New criteria for p-nilpotency of finite groups*

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Abstract

A subgroup H of a group G is said to be weakly s-supplementedly embedded in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se} \leq H$, where H_{se} is an S-permutably embedded subgroup of G. In this paper, we investigate the structure of G under the assumption that some subgroups of prime-power order are weakly S-supplementedly embedded in G, and some new criteria for p-nilpotency are obtained.

1 Introduction

Let G be a finite group. $|G|$ is the order of G, and $\pi(G) = \{p_1 > p_2 > \cdots > p_s\}$ is the set of prime divisors of $|G|$. For $p \in \pi(G)$, $\mathrm{Syl}_p(G)$ is the set of all Sylow p-subgroups of G, $O_p(G)$ is the maximal normal p-subgroup of G, and $O^p(G) = \langle Q \in Syl_q(G) \mid q \in \pi(G), q \neq p\rangle$. Let $[A]B$ denote the semidirect product of the groups A and B, where B is an operator group of A.
G is a Sylow-tower group if there exists a series: $1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_{s-1} \leq G_s = G$ such that $G_i \trianglelefteq G$ and $|G_{i+1}/G_i| = p_{i+1}^{\alpha_{i+1}}$, $i = 0, 1, \ldots, s-1$.
A subgroup H of G is $s$ubnormal in G if there exists a series: $H = H_1 \leq H_2 \leq \cdots \leq H_{s-1} \leq H_s = G$ such that $H_i \trianglelefteq H_{i+1}$, $i = 1, 2, \ldots, t-1$.

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A class of finite groups $\mathcal{F}$ is called a *f*ormation if the following conditions are satisfied:

(0) if $G \in \mathcal{F}$, then all groups isomorphic to G also belong to $\mathcal{F}$;

(1) if $G \in \mathcal{F}$ and N is a normal subgroup of G, then $G/N \in \mathcal{F}$;

(2) if N_i are normal subgroups of a group G (not necessarily belonging to $\mathcal{F}$ such that $G/N_i \in \mathcal{F}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in \mathcal{F}$.

Recall that the Frattini subgroup $\Phi(G) = \cap_{M \lessdot G} M$ of a group G is the intersection of all maximal subgroups of G. A formation $\mathcal{F}$ is *s*aturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. For example, the class of all p-nilpotent groups($p\mathcal{N}$) and the class of all supersolvable groups($\mathcal{U}$) are saturated formations.
All other notation and terminology is standard, following [7, 8].

Assume $\mathcal{F}$ is a class of groups and A/B is a chief factor of G. A/B is called *Frattini* provided $A/B \leq \Phi(G/B)$. Moreover, A/B is called *$\mathcal{F}$-central* if $[A/B](G/C_G(A/B)) \in \mathcal{F}$. Otherwise, A/B is called *$\mathcal{F}$-eccentric*. In 2009, Shemetkov and Skiba [15] introduced the concept of *$\mathcal{F}\Phi$-hypercentre* of G. The symbol $Z_{\mathcal{F}\Phi}(G)$ denotes the $\mathcal{F}\Phi$-hypercentre of G which is the product of all normal subgroups of G whose non-Frattini G-chief factors are $\mathcal{F}$-central in G. A (normal) subgroup E of G is called *$\mathcal{F}\Phi$-hypercentral* in G if $E \leq Z_{\mathcal{F}\Phi}(G)$. An important fact is that if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and $E \leq Z_{\mathcal{F}\Phi}(G)$, then $G \in \mathcal{F}$, for any *s*aturated formation $\mathcal{F}$. Especially, $G \leq Z_{\mathcal{F}\Phi}(G)$ is equal to the case that $G \in \mathcal{F}$.
Let $p \in \pi(G)$. Recall that a subgroup H of G is p-*l*ocal, if $H = N_G(S)$ for some nontrivial p-subgroup S of G. p-local subgroups play an important role in investigating the structure of finite groups. For example, Burnside's Theorem asserts that G is p-nilpotent if $N_G(P) = C_G(P)$ for some Sylow p-subgroup P of G and $p \in \pi(G)$. The following generalization of Burnside's Theorem is due to Hall [6]: if the p'-elements of $N_G(P)$ commute with the elements of P and the class size of P is smaller than p, then G is p-nilpotent. Huppert [7] showed that a group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent. The Frobenius Theorem asserts that G is p-nilpotent if and only if $N_G(S)$ is p-nilpotent, for every nontrivial p-subgroup S of G.
The idea behind these results (and other available in the literature) is to consider local properties of subgroups having prime-power order. The aim of this paper is to investigate whether it is possible to reduce the number of subgroups that is needed to characterize p-nilpotency.
Recall that a subgroup H of G is called *S-permutable* (or S-quasinormal or π-quasinormal) in G if $HP = PH$ for all Sylow subgroups P of G. H is called *S-permutably embedded* in G if each Sylow subgroup of H is a Sylow subgroup of some S-permutable subgroup of G.

