Existence and asymptotically stable solution of a Hammerstein type integral equation in a Hölder space

Somayeh Saiedinezhad

Abstract

The following nonlinear quadratic integral equation of Hammerstein type is studied.

$$x(t) = p(t) + x(t) \int_0^{q(t)} H(t, \tau, x(\tau)) \mathrm{d}\tau.$$

The methodology relies on the measure of noncompactness in the space of functions with tempered increments, namely the space of α-Hölder continuous functions. The results follow from the Darbo fixed point theorem. Some examples are included to show the applicability of the main results.

1 Introduction

Several applications of nonlinear integral equations in various fields of science and technology have attracted the attention of mathematicians to study various nonlinear integral equations, see for example [11] and the references therein. The analysis techniques, specially the fixed point techniques, which guarantee the existence of solutions of the nonlinear integral equation, in particular when the numerical methods failed, are more valuable (for example, see [9, 14, 15] and the references therein).

The concept of measure of noncompactness was introduced by Kuratowski in [16]. It is a powerful tool in fixed point theory, leading to a series of fixed

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point theorems, including the Schauder Fixed Point Theorem, which has been generalized to the Darbo Fixed Point Theorem.

For every bounded subset B of a given Banach space X, Kuratowski introduced $\alpha(B)$ as the infimum of the set of positive numbers ε such that B is covered by a finite number of sets of diameter less than ε. The function α, defined on the family of all bounded subsets of X, satisfies a series of properties, that served as the defining axioms of a measure of noncompactness (MNC). The axiomatic definition of MNC's is originally due to Sadovskii [18].

There are several known types of MNC's. Banaś [4] introduced μ_0 as a MNC on $BC(\mathbb{R}^+)$. We need some notation in order to give a precise definition of μ_0. Let B be a bounded subset of $BC(\mathbb{R}^+)$. For $x \in B$, $T > 0$ and $\varepsilon > 0$, set

$$\begin{aligned}
\omega^T(x,\varepsilon) &:= \sup\{|x(t)-x(s)|; t,s \in [0,T], |t-s| \leq \varepsilon\}; \\
\omega^T(B,\varepsilon) &:= \sup\{\omega^T(x,\varepsilon); x \in B\}; \\
\omega_0^T(B) &:= \lim_{\varepsilon \to 0} \omega^T(B,\varepsilon); \\
\omega_0(B) &:= \lim_{T \to \infty} \omega_0^T(B); \\
\mu_0(B) &:= \omega_0(B) + \limsup_{t \to \infty} \operatorname{diam} B(t)
\end{aligned}$$

with $\operatorname{diam} B(t) = \sup\{|x(t)-y(t)| \ ; \ x,y \in B\}$. One can readily check that the kernel of this MNC consists of nonempty and bounded subsets B of $BC(\mathbb{R}^+)$, such that the functions from B are locally equicontinuous on $\mathbb{R}^+$.

Several authors have studied the existence of solutions of integral equations using the Darbo Fixed Point Theorem. In [10] the kernel of the integral term has separable form $(k(t,\tau)g(\tau,x(\tau)))$; in [5, 13] the kernel of the integral term has separable and singular form $(\frac{x(\tau)}{(t-\tau)^{1-\alpha}})$; in [8] the kernel of the integral term is bounded by a separable type function $(H(t,\tau,x(\tau)) \leq a(t)b(\tau)$ where $\lim_{t\to\infty} a(t)\int_0^t b(\tau)\mathrm{d}\tau = 0$.

In this article, we examine an application of the measure of noncompactness as developed first by Banaś and Nalepa [6], in order to obtain the existence results of the following nonlinear quadratic integral equation of Hammerstein type, in the space $H^\alpha([0,T])$ where $\alpha \in (0,1]$:

$$x(t) = p(t) + x(t)\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau, \tag{1}$$

where $t \in [0,T]$, p and q are given functions which are γ-Hölder continuous (for some specified $\gamma \in (0,1]$) and $H : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies certain conditions, which will be introduced later. The main feature that distinguishes the subsequent results is the application of a measure of noncompactness on the Hölder spaces and the fact that the kernel of the integral term in (1) is not separable with respect to the components t and x, i.e., it is in general not of the form $k(t,\tau)g(\tau,x(\tau))$. Moreover, we present some explicit examples.

2 Preliminary Results

For a Banach space X, let M_X be the set of all nonempty and bounded subsets of X and let N_X be subset of M_X consisting of relatively compact sets.

Definition 2.1. [4] A function $\mu : M_X \to [0, +\infty]$ is called a measure of noncompactness (MNC) in the space X if the following conditions are satisfied;

(i) The family $\ker \mu := \{B \in M_X \,;\, \mu(B) = 0\}$ is nonempty and $\ker \mu \subseteq N_X$.

(ii) $B_1 \subseteq B_2 \Longrightarrow \mu(B_1) \leq \mu(B_2)$.

