Polish factorizations, cosmic spaces and domain representability*

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Abstract

We say that a space X is *cofinally Polish* if for every continuous onto map $f : X \to M$ of X onto a separable metrizable space M, there exists a Polish space P and continuous onto maps $g : X \to P$ and $h : P \to M$ such that $f = h \circ g$. We study general properties of cofinally Polish spaces and compare the property of being cofinally Polish with subcompactness and domain representability. It is established, among other things, that a space with a countable network is cofinally Polish if and only if it is domain representable. We also show that any G_δ-subset of an Eberlein compact space must be subcompact thus giving an answer to an open problem published in 2013.

1 Introduction

Subcompactness is the weakest of so called *Amsterdam properties*; it was introduced by de Groot in 1963 with the idea to express in general terms not only the completeness of metric spaces but also the fact that any Čech-complete space is an absolute G_δ. Subcompactness is preserved by arbitrary products, finite unions and open subspaces; in metrizable spaces it coincides with Čech-completeness. These nice categorical properties were the main motivation for an intensive study of the class of subcompact spaces during more than 50 years after their discovery.

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One of the most important lines of research here is to compare subcompactness with other completeness properties. It is easy to see that every subcompact space is pseudocomplete and hence has the Baire property. It is not difficult to find examples of subcompact spaces that are not Čech-complete, but it turned out to be a hard task to find out whether Čech-completeness implies subcompactness.

In this direction Önal and Vural proved in [11] that $X \backslash A$ is subcompact if X is compact and $A \subset X$ is countable; van Mill and Tkachuk showed that all separable Čech-complete spaces are subcompact; but it is still an open question whether every Čech-complete space is subcompact.

We will also deal with domain representability which is another completeness property that had its origin in computer science about 20 years ago (see [3] and [4]). Domain representability is weaker than subcompactness and its definition is cumbersome but it has nicer categorical properties and quite a few questions that are open for subcompactness are solved for domain representability. It is known that domain representability implies the Baire property, coincides with Čech-completeness in metrizable spaces, it is invariant under arbitrary products and G_δ-subsets so every Čech-complete space is domain representable. It was proved in [8] that domain representability of $C_p(X)$ implies that X discrete.

In this paper we introduce yet another completeness property based on factorizations through Polish spaces. We call the respective spaces *cofinally Polish* and compare them with subcompact spaces and domain representable spaces. We establish that a cosmic space is cofinally Polish if and only if it is domain representable. As a consequence, in countable spaces domain representability and subcompactness coincide. We also show that any G_δ-subset of an Eberlein compact space is subcompact thus giving a positive answer to Problem 3.11 from the paper [7].

2 Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. As usual, $\mathbb{R}$ is the set of reals and $\mathbb{Q}$ is the set of the rational numbers; we denote $\omega\backslash\{0\}$ by $\mathbb{N}$ and $\mathbb{D} = \{0,1\} \subset \mathbb{R}$.

If $(Q, \ll)$ is a partially ordered set, then a set $D \subset Q$ is *upward directed* if for any $p, q \in D$, there exists $r \in D$ such that $p \ll r$ and $q \ll r$. A space X is *domain representable* if there exists a partially ordered set $(Q, \ll)$ and a map $B : Q \to \tau^*(X)$ with the following properties:

(DR1) the family $\{B(q) : q \in Q\}$ is a base of X;
(DR2) if $p, q \in Q$ and $p \ll q$, then $B(q) \subset B(p)$;
(DR3) for each $x \in X$, the set $\{q \in Q : x \in B(q)\}$ is upward directed;
(DR4) if $D \subset Q$ and $(D, \ll)$ is upward directed, then $\bigcap\{B(p) : p \in D\} \neq \emptyset$.

In this case we say that *the triple* $(Q, \ll, B)$ *represents the space* X.

Given a space X, a family $\mathcal{F} \subset \tau^*(X)$ is *a regular filterbase*, if for any $U, V \in \mathcal{F}$, there exists $W \in \mathcal{F}$ such that $\overline{W} \subset U \cap V$. A base $\mathcal{B} \subset \tau^*(X)$ is *subcompact* if any regular filterbase $\mathcal{F} \subset \mathcal{B}$ has non-empty intersection. The space X is called subcompact if it has a subcompact base. A space P is Polish if it is Čech-complete and second countable.