Definition 1.1. A subgroup H of G is called *w*eakly S-supplementedly embedded in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se} \leq H$, where H_{se} is an S-permutably embedded subgroup of G.

Take $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$, and let P' be the derived subgroup of P. Let $\mathcal{H}(P) = \{H \leq P \mid P' \leq H \leq \Phi(P)\}$, and let $\mathcal{K}(P)$ be the set of subgroups $K \leq G$ such that K is p-closed and $\mathcal{H}(P)$ contains the Sylow p-subgroup of K. Obviously $\mathcal{H}(P) \subseteq \mathcal{K}(P)$ and each element in $\mathcal{H}(P)$ is normal in P.

Our main result consists of the following characterizations of hypercentre of a group G with normal subgroup E and Sylow p-subgroup P of E(see Theorems 3.1, 3.5 and 3.8): $E \leq Z_{p\mathcal{N}\Phi}(G)$ if there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(P)$ is p-nilpotent or if there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(H)$ is p-nilpotent or assume that $(|G|, p-1) = 1$ and if one of the following conditions is satisfied

(1) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly s-supplementedly embedded in $N_G(P)$;

(2) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order 4 if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in $N_G(P)$.

Further, we obtain the following characterizations of p-nilpotency of a group G with Sylow p-subgroup P: G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(P)$ is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(H)$ is p-nilpotent, see corollaries 3.2 and 3.9. These can be viewed as alternative versions of the Theorems of Burnside and Frobenius. In corollary 3.7, we give sufficient conditions for a group G to belong to a saturated formation that contains the class of all supersolvable groups.

2 Preliminaries

Lemma 2.1. *[9]*

(a) An S-permutable subgroup of G is subnormal in G.

(b) If $H \leq K \leq G$ and H is S-permutable in G, then H is S-permutable in K.

(c) Let $K \trianglelefteq G$. If H is S-permutable in G, then HK/K is S-permutable in G/K.

(d) If P is an S-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.

Lemma 2.2. *[1, Lemma 2.1] Suppose that U is S-permutably embedded in G, and that $H \leq G$ and $N \trianglelefteq G$.*

(a) If $U \leq H$, then U is S-permutably embedded in H.

(b) UN is S-permutably embedded in G and UN/N is S-permutably embedded in G/N.

Lemma 2.3. *[16, Lemma 2.5] Suppose that H is S-permutable in G, and that P is a Sylow p-subgroup of H, with $p \in \pi(G)$. If $H_G = 1$ or $P \leq O_p(G)$, then P is S-permutable in G.*

Lemma 2.4. *[11, Lemma 2.3] Let U be a weakly S-supplementedly embedded subgroup of G and N be a normal subgroup of G.*

(a) If $U \leq H \leq G$, then U is weakly S-supplementedly embedded in H.

(b) If $N \leq U$, then U/N is weakly S-supplementedly embedded in G/N.

(c) Let π be a set of primes, U a π-subgroup and N a π'-subgroup of G. Then UN/N is weakly S-supplementedly embedded in G/N.

Lemma 2.5. *Let P be a Sylow p-subgroup of G, with $p \in \pi(G)$. Assume that $K \leq G$, and let H be a Sylow p-subgroup of K such that $H \trianglelefteq K$ and $H \leq \Phi(P)$. If K is weakly S-supplementedly embedded in G, then H is S-permutably embedded in G.*

Proof. By hypothesis, there is a subgroup A of G and an S-permutably embedded subgroup K_{se} of G such that $G = KA$ and $K \cap A \leq K_{se} \leq K$. Since $H \trianglelefteq K$, there exists a Sylow p-subgroup P_1 of A such that $P = HP_1 \leq \Phi(P)P_1 \leq P_1$. Furthermore, $H \leq P \leq A$ and H is a Sylow p-subgroup of K_{se}. It follows from the definition of the S-permutably embedded subgroup that H is an S-permutably embedded subgroup of G. ∎

Lemma 2.6. *[12] Assume that P is a Sylow p-subgroup of G, with $p \in \pi(G)$, and that $N \trianglelefteq G$. If $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.7. *Assume that P is a normal Sylow p-subgroup of G, with $p \in \pi(G)$, and that $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. If every maximal subgroup of P or every cyclic subgroup of P with order p or order 4 (if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in G, then P is cyclic.*