(iii) $\mu(\overline{B}) = \mu(B)$.

(iv) $\mu(\text{conv} B) = \mu(B)$.

(v) $\mu(\lambda B_1 + (1-\lambda)B_2) \leq \lambda\mu(B_1) + (1-\lambda)\mu(B_2)$, for every $\lambda \in [0,1]$.

(vi) If $(B_n)_n$ is a sequence of closed sets in M_X such that $B_{n+1} \subseteq B_n$ and $\lim_{n\to\infty} \mu(B_n) = 0$ then the intersection $\bigcap_{n=1}^{\infty} B_n$ is nonempty.

For $T > 0$ and $\alpha \in (0,1)$, the space $H^\alpha([0,T])$ of α-Hölder continuous functions is the family of all continuous functions $x = x(t)$ on $[0,T]$ such that

$$\sup\{V_\alpha(x;t,s) \,;\, t,s \in [0,T], t \neq s\} < \infty;$$

where $V_\alpha(x;t,s) := \dfrac{|x(t) - x(s)|}{|t-s|^\alpha}$. It is known that $H^\alpha([0,T])$ is a Banach space under the norm $\|x\|_\alpha = |x(0)| + \sup\{V_\alpha(x;t,s) \,;\, t,s \in [0,T], t \neq s\}$, for every $x \in H^\alpha([0,T])$. It is obvious that $\|x\|_\infty \leq \|x\|_\alpha$, where $\|x\|_\infty = \sup\{|x(t)| \,;\, t \in [0,T]\}$. For further detail on $H^\alpha([0,T])$, we refer to [7].
For a bounded subset B of $H^\alpha([0,T])$, a given $\varepsilon > 0$ and $x \in B$ we consider the following quantities:

$$\begin{aligned} \beta_\alpha(x,\varepsilon) &:= \sup\{V_\alpha(x;t,s); t,s \in [0,T], t \neq s, |t-s| \leq \varepsilon\}; \\ \beta_\alpha(B,\varepsilon) &:= \sup\{\beta_\alpha(x,\varepsilon); x \in M\}; \\ \beta_\alpha^0(B) &:= \lim_{\varepsilon\to 0} \beta_\alpha(B,\varepsilon). \end{aligned}$$

Theorem 2.2. *[6] The function $\beta_\alpha^0 : M_{H^\alpha([0,T])} \to [0,+\infty)$ is a measure of noncompactness on $H^\alpha([0,T])$.*

We remark that the construction of a MNC on a Banach space X relies on the characterization of relative compactness of bounded subsets of X. For example, the Arzela-Ascoli Theorem is crucial in the construction of the MNC μ_0, and Theorem 2.3 plays a similar role for the construction of β_α^0.

Theorem 2.3. *[6]. Assume that B is a bounded subset of the space $H^\alpha([0,T])$. This means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x \in B$ and $t,s \in [0,T]$, we have that*

$$0 < |t-s| \leq \delta \implies \frac{|x(t)-x(s)|}{|t-s|^\alpha} \leq \varepsilon,$$

or, equivalently, the functions belonging to B are equicontinuous with respect to the modulus of continuity $w(r) = r^\alpha$. Then the set B is relatively compact in the space $H^\alpha([0,T])$.

Darbo's Fixed Point Theorem 2.4 is a generalization of the Schauder Fixed Point Theorem. Along with its generalizations, it plays an important role in the development of the theory of measures of noncompactness and their applications in operator theory, see for example [1, 2, 3].

Theorem 2.4. *[12]. Let B be a nonempty, bounded, closed and convex subset of a Banach space X and let $F : B \longrightarrow B$ be a continuous map. Assume that there exists a constant $\kappa \in [0,1)$ such that $\mu(FY) \leq \kappa\mu(Y)$ for any nonempty subset of X, where μ is a MNC on X. Then T has a fixed point in the set X and all of its fixed points belong to* $\operatorname{Ker}\mu$.

3 Main results

From now, let $X := H^\alpha([0,T])$. For every $x \in X$, we denote by Fx the function defined on the interval $[0,T]$, in the following way,

$$(Fx)(t) := p(t) + x(t)\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau.$$

Definition 3.1. We call a function ψ of Γ type if $\psi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing, continuous function at zero and $\lim_{t\to 0^+}\psi(t) = 0$.

Let us to consider the following assumptions, which are needed in the sequel.

(I) $\alpha \in (0,1)$ and $p, q \in H^\gamma([0,T])$ with $\gamma \in (\alpha, 1]$.

Moreover $H : [0,T] \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfying conditions (II)-(IV).

(II) $|H(t,\tau,x) - H(t,\tau,y)| \leq \rho(t)\psi(|x-y|)$ where ψ is a function of Γ type and $\rho \in L^\infty([0,T])$.

(III) There exists $K_1 \in L^\infty([0,T])$ such that for every $t,\tau \in [0,T]$ we have $|H(t,\tau,0)| \leq K_1(t)$.