A family $\mathcal{N}$ is a *network* of a space X if every open subset of X is a union of a subfamily of $\mathcal{N}$. The spaces with a countable network are called *cosmic*. In the realm of Tychonoff spaces, every cosmic space X,

(C1) has a countable network that consists of closed subsets of X;
(C2) is a continuous image of a separable metric space;
(C3) has a weaker separable metrizable topology.

Observe that a network in a space looks like a base whose elements are not necessarily open. However, there are plenty of spaces with a countable network (or even countable) without a countable base. For example, any countable dense subset D of $\mathbb{R}^{\mathfrak{c}}$ is cosmic as well as the space $C_p(\mathbb{R})$ but neither $C_p(\mathbb{R})$ nor D has a countable base.

If all G_δ-subsets of a space X are open in X, then X is called a *P-space.* For any space X the cardinal $c(X)$, called the *Souslin number of* X is the supremum of cardinalities of disjoint families of non-empty open subsets of X. The spaces whose Souslin number is countable are said to have the Souslin property.

Given spaces X and Y, a map $f : X \to Y$ is called a *condensation* if it is a continuous bijection. A family $\mathcal{B} \subset \tau^*(X)$ is a *π-base* of the space X if every non-empty open subset of X contains an element of $\mathcal{B}$. The cardinal $s(X)$, called *the spread of* X, is the supremum of cardinalities of discrete subspaces of X and $F \subset X$ is a cozero set of X if there exists a continuous function $f : X \to \mathbb{R}$ such that $F = f^{-1}(\mathbb{R}\backslash\{0\})$.

The rest of our notation is standard and can be found in [6]. All necessary facts and notions of C_p-theory can be consulted in the books [12] and [13].

3 Cofinally Polish and cosmic spaces

We will see that being cofinally Polish is a completeness property that coincides with Čech-completeness in metrizable spaces.

3.1 Definition. Say that a space X is *cofinally Polish* if for any continuous map f of X onto a second countable space M, there exists a Polish space P and continuous onto maps $g : X \to P$ and $h : P \to M$ such that $h \circ g = f$.

Any Polish space is trivially cofinally Polish. We will see that the converse is not true for general spaces but it is natural to expect that these two concepts coincide in the class of second countable spaces. This is, indeed, true.

3.2 Proposition. *A second countable cofinally Polish space is Polish.*

Proof. If X is second countable and cofinally Polish, then for the identity map $id : X \to X$ there exist maps $g : X \to P$ and $h : P \to X$ such that P is a Polish space and $h \circ g = id$. Clearly, both g and h are homeomorphisms, so X is Polish being homeomorphic to a Polish space P. ∎

Since every continuous image of a Polish space is analytic, we obtain the following property of cofinally Polish spaces.

3.3 Proposition. *If X is cofinally Polish, then any second countable continuous image of X is analytic.*

3.4 Corollary. *If X is a cosmic cofinally Polish space, then X is analytic.*

Proof. All second countable continuous images of the space X are analytic by Proposition 3.3 so X must be analytic by Theorem 1.3 of the paper [10]. ∎

3.5 Proposition. *A cosmic space is cofinally Polish if and only if for any continuous map $f : X \to M$ of X onto a second countable space M, there exist continuous onto maps $g : X \to P$ and $h : P \to M$ such that g is a condensation, the space P is Polish and $h \circ g = f$.*

Proof. The sufficiency being clear, assume that X is a cofinally Polish space and $f : X \to M$ is a continuous map of X onto a second countable space M. There exists a condensation $\varphi : X \to N$ of X onto a second countable space N. Let ξ be the diagonal product of f and φ; the space $M' = \xi(X) \subset M \times N$ is second countable and the map $\xi : X \to M'$ is a condensation. If $p : M' \to M$ is the projection, then $p \circ \xi = f$.