Proof. Let P_1 be a maximal subgroup of P. If P_1 is weakly S-supplementedly embedded in G, then we claim that $P_1 \leq \Phi(P)$. Let T be a supplement of P_1 in G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se} \leq P_1$. Then $G = P_1T$ and $P = P \cap G = P \cap P_1T = P_1(P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ and $(P \cap T)\Phi(P) \trianglelefteq G$. Since $P/\Phi(P)$ is a minimal normal Sylow p-subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P \cap T = P$. If $P \cap T \leq \Phi(P)$, then $P = P_1(P \cap T) = P_1$, which is a contradiction. Now we assume that $P \cap T = P$. Then $P_1 \leq P_1 \cap T \leq (P_1)_{se} \leq O_p(G) = P$. Hence P_1 is S-permutable in G by Lemma 2.3. Then $P_1\Phi(P)/\Phi(P)$ is S-permutable in $G/\Phi(P)$ by Lemma 2.1(c) and so $N_{G/\Phi(P)}(P_1\Phi(P)/\Phi(P)) \geq O^p(G/\Phi(P))$. Furthermore, $P_1\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. By the minimality of $P/\Phi(P)$ as a normal subgroup of $G/\Phi(P)$ again, $P_1 \leq \Phi(P)$. Hence P has a unique maximal subgroup by the above argument, which implies that P is cyclic.
If every cyclic subgroup of P with order p (and order 4 if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in G, then we also have $|P/\Phi(P)| = p$ and then P is cyclic. Otherwise, let $K/\Phi(P)$ be any non-trivial cyclic subgroup of $P/\Phi(P)$. Let $x \in K \setminus \Phi(P)$ such that $T = \langle x \rangle \Phi(P)$. Then by the above argument, $\langle x \rangle \leq \Phi(P)$ and so $T = \Phi(P)$, which is a contradiction. ∎

Lemma 2.8. *Let P be a normal Sylow p-subgroup of G, with $p \in \pi(G)$, and assume that $(|G|, p-1) = 1$. Then the following assertions are equivalent:*

(1) G is p-nilpotent;

(2) every maximal subgroup of P is weakly S-supplementedly embedded in G;

(3) every cyclic subgroup of P of order p is weakly S-supplementedly embedded in G, and, in the situation where $p = 2$ and P is non-abelian, every cyclic subgroup of P of order 2 or 4 is weakly S-supplementedly embedded in G.

Proof. $\underline{(1) \Rightarrow (2)}$. If G is p-nilpotent, then G has a normal p-complement T. If P_1 is a maximal subgroup of P, then P_1T is normal in G since $|G : P_1T| = p$, and it follows that P_1 is weakly S-supplementedly embedded in G.
$\underline{(1) \Rightarrow (3)}$. Let P_1 be a cyclic subgroup of P of order p. It follows that P_1 is a Sylow p-subgroup of P_1T. Let Q be a Sylow q-subgroup of G, where $q \neq p$ is a prime divisor of $|G|$. Then $Q \leq T$ and $P_1TQ = QP_1T = P_1T$. By hypothesis, P is normal in G hence $P_1TP = PT$ is a subgroup of G, which implies that P_1T is an S-permutable subgroup of G and P_1 is S-permutably embedded in G. Hence P_1 is weakly S-supplementedly embedded in G. Similar arguments apply to cyclic subgroups of order 4 in the case where $p = 2$ and P is non-abelian.
$\underline{(3) \Rightarrow (1)}$. Assume that G is not p-nilpotent. This means that the class of non-p-nilpotent groups G with order relatively prime to $p - 1$ and containing P as a normal p-subgroup is not empty, and we can take such a group G with minimal order.
Let M be a proper subgroup of G. Then $P \cap M$ is a normal Sylow p-subgroup of M, and it follows from Lemma 2.4 that every cyclic subgroup of P with order p or order 4 is weakly S-supplementedly embedded in M and so M is p-nilpotent. By [14, VI, Theorem 24.2], $P/\Phi(P)$ is a G-chief factor of P. Now by Lemma 2.7, P is cyclic and it follows from Burnside's Theorem that G is p-nilpotent, which is a contradiction.
$\underline{(2) \Rightarrow (1)}$. Assume that G is not p-nilpotent. This means that the class of non-p-nilpotent groups G with order relatively prime to $p - 1$ and containing P as a normal p-subgroup is not empty, and we can take such a group G with minimal order.
Let N be a minimal normal subgroup of G contained in P. It is easy to see that G/N is p-nilpotent. By a routine argument, we have that $N = P$. It follows from Lemma 2.7 that G is p-nilpotent, which is a contradiction. ■

Lemma 2.9. *Let q is a prime divisor of $|G|$, and let Q be a normal Sylow q-subgroup of G such that G/Q is supersolvable. G is supersolvable if one of the two following conditions is satisfied:*

(1) every maximal subgroup of Q is weakly S-supplementedly embedded in G;

(2) every subgroup of Q of order q, and in the situation where $q = 2$ and Q is non-abelian, every subgroup of order 2 or 4 is weakly S-supplementedly embedded in G.