(IV) $|H(t,\tau,x) - H(s,\tau,x)| \leq |t-s|^m\varphi(|x|)$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a nondecreasing continuous function and $m \in (\alpha, 1]$.

(V) There exists $r_0 > 0$ such that $\|q\|_\gamma \left((1+(2T)^{\gamma-\alpha})\left(\rho^+\psi(r_0)+k_1^+\right) + (2T)^{m-\alpha}\varphi(r_0)\right) < 1$; in which $\rho^+ = \|\rho\|_\infty$ and $k_1^+ = \|k_1\|_\infty$.

Proposition 3.2. *Assume that the conditions $(I)-(IV)$ are satisfied, then F is a self mapping function on X. Moreover, by supposing (V), F maps $B_{r_0} := \{x \in X, \|x\|_\alpha < r_0\}$ into B_{r_0}.*

Proof. Let $x \in X$, we will show that $\sup\{V_\alpha(Fx;t,s); t,s \in [0,T], t \neq s\} < \infty$. Indeed,

$$\begin{aligned} V_\alpha(Fx;t,s) &= V_\alpha(p;t,s) + \frac{1}{|t-s|^\alpha}[x(t)\int_0^{q(t)} H(t,\tau,x(\tau)\mathrm{d}\tau \\ &\quad -x(s)\int_0^{q(s)} H(s,\tau,x(\tau)\mathrm{d}\tau] \\ &= V_\alpha(p;t,s) + V_\alpha(x;t,s)|\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau| \\ &\quad +\frac{|x(s)|}{|t-s|^\alpha}\left[\left|\int_{q(s)}^{q(t)} H(s,\tau,x(\tau))\right|\right. \\ &\quad \left.+\left|\int_0^{q(s)}(H(t,\tau,x(\tau)) - H(s,\tau,x(\tau)))\mathrm{d}\tau\right|\right]. \end{aligned}$$

Applying (II) and (III), we find that

$$|\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau| \leq |\int_0^{q(t)} H(t,\tau,x(\tau)) - H((t,\tau,0)\mathrm{d}\tau + \int_0^{q(t)} H(t,\tau,0)\mathrm{d}\tau| \tag{2}$$

$$\leq q(t)\rho(t)\psi(\|x\|_\alpha) + q(t)k_1(t). \tag{3}$$

Moreover,

$$|\int_{q(s)}^{q(t)} H(s,\tau,x(\tau))\mathrm{d}\tau| \leq |\int_{q(s)}^{q(t)}(H(s,\tau,x(\tau)) - H(0,\tau,x(\tau)))\mathrm{d}\tau| + |\int_{q(s)}^{q(t)} H(0,\tau,x(\tau))\mathrm{d}\tau| \tag{4}$$

$$\leq |q(t)-q(s)|(\rho(s)\psi(\|x\|_\alpha) + k_1(s)). \tag{5}$$

Furthermore, in view of (IV) we deduce that

$$\int_0^{q(s)} |H(t,\tau,x(\tau)) - H(s,\tau,x(\tau))\mathrm{d}\tau \leq q(s)|t-s|^m|\varphi(\|x\|_\alpha). \tag{6}$$

Hence

$$\begin{aligned} V_\alpha(Fx;t,s) &\leq (2T)^{\gamma-\alpha}\|p\|_\gamma + \|x\|_\alpha q(t)(\rho(t)\psi(\|x\|_\alpha) + k_1(t)) \\ &\quad + \|x\|_\alpha \frac{|q(t)-q(s)|}{|t-s|^\gamma}|t-s|^{\gamma-\alpha}(\rho(s)\psi(\|x\|_\alpha) + k_1(s)) \\ &\quad + \|x\|_\alpha q(s)|t-s|^{m-\alpha}\varphi(\|x\|_\alpha) \\ &\leq (2T)^{\gamma-\alpha}\|p\|_\gamma + \|x\|_\alpha q^+(\rho^+\psi(\|x\|_\alpha) + k_1^+) \\ &\quad + \|x\|_\alpha\|q\|_\gamma(2T)^{\gamma-\alpha}(\rho^+\psi(\|x\|_\alpha) + k_1^+) \\ &\quad + \|x\|_\alpha q^+(2T)^{m-\alpha}\varphi(\|x\|_\alpha). \end{aligned}$$

Since ψ and φ are continuous functions, for $x \in X$ which $\sup\{V_\alpha(x;t,s); t,s \in [0,T], t \neq s\} < \infty$ and so $\|x\|_\alpha < \infty$, we have $\sup\{V_\alpha(Fx;t,s); t,s \in [0,T], t \neq s\} < \infty$ that is $Fx \in X$.