Since X is cofinally Polish, we can find a Polish space P together with continuous maps $g : X \to P$ and $h' : P \to M'$ such that $h' \circ g = \xi$. It follows from bijectivity of ξ that g is also a condensation. If $h = p \circ h'$, then $h \circ g = f$ so g and h witness necessity. ∎

3.6 Proposition. *Any Lindelöf P-space is cofinally Polish.*

Proof. Assume that X is a Lindelöf P-space and $f : X \to M$ is a continuous map of X onto a second countable space M. Since every $x \in M$ is a G_δ-subset of M, the family $\mathcal{U} = \{f^{-1}(x) : x \in M\}$ is a partition of X into clopen subsets. It follows from the Lindelöf property of X that $\mathcal{U}$ is countable and hence the space M is countable as well.

If P is the set M with the discrete topology then, letting $g = f$ and $h(x) = x$ for any $x \in M$, we obtain continuous onto maps $g : X \to P$ and $h : P \to M$ such that $f = h \circ g$. The space P is Polish being countable and discrete so the maps g and h witness that X is cofinally Polish. ∎

3.7 Example. Let $S = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| < \omega\}$ be the σ-product in the Cantor cube $\mathbb{D}^{\omega_1}$. Consider the space X whose underlying set is S and the topology is generated by all G_δ-subsets of S. It is trivial that X is a P-space; to see that it is Lindelöf, observe that every subspace $F_n = \{x \in X : |x^{-1}(1)| \leq n\}$ is scattered and compact as a subspace of S because it is a continuous image of a finite power of the one-point compactification of a discrete space.

For Lindelöf scattered spaces Z, it is a theorem of Uspenskij (published in 1982 in a non-translated paper in Russian), that the topology on Z generated by its G_δ-subsets is still Lindelöf (for a proof in English see, e.g., [13, Problem 128]). This shows that F_n is Lindelöf as a subspace of X. Since $X = \bigcup_{n\in\omega} F_n$, the space X is Lindelöf being the countable union of its Lindelöf subspaces. It is standard that every F_n is a closed nowhere dense subset of X so the space X is of first category in itself. Besides, X is cofinally Polish by Proposition 3.6 so a cofinally Polish space need not have the Baire property.

3.8 Theorem. *Suppose that X is a space and for every $U \in \tau^*(X)$ there exists a cozero set $V \subset U$ such that $U \subset \overline{V}$. If, additionally, X is cofinally Polish, then it has the Baire property.*

Proof. If X does not have the Baire property, then there is a non-empty open set $U \subset X$ and a family $\{U_n : n \in \omega\} \subset \tau(X)$ such that $U_n \subset U \subset \overline{U}_n$ for every $n \in \omega$ and $\bigcap_{n\in\omega} U_n = \emptyset$. Take a cozero set V_n such that $V_n \subset U_n \subset \overline{V}_n$ for every $n \in \omega$. It is clear that we also have the inclusions $V_n \subset U \subset \overline{V}_n$ for every $n \in \omega$ and $\bigcap_{n\in\omega} V_n = \emptyset$. Since any finite intersections of cozero sets is a cozero set, there is no loss of generality to assume that $V_{n+1} \subset V_n$ for every $n \in \omega$.

Take a continuous function $f_n : X \to \mathbb{R}$ such that $V_n = f_n^{-1}(\mathbb{R}\backslash\{0\})$ for each $n \in \omega$ and consider the diagonal product $f = \Delta_{n\in\omega} f_n : X \to \mathbb{R}^\omega$. If $M = f(X)$, then M is second countable, the set $W_n = f(V_n)$ is open in M and $V_n = f^{-1}(W_n)$ for each $n \in \omega$. There exist continuous onto maps $g : X \to P$ and $h : P \to M$ such that P is a Polish space and $f = h \circ g$. A moment's reflection shows that $h^{-1}(W_n) = g(V_n)$ and hence $G_n = g(V_n)$ is open in P for every $n \in \omega$. Since $\emptyset = \bigcap_{n\in\omega} V_n = \bigcap_{n\in\omega} f^{-1}(W_n) = f^{-1}(\bigcap_{n\in\omega} W_n)$, it follows from surjectivity of the mapping f that $\bigcap_{n\in\omega} W_n = \emptyset$. This implies that $\bigcap_{n\in\omega} G_n = h^{-1}(\bigcap_{n\in\omega} W_n) = \emptyset$.