Proof. Assume that G is not supersolvable.
If (2) holds, then it follows from Lemma 2.8 that G is minimal non-supersolvable, in the sense that every proper subgroup of G is supersolvable. By [2], G has

a normal Sylow p-subgroup P such that $G = PM$, where M is a supersolvable maximal subgroup of G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. If $P \neq Q$, then $G \lesssim G/P \times G/Q$ is supersolvable, which is a contradiction. Hence $P = Q$. Now Q is cyclic by Lemma 2.7 and then G is supersolvable, which is a contradiction.
Assume that (1) holds, and let N be a minimal normal subgroup of G contained in Q. Assume that Q_1/N is a maximal subgroup of Q/N, then Q_1 is maximal in Q. By the hypothesis and Lemma 2.4, Q_1/N is weakly S-supplementedly embedded in G/N. So G/N satisfies the hypothesis of the Lemma and G/N is supersolvable by the choice of G. It follows that N is the unique minimal normal subgroup of G contained in Q and $N \nleq \Phi(G)$. Hence $N = Q$. It follows from Lemma 2.7 that Q is cyclic and G is supersolvable, which is a contradiction. ∎

Lemma 2.10. *Let G be a group and $P \in Syl_p(G)$ where $p \in \pi(G)$. If P is abelian and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Since $N_G(P)$ is p-nilpotent, $N_G(P) = P \times H$, where H is a normal p-complement of P in $N_G(P)$, and $H \leq C_G(P)$. On the other hand P is abelian and $P \leq C_G(P)$, hence $N_G(P) = C_G(P)$ and G is p-nilpotent by Burnside's Theorem. ∎

Lemma 2.11. *[5, Theorem 1.8.17] Let N be a nontrivial solvable normal subgroup of G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.12. *[3, Lemma A.9.11] Let K and N be the normal subgroups of G with $N \leq K$ and K is nilpotent. If $K/N \leq \Phi(G/N)$, then $K \leq \Phi(G)N$.*

Lemma 2.13. *[13, Lemma 2.4] Suppose that P is a p-subgroup of G contained in $O_p(G)$. If P is S-permutably embedded in G, then P is S-permutable in G.*

Lemma 2.14. *Suppose that R is a minimal normal subgroup of G and $R \leq O_p(G)$, where $p \in \pi(G)$. $|R| = p$ if one of the following conditions is satisfied*

(1) every maximal subgroup of P is S-supplementedly embedded in G;

(2) every cyclic subgroup of P of order p (and of order 4 if P is non-abelian and $p = 2$) is S-supplementedly embedded in G.

Proof. Since R is a minimal normal subgroup of G and $R \leq O_p(G)$, we may let R_1 be the maximal subgroup or cyclic subgroup of P of order p (and of order 4 if P is non-abelian and $p = 2$) of R such that $R_1 \trianglelefteq P$, where P is a Sylow p-subgroup of G. By the hypothesis, R_1 is S-permutably embedded in G and Lemma 2.13, then R_1 is S-permutable in G. Further, by Lemma 2.1, $N_G(R_1) \geq O^p(G)$. Then $N_G(R_1) = G$ and $R_1 \trianglelefteq G$ by the choice of R_1. Hence $R_1 = 1$ and $|R| = p$ since R is a minimal normal subgroup of G. ∎

3 Main Results

Theorem 3.1. *Let $E \trianglelefteq G$, $p \in \pi(E)$, and let P be a Sylow p-subgroup of E. If there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(P)$ is p-nilpotent, then $E \leq Z_{p\mathcal{N}\Phi}(G)$.*

Proof. Suppose that there exists G, E, P satisfying the conditions of the Theorem such that $E \nleq Z_{p\mathcal{N}\Phi}(G)$. Fixing P the class of all couples (G, E) satisfying the conditions of the Theorem such that $E \nleq Z_{p\mathcal{N}\Phi}(G)$ is not empty, and we can choose a (G, E) in such a way that $|G| + |E|$ is minimal. In several steps, we show that this leads to a contradiction.

Step 1. $O_{p'}(E) = 1$.

Now, we consider the couple $(\overline{G}, \overline{E}) = (G/O_{p'}(E), E/O_{p'}(E))$. Then $\overline{P} = PO_{p'}(E)/O_{p'}(E)$ is a Sylow p-subgroup of $\overline{E}$. Certainly, $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ and $(\overline{P})' \leq \overline{P'} \leq \overline{H} \leq \overline{\Phi(P)} \leq \Phi(\overline{P})$. It follows that $(\overline{P})' \leq \overline{H} \leq \Phi(\overline{P})$. Hence $\overline{H} \in \mathcal{H}(\overline{P})$. By Lemma 2.4, it is easy to see that $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the conditions of the Theorem, and $E/O_{p'}(E) \leq Z_{p\mathcal{N}\Phi}(G/O_{p'}(E))$ by the choice of (G, E). Further, $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction.