Furthermore,

$$\|Fx\|_\alpha \leq (2T)^{\gamma-\alpha}\|p\|_\gamma + \|x\|_\alpha\|q\|_\gamma \left(\left(1+(2T)^{\gamma-\alpha}\right)\left(\rho^+\psi(\|x\|_\alpha)+k_1^+\right) + (2T)^{m-\alpha}\varphi(\|x\|_\alpha)\right).$$

Thus by considering the assumption (V), for $\|x\| = r_0$ we obtain $\|Fx\|_\alpha \leq r_0$ which shows that F maps B_{r_0} in B_{r_0}. ■

Proposition 3.3. *Assume that the conditions (I)-(IV) are satisfied. For every $x,y \in X$ and $\tau \in [0,T]$ denote the function $L_{x,y;\tau} : [0,T] \longrightarrow \mathbb{R}$ by $L_{x,y;\tau}(t) := H(t,\tau,x) - H(t,\tau,y)$. Suppose that there exists functions η of Γ type and a nonnegative function*

$$w : [0,T] \times [0,T] \times [0,T] \to [0,\infty),$$

for which the estimate

$$\text{(VI)} \qquad |L_{x,y;\tau}(t) - L_{x,y;\tau}(s)| \leq w(t,s,\tau)\eta(|x-y|),$$

is satisfied and $C_0 := \sup\{\int_0^{\|q\|_\alpha} \frac{|w(t,s,\tau)|}{|t-s|^\alpha}\mathrm{d}\tau; t \neq s, t,s \in [0,T]\} < \infty$. Then, for every $r > 0$, F is a continuous map on B_r.

Proof. For an arbitrary $r > 0$, let $x \in B_r$ and fix an arbitrary $\varepsilon > 0$. Take $y \in B_r$ such that $\|x-y\|_\alpha \leq \varepsilon$; we will show that, $\|Fx - Fy\|_\alpha \leq \zeta(\varepsilon)$, which $\zeta(\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$. Indeed,

$$\begin{aligned}
V_\alpha(Fx-Fy;t,s) &= \frac{|(Fx-Fy)(t)-(Fx-Fy)(s)|}{|t-s|^\alpha} \\
&= \frac{1}{|t-s|^\alpha}|x(t)\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau - y(t)\int_0^{q(t)} H(t,\tau,y(\tau))\mathrm{d}\tau \\
&\quad -x(s)\int_0^{q(s)} H(s,\tau,x(\tau))\mathrm{d}\tau + y(s)\int_0^{q(s)} H(s,\tau,y(\tau))\mathrm{d}\tau)| \\
&= \frac{1}{|t-s|^\alpha}|[(x(t)-y(t)]\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau \\
&\quad +y(t)\int_0^{q(t)}(H(t,\tau,x(\tau)-H(t,\tau,y(\tau))\mathrm{d}\tau \\
&\quad -[x(s)-y(s)]\int_0^{q(s)} H(s,\tau,x(\tau))\mathrm{d}\tau \\
&\quad +y(s)\int_0^{q(s)}(H(s,\tau,x(\tau)-H(s,\tau,y(\tau))\mathrm{d}\tau| \\
&\leq V_\alpha(x-y;t,s)|\int_0^{q(t)} H(t,\tau,x(\tau)\mathrm{d}\tau| \\
&\quad +\frac{|x(s)-y(s)|}{|t-s|^\alpha}|\int_0^{q(t)} H(t,\tau,x(\tau)\mathrm{d}\tau - \int_0^{q(s)} H(s,\tau,x(\tau)\mathrm{d}\tau|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|y(t)-y(s)|}{|t-s|^{\alpha}}|\int_0^{q(t)}(H(t,\tau,x(\tau)-H(t,\tau,y(\tau))\mathrm{d}\tau| \\
&+\frac{|y(s)|}{|t-s|^{\alpha}}|\int_{q(s)}^{q(t)}(H(t,\tau,x(\tau)-H(t,\tau,y(\tau))\mathrm{d}\tau)| \\
&+\frac{|y(s)|}{|t-s|^{\alpha}}|\int_0^{q(s)}(L_{x,y;\tau}(t)-L_{x,y;\tau}(s))\mathrm{d}\tau|.
\end{aligned}
$$

Applying the estimates (3), (5), (6) and (VI) to this expression we obtain that

$$
\begin{aligned}
V_\alpha(Fx-Fy;t,s) &\le \|x-y\|_\alpha q^+(\rho^+\psi(\|x\|_\alpha)+k_1^+) \\
&+ \|x-y\|_\alpha\left((2T)^{m-\alpha}q^+\varphi(\|x\|_\alpha)+\|q\|_\gamma(2T)^{\gamma-\alpha}(\rho^+\psi(\|x\|_\alpha)+k_1^+)\right) \\
&+ \|y\|_\alpha q^+\rho^+\psi(\|x-y\|_\alpha)+\|y\|_\alpha\|q\|_\gamma(2T)^{\gamma-\alpha}\rho^+\psi(\|x-y\|_\alpha) \\
&+ C_0q^+\|y\|_\alpha\eta(\|x-y\|_\alpha).
\end{aligned}
$$

