basic blocks Δ_3 and Δ_4 are

$$\Delta_3 = \{6, 17, 23, 28, 43, 61, 73, 84, 85, 87, 88, 90, 112, 118, 150, 165\},$$
$$\Delta_4 = \{19, 30, 50, 51, 63, 67, 72, 79, 93, 104, 105, 121, 135, 139, 157, 158\}.$$

For the other transitive permutation representation of degree 176, we also get two 2-$(176, 16, 9)$ designs, isomorphic to $\mathcal{D}_3$ and $\mathcal{D}_4$ respectively. Obviously, $\mathrm{M}_{22} \leq \mathrm{Aut}(\mathcal{D}_3)$ and $\mathrm{M}_{22} \leq \mathrm{Aut}(\mathcal{D}_4)$ are both flag-transitive and point-primitive, but not antiflag-transitive. ■

Similarly, we can construct 10 other designs admitting a block-transitive point-primitive automorphism group G from the remaining 496 cases. They are listed with their automorphism groups, point stabilizers, parameters, and basic blocks in Table 2.

Table 2: Ten 2-designs

No.	G	G_α	(v, k, λ)	Basic block
1	M_{11}	M_{10}	$(11, 3, 9)$	$\{1, 4, 11\}$
2		$\mathrm{L}_2(11)$	$(12, 6, 5)$	$\{1, 4, 6, 8, 9, 11\}$
3			$(12, 3, 10)$	$\{3, 4, 10\}$
4		$\mathrm{M}_9 : 2$	$(55, 3, 4)$	$\{15, 37, 49\}$
5			$(55, 3, 8)$	$\{25, 27, 52\}$
6			$(55, 4, 8)$	$\{3, 8, 12, 55\}$
7			$(55, 9, 8)$	$\{1, 6, 10, 11, 20, 26, 32, 46, 47\}$
8			$(55, 6, 10)$	$\{2, 11, 28, 31, 47, 49\}$
9	M_{12}	M_{11}	$(12, 3, 10)$	$\{4, 6, 11\}$
10	$\mathrm{M}_{22} : 2$	$\mathrm{L}_3(4) : 2_2$	$(22, 6, 5)$	$\{1, 2, 6, 10, 12, 22\}$

Remark 3. Both $\mathrm{M}_{22}: 2$ and M_{22} can act block-transitively and point-primitively on a 2-$(22, 6, 5)$ design. Although the basic blocks of these two actions are different, the two 2-$(22, 6, 5)$ designs are isomorphic, as the command Γ `in` Δ^G returns true (where $G = \mathrm{M}_{22}$, $\Delta = \{6, 8, 10, 11, 16, 17\}$ and $\Gamma = \{1, 2, 6, 10, 12, 22\}$ are the corresponding basic blocks of M_{22} and $\mathrm{M}_{22}: 2$ respectively), which means that Γ lies in $\Delta^{\mathrm{M}_{22}}$.

This completes the proof of Theorem 1. ∎

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