Step 2. $E = G$.

If $E < G$, then we consider the couple (E, E). By Lemma 2.4, (E, E) satisfies the conditions of the Theorem, and $E \leq Z_{p\mathcal{N}\Phi}(E)$ by the choice of (G, E). Further, E is p-nilpotent and $E = P \trianglelefteq G$ by Step 1. Then $N_G(P) = G$ is p-nilpotent and $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction.

Step 3. H is a non-trivial S-permutably embedded subgroup of G and G is not a non-abelian simple group.

It follows from Lemma 2.5 that H is an S-permutably embedded subgroup of G. If $H = 1$, then $P' \leq H = 1$ implies that P is abelian. It follows from Lemma 2.10 that G is p-nilpotent, which is a contradiction. Let A be an S-permutable subgroup of G such that H is a Sylow p-subgroup of A. Then $A \neq 1$. Since $H < P$ and $A < G$, A is a non-trivial subnormal subgroup of G, which implies that G is not a non-abelian simple group.

Step 4. G has a unique minimal normal subgroup N and G/N is p-nilpotent. Furthermore, $O_{p'}(G) = 1$ and $N \nleq \Phi(G)$.

Let N be a minimal normal subgroup of G, and consider the quotient group $\overline{G} = G/N$. Then $\overline{P} = PN/N$ is a Sylow p-subgroup of $\overline{G}$. Certainly, $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ and $(\overline{P})' \leq \overline{P'} \leq \overline{H} \leq \overline{\Phi(P)} \leq \Phi(\overline{P})$. It follows that $(\overline{P})' \leq \overline{H} \leq \Phi(\overline{P})$. Hence $\overline{H} \in \mathcal{H}(\overline{P})$. By Step 1 and Lemma 2.2, it is easy to see that G/N satisfies the hypothesis, and G/N is p-nilpotent by the choice of G. Obviously N is the unique minimal normal subgroup of G. Furthermore, $O_{p'}(G) = 1$ and $N \nleq \Phi(G)$.

Step 5. $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Then $N \leq O_p(G)$ and $N \cap \Phi(G) = 1$ by Step 4. It follows that $O_p(G) \cap \Phi(G) = 1$, and $N = O_p(G)$ by [10, Lemma 2.6].

Now we claim that $N \leq \Phi(P)$. Let A be an S-permutable subgroup of G such that H is a Sylow p-subgroup of A. If $A_G \neq 1$, then $O_p(G) = N \leq H \leq \Phi(P)$. If $A_G = 1$, then H is an S-permutable subgroup of G by Lemma 2.3. It follows from Lemma 2.1(d) that $O^p(G) \leq N_G(H)$ and so $G = PO^p(G) \leq N_G(H)$, which implies that $H \trianglelefteq G$. Hence either $H = 1$ or $N \leq H$. If $H = 1$, then $P' = 1$ and G is p-nilpotent by Lemma 2.10, a contradiction. Hence $N \leq H$ and it follows that $N \leq \Phi(P)$. Then $N \leq \Phi(G)$, which contradicts Step 4. So $O_p(G) = 1$.

Step 6. Final contradiction.

If $NP < G$, then NP satisfies the hypothesis and NP is p-nilpotent by the choice of G. Therefore N is p-nilpotent, which contradicts Step 5. Hence $G = NP$.

By Step 3, H is a Sylow p-subgroup of an S-permutable subgroup A of G. If $A_G = 1$, then by Lemma 2.3 H is S-permutable in G and so $H \leq O_p(G)$, which contradicts Step 5. So $A_G \neq 1$. It follows from the uniqueness of N that $N \leq A_G \leq A$ and so $H \cap N$ is a Sylow p-subgroup of N. Since $P \cap N$ is also a Sylow p-subgroup of N and $H \cap N \leq P \cap N$, $P \cap N = H \cap N \leq H \leq \Phi(P)$. By Lemma 2.6, N is p-nilpotent, which contradicts Step 5. ∎

Corollary 3.2. *Let $p \in \pi(G)$, and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(P)$ is p-nilpotent.*

Proof. The sufficency follows easily from Theorem 3.1. Next, we consider the necessity.

If G is p-nilpotent, then $N_G(P)$ is p-nilpotent and G has a normal p-complement T such that $G = PT$. It follows that $P'T$ is normal in G and P' is a Sylow p-subgroup of $P'T$, which implies that P' is a weakly S-supplementedly embedded subgroup of G. It is obvious that $P' \in \mathcal{H}(P)$. ∎

Corollary 3.3. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ satisfying $(|G|, p-1) = 1$. The following statements are equivalent*

(1) G is p-nilpotent;

(2) there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly s-supplementedly embedded in $N_G(P)$;

(3) there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p and of order 2 or 4 is weakly S-supplementedly embedded in $N_G(P)$.