Moreover, by the estimate (3) we have,

$$
\begin{aligned}
|(Fx-Fy)(0)| &= |x(0)-y(0)||\int_0^{q(0)}H(0,\tau,x(\tau))\mathrm{d}\tau| \\
&\le \|x-y\|_\alpha(q(0)\rho(0)\psi(\|x\|_\alpha)+q(0)k_1(0)) := \|x-y\|_\alpha q(0)(C_1\psi(\|x\|_\alpha)+C_2).
\end{aligned}
$$

Since $\|x-y\|_\alpha \le \varepsilon$ and $\|x\|_\alpha = \|y\|_\alpha = r$ we have that $\|Fx-Fy\|_\alpha \le \zeta(\varepsilon)$ with

$$
\begin{aligned}
\zeta(\varepsilon) &= \varepsilon\|q\|_\gamma[C_1\psi(r)+C_2+\rho^+\psi(r)+k_1^++(2T)^{m-\alpha}\varphi(r)+(2T)^{\gamma-\alpha}\rho^+\psi(r) \\
&+ (2T)^{\gamma-\alpha}k_1^+]+r\rho^+\psi(\varepsilon)\|q\|_\gamma(1+(2T)^{\gamma-\alpha})+C_0q^+r\eta(\epsilon).
\end{aligned}
$$

It is obvious that $\lim_{\varepsilon\to 0}\zeta(\varepsilon)=0$ and this completes the proof. ∎

Proposition 3.4. *If conditions (I)-(V) are satisfied, then there exists $\kappa < 1$ such that for every nonempty subset Y of B_{r_0}, $\beta_\alpha^0(FY) \le \kappa\beta_\alpha^0(Y)$. This means that F satisfies the contraction principle of Theorem 2.4, with β_α^0 as a MNC on B_{r_0}*

Proof. From the estimates (3), (5) and (6), for every $x \in B_{r_0}$, we insert

$$
\begin{aligned}
V_\alpha(Fx;t,s) &\le V_\gamma(p;t,s)|t-s|^{\gamma-\alpha}+V_\alpha(x;t,s)\left(q^+\rho^+\psi(r_0)+q^+k_1^+\right) \\
&+ r_0\left(V_\gamma(q;t,s)|t-s|^{\gamma-\alpha}(\rho^+\psi(r_0)+k_1^+)+q^+|t-s|^{m-\alpha}\varphi(r_0)\right).
\end{aligned}
$$

Taking the supremum over all $t,s \in [0,T]$ with $|t-s| \le \varepsilon$ we deduce

$$
\begin{aligned}
\beta_\alpha(Fx,\varepsilon) &\le \beta_\gamma(p,\varepsilon)\varepsilon^{\gamma-\alpha}+\beta_\alpha(x,\varepsilon)\left(q^+\rho^+\psi(r_0)+q^+k_1^+\right) \\
&+ r_0\left(\beta_\gamma(q,\varepsilon)\varepsilon^{\gamma-\alpha}(\rho^+\psi(r_0)+k_1^+)+q^+\varepsilon^{m-\alpha}\varphi(r_0)\right).
\end{aligned}
$$

Now by taking supremum over all x, which belongs to a bounded subset Y of B_{r_0} and tending ε to zero we obtain,

$$
\beta_\alpha^0(Fy) \le \|q\|_\gamma(\rho^+\psi(r_0)+k_1^+)\beta_\alpha^0(Y).
$$

By regarding the condition (V), we know that $\|q\|_\gamma(\rho^+\psi(r_0)+k_1^+) < 1$, so by letting $\kappa := \|q\|_\gamma(\rho^+\psi(r_0)+k_1^+)$ the proof is completed. ∎

Theorem 3.5. *Assume that the conditions $(I)-(V)$ and the inequality (VI) of the Proposition 3.3 are satisfied. Then the nonlinear quadratic integral equation*

$$x(t) = p(t) + x(t)\int_0^{q(t)} H(t,\tau,x(\tau))\mathrm{d}\tau \tag{7}$$

has at least one solution in $H^\alpha([0,T])$.

Proof. This is a straightforward application of Theorem 2.4, in view of Propositions 3.2, 3.3 and 3.4. ∎

In Theorem 3.6, we consider (7) in the case where its integral term has a separable kernel of the form $k(t,\tau)g(\tau,x(\tau))$. Then we can remove the restrictive condition (VI). Indeed, (VI) comes from (2) and (4) in the following and the assumptions (2) and (4) of Theorem 3.6 are in agreement with the assumptions (II) and (IV) of Theorem 3.5.

Theorem 3.6. *Consider the nonlinear quadratic integral equation*

$$x(t) = p(t) + x(t)\int_0^{q(t)} k(t,\tau)g(\tau,x(\tau))\mathrm{d}\tau \tag{8}$$

under the following assumptions,

(1) $p, q \in H^\gamma([0,T])$; $\gamma \in (\alpha, 1]$.