Observe that every V_n is dense in V_0 so G_n is dense in G_0 for each $n \in \omega$. Therefore the sequence $\{G_n : n \in \omega\} \subset \tau(P)$ witnesses that a non-empty open set $G_0 \subset P$ is of first category in P; this contradiction with Čech-completeness of P shows that X has the Baire property. ■

3.9 Corollary. *If X is a cofinally Polish space with $c(X) \le \omega$, then X has the Baire property.*

Proof. If $U \in \tau^*(X)$, then let $\mathcal{V}$ be a maximal disjoint family of cozero subsets of X contained in U. It is standard that $V = \bigcup \mathcal{V}$ is dense in U. Besides, $\mathcal{V}$ is countable so V is a cozero set being the countable union of cozero sets. Therefore every non-empty open subset of X has a dense cozero set and hence we can apply Theorem 3.8 to conclude that X has the Baire property. ■

3.10 Corollary. *If $C_p(X)$ is cofinally Polish for some space X, then $C_p(X)$ has the Baire property.*

Proof. Just observe that $c(C_p(X)) = \omega$ and apply Corollary 3.9. ■

3.11 Corollary. *If X is a perfectly normal cofinally Polish space, then X has the Baire property.*

Proof. This is because every open subset of X is a cozero set so Theorem 3.8 is applicable. ■

3.12 Proposition. *If X is a perfectly normal cofinally Polish space, $F \subset X$ is closed in X and $f : X \to M$ is a continuous map of X onto a second countable space M, then the set $f(F)$ is analytic.*

Proof. Take a continuous function $q : X \to \mathbb{R}$ such that $F = q^{-1}(0)$ and consider the diagonal product $\varphi = f \Delta q : X \to M \times \mathbb{R}$. The space $N = \varphi(X)$ is second countable, the set $G = \varphi(F)$ is closed in N and $F = \varphi^{-1}(G)$. If $\pi : N \to M$ is the projection, then $\pi \circ \varphi = f$. There exist continuous onto maps $g : X \to P$ and $h : P \to N$ such that P is a Polish space and $\varphi = h \circ g$.

It is standard that $g(F) = h^{-1}(G)$ and hence the set $g(F)$ is Polish being closed in P. Consequently, the set $\varphi(F) = h(g(F))$ is analytic being a continuous image of a Polish space. Finally, the set $f(F) = \pi(\varphi(F))$ is also analytic because it is a continuous image of an analytic space $\varphi(F)$. ∎

3.13 Corollary. *If X is a perfectly normal cofinally Polish space, then X has countable spread.*

Proof. If $s(X) > \omega$, then fix an uncountable discrete subset $D \subset X$. Since $F = \overline{D} \backslash D$ is a G_δ-subset of X, we can find a family $\{U_n : n \in \omega\} \subset \tau(X)$ such that $F = \bigcap_{n\in\omega} U_n$ and hence $D = \bigcup_{n\in\omega}(D \backslash U_n)$

There exists $n \in \omega$ such that $E = D \backslash U_n$ is uncountable; it is easy to see that E is closed in X. It is well known that there exists an uncountable set $B \subset \mathbb{R}$ such that every compact subset of B is countable. Let $q : E \to B$ be an injection. By normality of X, there exists a continuous function $f : X \to \mathbb{R}$ such that $f|E = q$. Proposition 3.12 implies that the set $f(D) = q(D)$ is analytic which is impossible because any uncountable analytic space contains a copy of the Cantor set (see [13, Problem 353]). This contradiction shows that $s(X) \leq \omega$. ∎

3.14 Corollary. *A metrizable space is cofinally Polish if and only if it is Polish.*

Proof. Since sufficiency is trivial, assume that a metrizable space X is cofinally Polish. Then $s(X) \leq \omega$ by Corollary 3.13. This implies that X is second countable and hence Polish by Proposition 3.2. ∎

3.15 Example. A continuous image of a Polish space is not necessarily cofinally Polish. For example, a countably infinite discrete space X maps continuously onto $\mathbb{Q}$ which is not Polish and hence not cofinally Polish by Proposition 3.2.