Proof. This result follows from Lemma 2.8 and corollary 3.2. ∎

Corollary 3.4. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ satisfying $(|G|, p-1) = 1$. The following statements are equivalent*

(1) G is p-nilpotent;

(2) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly S-supplemently embedded in $N_G(P)$;

(3) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every cyclic subgroup of P with order p (and every cyclic subgroup of order 4 in the case where $p = 2$ and P is non-abelian) is weakly S-supplementedly embedded in $N_G(P)$.

Proof. This result follows from Lemma 2.5 and corollary 3.3. ∎

Theorem 3.5. *Let G be a group and let p be a prime divisor of $|G|$ satisfying $(|G|, p-1) = 1$. Suppose that E is a normal subgroup of G. Let P be a Sylow p-subgroup of E. $E \leq Z_{p\mathcal{N}\Phi}(G)$ if one of the following conditions is satisfied*

(1) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly s-supplementedly embedded in $N_G(P)$;

(2) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order 4 if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in $N_G(P)$.

Proof. Suppose that there exists G, E, P satisfying the conditions of the Theorem such that G is not p-nilpotent. Fixing P the class of all couples (G, E) satisfying the conditions of the Theorem such that G is not p-nilpotent is not empty, and we can choose a (G, E) in such a way that $|G| + |E|$ is minimal. It follows from Lemma 2.4 and corollary 3.4 that E is p-nilpotent. Let T be the normal p-complement of E. Then $T \trianglelefteq G$.

If $T \neq 1$, then we consider G/T with normal subgroup E/T. It is easy to see that $E = PT$ and $(|P|, |T|) = 1$. An argument similar to Step 4 in Theorem 3.1 shows that the $(G/T, E/T)$ satisfies the conditions of the Theorem, and this implies that $E/T \leq Z_{p\mathcal{N}\Phi}(G/T)$, by the minimality of $|G| + |E|$. Then $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction.

If $T = 1$, then $E = P$ is a p-group and $N_G(P) = G$. Assume that (1) holds. For every minimal normal subgroup N of G contained in P, by Lemma 2.2, Lemma 2.4, the argument similar to Step 4 in Theorem 3.1, then $E/N \leq Z_{p\mathcal{N}\Phi}(G/N)$. Next, we assert that $P \cap \Phi(G) = 1$. Otherwise, $P \cap \Phi(G) \neq 1$ and we may choose a minimal normal subgroup N of G such that $N \leq P \cap \Phi(G)$. By the discussion above, $E/N \leq Z_{p\mathcal{N}\Phi}(G/N)$ and $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction. Further, by Lemma 2.11, P is the direct product of minimal normal subgroups of G which are contained in P. We assert that P is a minimal normal subgroup of G. Otherwise, we may choose different minimal normal subgroups N_1 and N_2 of G contained in P. By the discussion above, $E/N_i \leq Z_{p\mathcal{N}\Phi}(G/N_i)$, $i = 1, 2$. By Lemma 2.12, $N_1N_2/N_2 \nleq \Phi(G/N_2)$ and $N_1N_2/N_2 \leq Z(G/N_2)$. Then $N_1 \leq Z(G)$ and $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction. Further, $|P| = p$ by Lemma 2.14 and $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction.
Assume that (2) holds. If every cyclic subgroup of P of order p (and of order 4 if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in G, then we assert that every cyclic subgroup of P of order p is S-permutably embedded in G. Otherwise, assume that there exists a subgroup L of P of order p is complemented in G. Then there exists a maximal subgroup of M of G such that $G = LM$ and $L \cap M = 1$. Further, $M \trianglelefteq G$, $P \cap M \trianglelefteq G$ and $P/P \cap M$ is a minimal normal subgroup of $G/P \cap M$. Next, we consider $(G, P \cap M)$. By Lemma 2.4 and the choice of (G, E), $P \cap M \leq Z_{p\mathcal{N}\Phi}(G)$ and $P \leq Z_{p\mathcal{N}\Phi}(G)$ since $|P/P \cap M| = p$, which is a contradiction. By Lemma 2.13, every cyclic subgroup of P of order p is

S-permutable in G. By the argument similar to the case on the maximal subgroups, every minimal normal subgroup N of G contained in P is of order p. By Lemma 2.2, Lemma 2.4, the argument similar to Step 4 in Theorem 3.1 and Lemma 2.11, it easy to see that P is a minimal normal subgroup of G. Then $|P| = p$ by Lemma 2.14. Further, $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction. ∎

Corollary 3.6. *Let G be a group and let p be a prime divisor of $|G|$ satisfying $(|G|, p-1) = 1$. Suppose that E is a normal subgroup of G such that G/E is p-nilpotent. Let P be a Sylow p-subgroup of E. G is p-nilpotent if one of the following conditions is satisfied*

(1) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly s-supplementedly embedded in $N_G(P)$;

(2) there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order 4 if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in $N_G(P)$.