(2) $k : [0,T]\times[0,T] \longrightarrow \mathbb{R}$ *is a continuous function and there exist constants* $k_0 > 0$ *and* $n \geq \alpha$ *such that for every* $t,s,\tau \in [0,T]$; $|k(t,\tau)-k(s,\tau)| \leq k_0|t-s|^n$.

(3) $g : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function and there exists a function G of Γ type such that for every* $t \in [0,T]$ *and* $x,y \in \mathbb{R}$; $|g(t,x)-g(t,y)| \leq G(|x-y|)$.

(4) *There exists $r_0 > 0$ such that*

$$\|q\|_\gamma \left(\overline{k}\left(1+(2T)^{\gamma-\alpha}\right) + k_0(2T)^{n-\alpha}\right)\left(G(r_0)+\overline{g}\right) < 1;$$

in which $\overline{k} := \sup_{t,\tau\in[0,T]} k(t,\tau)$ *and* $\overline{g} := \sup_{\tau\in[0,T]} g(\tau,0)$.

Then (8) has at least one solution in $H^\alpha([0,T])$.

Proof. Let $\rho(t) := \sup_{\tau\in[0,T]}|k(t,\tau)|$, $\psi(t) = G(t)$, $k_1(t) := \rho(t)\overline{g}$ and $\psi(t) = k_0(G(t)+\overline{g})$. We can check that $(II)-(IV)$ are satisfied. Condition (V) follows from assumption (4). Moreover, the inequality (VI) is satisfied by letting $w(t,s,\tau) := |t-s|^n$ and $\eta(t) := G(t)$. Then the result is an application of Theorem 3.5. ∎

Remark 3.7. Theorem 5.1 in [6] is a particular case of Theorem 3.6: take $q(t) = 1$ and $n = \gamma \in (\alpha, 1]$.

Example 3.8. Consider the following nonlinear integral equation

$$x(t) = \frac{t}{\sqrt{1+t^2}} + \frac{1}{50}x(t)\int_0^{\sin t}\left(x(\tau)e^{t+\tau} + e^{-t\cos x(\tau)}\right)\mathrm{d}\tau; \qquad (9)$$

where $t \in [0,1]$.
It is easy to see that this equation is a special case of (7): take $p(t) := \frac{t}{\sqrt{1+t^2}}$, $q(t) := \sin t$ and $H(t,\tau,x) := \frac{1}{50}\left(x(\tau)e^{t+\tau} + e^{-t\cos x(\tau)}\right)$. Let us verify assumptions (I)-(V) and the inequality (VI). Since $|p'(t)| = |\frac{1}{(1+t^2)\sqrt{1+t^2}}| \leq \frac{1}{2}$, so p is a Lipschitz function and thus it belongs to $H^1([0,1])$. Similarly $q \in H^1([0,1])$.
Denote $f(t,x) := e^{-t\cos x}$ then $|f_x(t,x)| = |t\sin x e^{-t\cos x}| \leq e$, so f is Lipschitz with respect to x and thus $|e^{-t\cos x(\tau)} - e^{-t\cos y(\tau)}| \leq e|x(\tau)-y(\tau)|$. Hence

$$\begin{aligned}|H(t,\tau,x(\tau)) - H(t,\tau,y(\tau))| &\leq \frac{1}{50}\left(e^{t+\tau}|x(\tau)-x(\tau)| + |e^{-t\cos x(\tau)} - e^{-t\cos x(\tau)}|\right)\\ &\leq \frac{1}{50}\left(e^2|x(\tau)-x(\tau)| + e|x(\tau)-x(\tau)|\right) = \frac{1}{50}(e^2+e)|x(\tau)-x(\tau)|.\end{aligned}$$

e^t and $e^{-t\cos x}$ are Lipschitz functions with respect to t and $|f_t(t,x)| \leq e$, hence

$$|H(t,\tau,x(\tau)) - H(s,\tau,x(\tau))| \leq \frac{1}{50}(e^2|x(\tau)|+e)|t-s|.$$

On the other hand, $|H(t,\tau,0)| = |\frac{1}{50}e^{-t}| \leq \frac{1}{50}$. Thus, for this example in the correspondence with Theorem 3.5, let $T=1$, $\rho^+ = \frac{1}{50}(e^2+e)$, $k_1^+ = \frac{1}{50}$, $\|q\|_\gamma \leq 1$, $\psi(r) = r$ and $\varphi(r) = e^2r+e$. Therefore, the inequality assumption (V) takes the form

$$\frac{1}{50}\left((1+2^{1-\alpha})((e^2+e)r+1) + 2^{1-\alpha}(e^2r+e)\right) < 1.$$