We already saw that cofinally Polish spaces often have the Baire property so there is no doubt that being cofinally Polish is a kind of completeness. Our purpose now is to find out how cofinally Polish spaces are situated with respect to other classical properties.

3.16 Example. Let X be an uncountable discrete space. Clearly, X is a locally compact (and hence Čech-complete) space which is not cofinally Polish by Corollary 3.14.

Example 3.16 shows that general Čech-complete spaces need not be cofinally Polish. However, in function spaces much less suffices for being cofinally Polish. Recall that a space X is *pseudocomplete* if there exists a sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases in X such that for any family $\{U_n : n \in \omega\}$ of open subsets of X such that $\overline{U}_{n+1} \subset U_n$ and $U_n \in \mathcal{B}_n$ for every $n \in \omega$, we have $\bigcap_{n\in\omega} U_n \neq \emptyset$.

3.17 Proposition. *If $C_p(X)$ is pseudocomplete, then it is cofinally Polish.*

Proof. For any set $A \subset X$, let $\pi_A : C_p(X) \to C_p(A)$ be the restriction map. Given any continuous onto map $\varphi : C_p(X) \to M$ for some second countable space M, there exists a countable set $A \subset X$ and a continuous map $\xi : \pi_A(C_p(X)) \to M$ such that $\xi \circ \pi_A = \varphi$ (see [12, Problem 300]). All countable subsets of X are closed and C-embedded by [12, Problem 485] so $P = \pi_A(C_p(X)) = \mathbb{R}^A$ is a Polish space which witnesses that $C_p(X)$ is cofinally Polish. ■

Since any second countable image of a pseudocompact space is compact, we have the following trivial observation.

3.18 Proposition. *Every pseudocompact space is cofinally Polish.*

Observe that the cofinally Polish space X constructed in Example 3.7 is not subcompact because it does not have the Baire property. We will show next that Proposition 3.18 implies that a function space can be cofinally Polish and countably compact but not subcompact.

3.19 Example. Let X be the space constructed in Example 3.7. Being uncountable and Lindelöf, X is a non-discrete P-space. Therefore $C_p(X, \mathbb{I})$ is countably compact by [12, Problem 397] but not subcompact because X is not discrete (see Corollary 2.8 of the paper [9]). In particular, $Y = C_p(X, \mathbb{I})$ is a cofinally Polish countably compact space that is not subcompact.

4 Domain representability vs cofinally Polish

The main purpose of this section is to establish that a cosmic space is cofinally Polish if and only if it is domain representable.

4.1 Proposition. *Given a hereditarily Lindelöf space X assume that $(Q, \ll)$ is a partially ordered set and $B : Q \to \tau^*(X)$ is a map that witnesses domain representability of X. Then, for any $p, q \in Q$ we can find a countable set $C(p,q) \subset Q$ such that $\bigcup\{B(r) : r \in C(p,q)\} = B(p) \cap B(q)$ while $p \ll r$, $q \ll r$ and $\overline{B(r)} \subset B(p) \cap B(q)$ for any $r \in C(p,q)$. We will call the set $C(q,p)$ regular refinement for the pair (p,q).*

Proof. For any point $x \in B(p) \cap B(q)$ we can find an element $a_x \in Q$ such that $x \in B(a_x) \subset \overline{B(a_x)} \subset B(p) \cap B(q)$. It is easy to find a point $r_x \in Q$ such that $x \in B(r_x)$ while $r_x \gg p$, $r_x \gg q$ and $r_x \gg a_x$. It follows from $B(r_x) \subset B(a_x)$ that $\overline{B(r_x)} \subset B(p) \cap B(q)$. There exists a countable set $A \subset B(p) \cap B(q)$ such that $\bigcup\{B(r_x) : x \in A\} = B(p) \cap B(q)$. It is immediate that the set $C(p,q) = \{r_x : x \in A\}$ is as promised. ■