Corollary 3.7. *Assume that $\mathcal{F}$ is a saturated formation containing the class of all supersolvable groups $\mathcal{U}$, $E \trianglelefteq G$ and $G/E \in \mathcal{F}$ for any prime p dividing $|E|$. Take $P \in Syl_p(E)$, and suppose that there exists $K \in \mathcal{K}(P)$ such that K is weakly S-supplementedly embedded in G. $G \in \mathcal{F}$ if one of the following conditions is satisfied:*

(1) every maximal subgroup of P is weakly S-supplementedly embedded in $N_G(P)$;

(2) every cyclic subgroup of P with order p and order 4 (if P is non-abelian and $p = 2$) is weakly S-supplementedly embedded in $N_G(P)$.

Proof. Assume that (G, E, K, P) satisfy the conditions of the Theorem and that $G \notin \mathcal{F}$. Fix (E, K, P). The class of groups G such that (G, E, K, P) satisfies the conditions of the Theorem and $G \notin \mathcal{F}$ is not empty, so we can find G of minimal order in this class.
Let q be the largest prime divisor of $|G|$ and $Q \in Syl_q(G)$. By corollary 3.6, G is a Sylow-tower group and $Q \trianglelefteq G$. Applying Lemmas 2.4 and 2.5, it is easy to see that G/Q satisfies the conditions of the corollary, and G/Q is supersolvable by minimality of G. It follows from Lemma 2.9 that G is supersolvable. ∎

Theorem 3.8. *Let $E \trianglelefteq G$, $p \in \pi(E)$, and let P be a Sylow p-subgroup of E. If there exists $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(H)$ is p-nilpotent, then $E \leq Z_{p\mathcal{N}\Phi}(G)$.*

Proof. Suppose that there exists G, E, P satisfying the conditions of the Theorem such that $E \nleq Z_{p\mathcal{N}\Phi}(G)$. Fixing P the class of all couples (G, E) satisfying the conditions of the Theorem such that $E \nleq Z_{p\mathcal{N}\Phi}(G)$ is not empty, and we can choose a (G, E) in such a way that $|G|$ + $|E|$ is minimal. In several steps, we show that this leads to a contradiction.
Step 1. $O_{p'}(E) = 1$.
Now, we consider the couple $(\overline{G}, \overline{E}) = (G/O_{p'}(E), E/O_{p'}(E))$. Then $\overline{P} = PO_{p'}(E)/O_{p'}(E)$ is a Sylow p-subgroup of $\overline{E}$. Certainly, $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ and $(\overline{P})' \leq \overline{P'} \leq \overline{H} \leq \overline{\Phi(P)} \leq \Phi(\overline{P})$. It follows that $(\overline{P})' \leq \overline{H} \leq \Phi(\overline{P})$. Hence

$\overline{H} \in \mathcal{H}(\overline{P})$. By Lemma 2.4, it is easy to see that $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the conditions of the Theorem, and $E/O_{p'}(E) \leq Z_{p\mathcal{N}\Phi}(G/O_{p'}(E))$ by the choice of (G, E). Further, $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction.

Step 2. $E = G$.

If $E < G$, then we consider the couple (E, E). By Lemma 2.4, (E, E) satisfies the conditions of the Theorem, and $E \leq Z_{p\mathcal{N}\Phi}(E)$ by the choice of (G, E). Further, E is p-nilpotent and $E = P \trianglelefteq G$ by Step 1. Then $N_G(P) = G$ is p-nilpotent and $E \leq Z_{p\mathcal{N}\Phi}(G)$, which is a contradiction.

Step 3. A_G, the largest normal subgroup of G contained in A, is not trivial.

By Lemma 2.5, H is an S-permutably embedded subgroup of G. If $A_G = 1$, then H is S-permutable in G by Lemma 2.3 and so $O^p(G) \leq N_G(H)$. Since H is normal in P, $G = PO^p(G) \leq N_G(H)$ is p-nilpotent, which is a contradiction.

Step 4. Final contradiction.

Since H is a Sylow p-subgroup of HA_G, it follows from [4, Lemma 3.6.10] that $N_{G/A_G}(HA_G/A_G) = N_G(H)A_G/A_G$ and $HA_G/A_G \in \mathcal{H}(PA_G/A_G)$. It is easy to see that G/A_G satisfies the hypothesis of the Theorem and G/A_G is p-nilpotent by Step 3 and the minimality of G.