We have a solution r_0 for every $\alpha \in (0,1)$. For example, if $\alpha = 1/2$, then every $r \in (0, 1.25)$ is admissible.
We now investigate assumption (VI) in Proposition 3.3. To this end, we apply the mean value theorem to f: for every $x,y \in C([0,1])$ and $\nu,\tau \in [0,1]$ there exists $\varsigma(\nu) = \varsigma_{x,y,\tau}(\nu) \in (x(\tau),y(\tau))$ (or $(y(\tau),x(\tau))$) such that

$$\frac{f(\nu,x(\tau)) - f(\nu,y(\tau))}{x(\tau)-y(\tau)} = f_x(\nu,\varsigma(\nu)), \qquad (10)$$

or, explicitly,

$$e^{-t\cos x(\tau)} - e^{-t\cos y(\tau)} = t\sin(\varsigma(t))e^{-t\cos(\varsigma(t))}(x(\tau)-y(\tau)).$$

Hence

$$\begin{aligned}&|H(t,\tau,x(\tau))) - H(t,\tau,y(\tau)) - H(s,\tau,x(\tau)) + H(s,\tau,y(\tau))|\\ &= \frac{1}{50}\left(\left|t\sin\varsigma(t)e^{-t\cos\varsigma(t)} - s\sin\varsigma(s)e^{-s\cos\varsigma(s)}\right| + |e^{t+\tau} - e^{s+\tau}|\right)|x(\tau)-y(\tau)|\end{aligned}$$

$$= \frac{1}{50}\left(\left|\int_s^t \frac{dh(\nu)}{d\nu}d\nu\right| + |e^{t+\tau} - e^{s+\tau}|\right)|x(\tau) - y(\tau)|, \tag{11}$$

with $h(\nu) := \nu \sin \varsigma(\nu) e^{-\nu \cos \varsigma(\nu)}$. Indeed,

$$h'(\nu) = \left(\sin \varsigma(\nu) - \frac{\nu}{2}\sin(2\varsigma(\nu)) + \varsigma'(\nu)(\nu \cos \varsigma(\nu) + \nu^2 \sin^2 \varsigma(\nu))\right) e^{-\nu \cos \varsigma(\nu)} \tag{12}$$

It follows from (10) that

$$\frac{f_\nu(\nu, x(\tau)) - f_\nu(\nu, y(\tau))}{x(\tau) - y(\tau)} = f_{x\nu}(\nu, \varsigma(\nu)) + \varsigma'(\nu) f_{xx}(\nu, \varsigma(\nu)). \tag{13}$$

Now $f(t, x) = e^{-t \cos x}$ entails that

$$\begin{aligned} f_\nu(\nu, x(\tau)) &= -\cos x(\tau) e^{-\nu \cos x(\tau)} \\ f_\nu(\nu, y(\tau)) &= -\cos y(\tau) e^{-\nu \cos y(\tau)} \\ f_{x\nu}(\nu, \varsigma(\nu)) &= \left(\sin \varsigma(\nu) - \frac{\nu}{2}\sin(2\varsigma(\nu))\right) e^{-\nu \cos \varsigma(\nu)} \\ f_{xx}(\nu, \varsigma(\nu)) &= \left(t \cos \varsigma(\nu) + t^2 \sin^2 \varsigma(\nu)\right) e^{-\nu \cos \varsigma(\nu)}. \end{aligned}$$

Using (13), we obtain that

$$\begin{aligned} &\varsigma'(\nu)\left(\nu \cos \varsigma(\nu) + \nu^2 \sin^2 \varsigma(\nu)\right) \\ &\quad = \left(\left[\frac{-\cos x(\tau) e^{-\nu \cos x(\tau)} + \cos y(\tau) e^{-\nu \cos y(\tau)}}{x(\tau) - y(\tau)}\right] e^{\nu \cos \varsigma(\nu)} - \right. \\ &\qquad\qquad \left. \sin \varsigma(\nu) + \frac{\nu}{2}\sin(2\varsigma(\nu))\right). \end{aligned}$$

Now

$$|(\cos x e^{-\nu \cos x})_x| = |(-\sin x + \frac{1}{2}\sin 2x) e^{-\nu \cos x}| \le \frac{3}{2}e,$$

hence

$$|-\cos x(\tau) e^{-\nu \cos x(\tau)} + \cos y(\tau) e^{-\nu \cos y(\tau)}| \le \frac{3}{2}e|x(\tau) - y(\tau)|$$

and therefore

$$|\varsigma'(\nu)\left(\nu \cos \varsigma(\nu) + \nu^2 \sin^2 \varsigma(\nu)\right)| \le \frac{3}{2}(e^2 + 1). \tag{14}$$

It follows from (14) and (12) that

$$|h'(\nu)| \le (3 + \frac{3}{2}e^2)e. \tag{15}$$

Substituting (15) in (11) we find

$$L_{x,y;\tau}(t) - L_{x,y;\tau}(s) \le \frac{1}{50}\left[(3 + \frac{3}{2}e^2)e + e^2\right]|t - s||x(\tau) - y(\tau)|;$$

finally $C_0 < \infty$ since $\alpha < 1$.