4.2 Proposition. *Given a space X assume that $(Q, \ll)$ is a partially ordered set and $B : Q \to \tau^*(X)$ is a map such that the triple $(Q, \ll, B)$ represents the space X. Suppose that $R \subset Q$ and the family $\{B(q) : q \in R\}$ is a base for a topology σ on X; let $X' = (X, \sigma)$. If the set $\{q \in R : x \in B(q)\}$ is upward directed for any $x \in X$, then the triple $(R, \ll, B|R)$ represents the space X' and hence X' is also domain representable.*

Proof. Let $B' = B|R$. If $D \subset R$ is an upward directed set in R, then D is upward directed in Q and hence $\bigcap\{B'(q) : q \in D\} = \bigcap\{B(q) : q \in D\} \neq \emptyset$; the last intersection is non-empty because the property (DR4) holds for the triple $(Q, \ll, B)$. Therefore (DR4) also holds for the triple $(R, \ll, B')$. Since the conditions (DR1)–(DR3) are trivially satisfied, the triple $(R, \ll, B')$ indeed, represents the space X'. ■

4.3 Theorem. *Given a space X, assume that for any countable family $\mathcal{U}$ of open subsets of X, there exists a Polish topology σ on X such that $\mathcal{U} \subset \sigma$ and $\sigma \subset \tau(X)$. Then X is domain representable.*

Proof. Let $\mathcal{R}$ be the collection of non-empty finite sequences of elements of $\tau^*(X)$. We consider a sequence $S \in \mathcal{R}$ as a function from the set of natural numbers $\{1, 2, \ldots, n(S)\}$ into $\tau^*(X)$; here $n(S)$ is the length of S. If $i \leq n(S)$, then S_i is the function S restricted to $\{1, 2, \ldots, i\}$. We write $\operatorname{ran} S$ for the range of S (i.e. the collection of open sets in the sequence) and the expression $S^\frown U$ stands for $S \cup \{(n(S)+1, U)\}$. For each $n \in \mathbb{N}$, the family $\mathcal{R}_n \subset \mathcal{R}$ is the collection of all sequences of length n. Our first step is to define a Polish topology $\tau_S \subset \tau(X)$ for every $S \in \mathcal{R}$; we will do this by induction on the length of S.

If $S \in \mathcal{R}_1$, then there is a Polish topology $\tau_S \subset \tau(X)$ with $S(1) \in \tau_S$ and hence $\operatorname{ran} S \subset \tau_S$; fix any complete metric μ_S generating τ_S. Now, suppose that τ_S and μ_S have been defined for each $S \in \mathcal{R}_n$ and take $S \in \mathcal{R}_{n+1}$. By the induction hypothesis, we have defined a Polish topology τ_{S_n} containing $\operatorname{ran} S_n$. Take a countable base $\mathcal{B}$ for τ_{S_n} and choose a Polish topology $\tau_S \subset \tau(X)$ such that $\mathcal{B} \cup \{S(n+1)\} \subset \tau_S$. Fixing any complete metric μ_S that generates the topology τ_S, we wrap up the induction. Notice that if $\hat{S}$ extends S, then $\tau_S \subset \tau_{\hat{S}}$.

Let Q be the collection of all finite sets F of elements of $\mathcal{R}$ such that

(a) $S(n(S)) = S'(n(S'))$ for all $S, S' \in F$;

(b) if $S \in F$, then $\operatorname{diam}_{\mu_{S_j}}(S(i)) \leq 2^{j-i} \operatorname{diam}_{\mu_{S_j}}(S(j))$ for all $j \leq i \leq n(S)$;

(c) if $S \in F$, then $\overline{S(i+1)}^{\tau_{S_i}} \subset S(i)$ and $\overline{S(i+1)}^{\tau_{S_i}} \neq S(i)$ for all $i \in \mathbb{N}$ with $i < n(S)$.

Given $F, \hat{F} \in Q$, let $F \ll \hat{F}$ if for every $S \in F$ there is $\hat{S} \in \hat{F}$ such that $\hat{S}$ is an extension of S. Now, if $F \in Q$, then take an element $S \in F$ and let $B(F) = S(n(S))$. Condition (a) guarantees that the map $B : Q \to \tau^*(X)$ is defined consistently.