$P \cap A_G = H \cap A_G \leq \Phi(P)$, A_G is p-nilpotent by Lemma 2.6. By Step 1, $A_G \leq H \leq \Phi(P)$ and so $A_G \leq \Phi(G)$, which implies that G is p-nilpotent, which is a contradiction. ■

Corollary 3.9. *Let G be a group and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if there exists a subgroup $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(H)$ is p-nilpotent.*

Proof. The necessity follows easily from Frobenius Theorem and corollary 3.2. Conversely, we assume $E = G$ and it follows from Theorem 3.8. ■

Corollary 3.10 follows as an immediate application of corollary 3.9.

Corollary 3.10. *A group G is nilpotent if and only if for every $p \in \pi(G)$, there exists a Sylow p-subgroup P of G and a subgroup $H \in \mathcal{H}(P)$ such that H is weakly S-supplementedly embedded in G and $N_G(H)$ is p-nilpotent.*

4 Applications

Let P be a Sylow p-subgroup of G. It is easy to see that p-nilpotency of $N_G(P)$ implies that $P' \in Syl_p((N_G(P))')$ and $\Phi(P) \in Syl_p(\Phi(N_G(P)))$. Therefore Theorem 3.3 has the following corollaries.

Corollary 4.1. *Assume that $(|G|, p-1) = 1$ and let P be a Sylow p-subgroup of G. The following assertions are equivalent.*

(1) G is p-nilpotent;

(2) P' is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly S-supplementedly embedded in $N_G(P)$;

(3) P' *is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order* 4 *if* $p = 2$ *and P is non-abelian) is weakly S-supplementedly embedded in* $N_G(P)$*;*

(4) $\Phi(P)$ *is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly S-supplementedly embedded in* $N_G(P)$*;*

(5) $\Phi(P)$ *is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order* 4 *if* $p = 2$ *and P is non-abelian) is weakly S-supplementedly embedded in* $N_G(P)$*;*

(6) $(N_G(P))'$ *is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly S-supplementedly embedded in* $N_G(P)$*;*

(7) $(N_G(P))'$ *is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order* 4 *if* $p = 2$ *and P is non-abelian) is weakly S-supplementedly embedded in* $N_G(P)$*;*

(8) $\Phi(N_G(P))$ *is weakly S-supplementedly embedded in G and every maximal subgroup of P is weakly S-supplementedly embedded in* $N_G(P)$*;*

(9) $\Phi(N_G(P))$ *is weakly S-supplementedly embedded in G and every cyclic subgroup of P of order p (and of order* 4 *if* $p = 2$ *and P is non-abelian) is weakly S-supplementedly embedded in* $N_G(P)$*.*

Finally, [11, Theorem 3.1] follows as a consequence of Theorem 3.3.

Corollary 4.2. *[11, Theorem 3.1] Assume that* $(|G|, p-1) = 1$ *and let P be a Sylow p-subgroup of G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly S-supplementedly embedded in* $N_G(P)$ *and if* P' *is S-permutable in G, then G is p-nilpotent.*

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References

[1] A. Ballester-Bolinches, M. C. Pedraza-Aguilera, Sufficient conditions for supersolvability of finite groups, *J, Pure Appl. Algebra* **127** (1998), 113–118.

[2] K. Doerk, Minimal nicht Überauflösbare endliche Gruppen, *Math. Z.* **91** (1996), 198–205.

[3] K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin/New York, 1992.

[4] D. Gorenstein, Finite Groups, Harper & Row, New York-London, 1968.

[5] W. Guo, The Theory of Classes of Groups, Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

[6] P. Hall, On a theorem of Frobenius, *Proc. London Math. Soc.* **40** (1936), 468–501.

[7] B. Huppert, Endliche Gruppen I, Springer, New York, Berlin, 1967.

[8] I. M. Isaacs, Finite Group Theory, Amer. Math. Soc., Providence, 2008.

[9] O. H. Kegel, Sylow-Gruppen and Subnoramlteiler Endlicher Gruppen, *Math. Z.* **78** (1962), 205–221.

[10] Y. Li, Y. Wang, H. Wei, The influence of π-quasinormality of some subgroups of a finite group, *Arch. Math. (Basel)*, **81** (2003), 245–252.

[11] X. Li, T. Zhao, Y. Xu, A new kind generalized normality in finite groups, *Indag. Math.* **22** (2011), 69–76.

[12] J. Tate, Nilpotent quotient groups, *Topology* **3** (1964), 109–111.

[13] S. Qiao, Y. Wang, S-Permutably embedded subgroups and p-supersolvable groups. *Sci. China Math.*, **57** (2014), 729–736.

[14] L. A. Shemetkov, Formations of Finite Groups, Nauka, Main Editorial Board for Physical and Mathematical Literature, Moscow, 1978.

[15] L. A. Shemetkov, A. N. Skiba, On the $\mathcal{X}\Phi$-hypercentre of finite groups, *J. Algebra*, **322**(2009), 2106–2117.

[16] H. Wei, Y. Wang, On c^*-normality and its properties, *J. Group Theory* **10** (2007), 211–223.

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