Remark 3.9. Consider the integral equation (9) in the situation where $t \in [0, T]$. Then the inequality (V) takes the form,

$$\frac{1}{50}\Big((1 + (2T)^{1-\alpha})((e^{2T} + Te^{T})r + 1) + (2T)^{1-\alpha}(e^{2T}r + e^{T})\Big) < 1.$$

It has a positive solution r_0 if $(2T)^{1-\alpha}(1 + e^{T}) < 49$. Thus, if $T \in (\frac{1}{2}, \log 48]$ we can only choose $\alpha \in (1 - \dfrac{\log \frac{49}{1+e^T}}{\log 2T}, 1)$ and if $T > \log 48$, there is no admissible $\alpha \in (0, 1)$ and finally, if $T \leq \frac{1}{2}$ every $\alpha \in (0, 1)$ is admissible, i.e., $(2T)^{1-\alpha}(1 + e^{T}) < 49$.

Example 3.10. Consider the nonlinear integral equation

$$x(t) = \sqrt{t+1} + \frac{1}{10}x(t)\int_0^{(t^2+1)^{\frac{1}{3}}} (\tau + t^3)^{\frac{1}{7}} \arctan x(\tau)\mathrm{d}\tau; \tag{16}$$

where $t \in [0, 1]$. Obviously (16) is a special case of (8): take $p(t) = \sqrt{t+1}$, $q(t) = (t^2+1)^{\frac{1}{3}}$, $k(t, \tau) = \frac{1}{10}(\tau + t^3)^{\frac{1}{7}}$ and $g(\tau, x(\tau)) = \arctan(x(\tau))$. Elementary computations show that $|t^{\frac{1}{p}} - s^{\frac{1}{p}}| \leq |t - s|^{\frac{1}{p}}$ if $p > 1$ and $t \geq s > 0$. Hence, $\sqrt{t+1} - \sqrt{s+1} \leq |t - s|^{\frac{1}{2}}$ and also $|(t^2+1)^{\frac{3}{2}} - |(s^2+1)^{\frac{3}{2}}| \leq |t^2 - s^2| \leq 2^{\frac{1}{3}}|t - s|^{\frac{1}{3}}$, for $t, s \in [0, 1]$. Hence $p \in H^{\frac{1}{2}}([0, 1])$ and $q \in H^{\frac{1}{3}}([0, 1])$. Since $H^{\frac{1}{2}}([0, 1]) \subset H^{\frac{1}{3}}([0, 1])$ let $\gamma = \frac{1}{3}$. On the other hand,

$$|k(t, \tau) - k(s, \tau)| = \frac{1}{10}|(\tau + t^3)^{\frac{1}{7}} - (\tau + s^3)^{\frac{1}{7}}| \leq \frac{1}{10}|t^3 - s^3|^{\frac{1}{7}} \leq \frac{3^{\frac{1}{7}}}{10}|t - s|^{\frac{1}{7}}.$$

Thus in line with the assumption (2) in Theorem 3.6 we have $k_0 = \frac{1}{10}3^{\frac{1}{7}}$ and $n = \frac{1}{7}$. Further, since $|\arctan x - \arctan y| \leq |x - y|$, set $G(r) = r$ and so $\widehat{g} = 0$, $\overline{k} = \frac{1}{10}2^{\frac{1}{7}}$, $\|g\|_{\frac{1}{3}} = 2^{\frac{1}{3}}$. With these choices, the inequality of the assumption (4) in Theorem 3.6 takes the form,

$$\frac{1}{10}2^{\frac{1}{3}}\left((1 + 2^{\frac{1}{3}-\alpha})2^{\frac{1}{7}} + 2^{\frac{1}{7}-\alpha}3^{\frac{1}{7}}\right) r_0 < 1. \tag{17}$$

For every value of α, for example $\alpha = 1/21$, we can calculate which r_0 satisfies (17). Hence by applying the Theorem 3.6, Eq.(16) has at least a solution on $H^{\frac{1}{21}}(0, 1)$.

Definition 3.11. Let Ω be a nonempty subset of $H^{\alpha}([0, 1])$ and let F be an operator defined on Ω with values in $H^{\alpha}([0, 1])$. Consider the equation

$$x(t) = (Fx)(t), \tag{18}$$

The function x is called an asymptotically stable solution of (18) if for every $\varepsilon > 0$ there exists $T^0 = T^0(\varepsilon) > 0$ such that for every $t \geq T^0$ and for every other solution y of (18) we have that $|x(t) - y(t)| < \varepsilon$.

Corollary 3.12. *Observe that the solutions of the integral equations considered in Theorems 3.5 and 3.6 are the fixed points of their corresponding operator F and belong to* $\ker\beta_\alpha^0$*. Moreover in view of the definition of the MNC* β_α^0*, we conclude that all solutions of the equations which are considered in this article are asymptotically stable in the sense of Definition 3.11.*

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School of Mathematics, Iran University of Science and Technology,
Narmak, Tehran, Iran
email :ssaiedinezhad@iust.ac.ir