It is clear that the relation $\ll$ is reflexive. If $F \ll F' \ll F''$, then take any $S \in F$. There exists $S' \in F'$ extending S and $S'' \in F''$ extending S'. Therefore S'' extends S which shows that $F \ll F''$, i.e., $\ll$ is transitive.

Now, if $F \ll F'$ and $F' \ll F$, then take any $S \in F$ and $S' \in F'$ which extends F. Then there exists $S'' \in F$ that extends S' so S'' also extends S. If $n(S) \neq n(S'')$ then $S''(n(S'')) \neq S(n(S))$ by (c); this contradiction with (a) shows that $S = S' = S''$ and hence $F \subset F'$. An analogous argument shows that $F' \subset F$, i.e., $F = F'$ and hence $\ll$ is a partial order on Q.

Given any $U \in \tau^*(X)$ let $S(1) = U$; then $S \in \mathcal{R}_1$ and $F = \{S\}$ belongs to Q. As $B(F) = U$, we proved that $B(Q) = \tau^*(X)$ which is certainly a base in X so the triple $(Q, \ll, B)$ has the property (DR1).

If $F \ll F'$, then take any $S \in F$; we have the equality $B(F) = S(n(S))$. Some element $S' \in F'$ extends S. Therefore $n(S') \geq n(S)$ and it follows from property

(c) that $B(F') = S'(n(S')) \subset S'(n(S)) = S(n(S)) = B(F)$. This shows that the triple $(Q, \ll, B)$ has the property (DR2).

To verify the condition (DR3) fix any point $x \in X$ and take $F, F' \in Q$ such that $x \in B(F) \cap B(F')$. If the point x is isolated in X, then for any $S \in F \cup F'$ let $S^* = S$ if $\{x\} = S(n(S))$; otherwise let $S^* = S^\frown\{x\}$. It is straightforward that the set $\hat{F} = \{S^* : S \in F \cup F'\}$ belongs to Q and $x \in B(\hat{F})$. Since also $F \ll \hat{F}$ and $F' \ll \hat{F}$, we have completed the verification of (DR3) for this case.

If x is not isolated in X, then it is not isolated in τ_S for all $S \in \mathcal{R}$; take any $S \in F \cup F'$. For every $i \leq n(S)$ it is easy to find a set $V(S,i) \in \tau_{S_i}$ such that $x \in V(S,i) \subset \overline{V(S,i)}^{\tau_{S_i}} \subset S(i)$ while $\overline{V(S,i)}^{\tau_{S_i}} \neq S(i)$ and the set $V(S,i)$ has μ_{S_i}-diameter bounded by $2^{i-n(S)-1}\operatorname{diam}_{\mu_{S_i}}(S(i))$. Observe that the set $U = \bigcap\{V(S,i) : S \in F \cup F' \text{ and } i \leq n(S)\}$ is an open neighborhood of x in X; we claim that $\hat{F} = \{S^\frown U : S \in F \cup F'\}$ witnesses that (DR3) holds in this case.

By our construction, $\hat{S}(n(\hat{S})) = U$ for all $\hat{S} \in \hat{F}$ so (a) holds for $\hat{F}$.

To verify (b) for the set $\hat{F}$ fix $\hat{S} \in \hat{F}$; there is no loss of generality to assume that $\hat{S} = S^\frown U$ for some $S \in F$. Besides, it suffices to prove that (b) is satisfied for $j < i = n(\hat{S}) = n(S) + 1$ because its validity for the case $j < i < n(\hat{S}) = n(S) + 1$ follows from the fact that $S \in F \in Q$. So, observe that for $j \leq n(S)$, we have

$$\operatorname{diam}_{\mu_{\hat{S}_j}}(\hat{S}(n(\hat{S}))) = \operatorname{diam}_{\mu_{\hat{S}_j}}(U) = \operatorname{diam}_{\mu_{S_j}}(U) \leq \operatorname{diam}_{\mu_{S_j}}(V(S,j))$$

$$\leq 2^{j-n(S)-1}\operatorname{diam}_{\mu_{S_j}}(S(j)) = 2^{j-n(\hat{S})}\operatorname{diam}_{\mu_{\hat{S}_j}}(\hat{S}(j)),$$

and therefore condition (b) is satisfied for the set $\hat{F}$ because $F \ll \hat{F}$, $F' \ll \hat{F}$ and $x \in U = B(\hat{F})$.

As for condition (c), if $i < n(S) = n(\hat{S}) - 1$, then $\overline{S(i+1)}^{\tau_{S_i}} \subset S(i)$ and $\overline{S(i+1)}^{\tau_{S_i}} \neq S(i)$ since $S \in F \in Q$. If $i = n(S)$ then

$$\overline{\hat{S}(i+1)}^{\tau_{\hat{S}_i}} = \overline{U}^{\tau_{S_i}} \subset \overline{V(S,i)}^{\tau_{S_i}} \subset S(i) = \hat{S}(i),$$

and it follows from $\overline{V(S,i)}^{\tau_{S_i}} \neq S(i) = \hat{S}(i)$ that $\overline{\hat{S}(i+1)}^{\tau_{\hat{S}_i}} \neq \hat{S}(i)$, i.e., condition (c) is also satisfied for $\hat{F}$. Finally, it is straightforward that $F \ll \hat{F}$ and $F' \ll \hat{F}$ so (DR3) is verified for the triple $(Q, \ll, B)$.

To prove that the triple $(Q, \ll, B)$ has (DR4) fix an upward directed set $J \subset Q$. If the family $\mathcal{W} = \{B(F) : F \in J\}$ has a minimal element W with respect to inclusion, then $\bigcap\{B(F) : F \in J\} = W \neq \emptyset$ so we can assume, without loss of generality, that for any $F \in J$, there exists $F' \in J$ such that $F \ll F'$ and $B(F') \neq B(F)$; whence $B(F')$ is strictly contained in $B(F)$. The core of the proof is the following property:

(1) for any set $\{F^k : k \in \omega\} \subset J$ such that $F^k \ll F^{k+1}$ and $B(F^{k+1}) \neq B(F^k)$ for any $k \in \omega$, there exists $x \in X$ such that $\bigcap\{B(F^k) : k \in \omega\} = \{x\}$.

To prove (1), choose inductively a sequence $\{S^k : k \in \omega\}$ such that $S^k \in F_k$ and S^{k+1} is an extension of S^k for any $k \in \omega$. Observe first that $B(F^{k+1}) \subset B(F^k)$ and $B(F^{k+1}) \neq B(F^k)$; this shows that S^{k+1} is a proper extension of S^k and therefore $n(S^k) \geq k$ for every $k \in \omega$. To simplify the notation we are going to denote the set $B(F^k)$ by V_k and write μ_k instead of μ_{S_k} for all $k \in \omega$.

If $k, m \in \omega$ and $k > m$, then we can take $j = n(S^m)$ and $i = n(S^k)$ in property (b) for S^k to conclude that

$$\operatorname{diam}_{\mu_m}(V_k) = \operatorname{diam}_{\mu_m}(S^k(n(S^k))) \leq 2^{n(S^m)-n(S^k)} \operatorname{diam}_{\mu_m}(S^k(n(S^m)))$$
$$= 2^{n(S^m)-n(S^k)} \operatorname{diam}_{\mu_m}(V_m) \leq 2^{n(S_m)-k} \operatorname{diam}_{\mu_{S_m}}(V_m).$$

Keeping m fixed in the above inequality, we see that the μ_{S_m}-diameter of V_k approaches zero as k approaches infinity. Since (X, μ_m) is a complete metric space, there exists a point $x_m \in X$ such that $\bigcap\{\overline{V_k}^{\tau_{S_m}} : k > m\} = \{x_m\}$. Besides, it follows from property (c) for S^k that

$$V_m = S^k(n(S^m)) \supseteq \overline{S^k(n(S^m)+1)}^{\tau_{S^m}} \supseteq \overline{S^k(n(S^k))}^{\tau_{S^m}} = \overline{V_k}^{\tau_{S^m}}$$

for any $k > m$ and hence $x_m \in V_m$. Therefore we established that for any $m \in \omega$,

(2) there exists $x_m \in V_m$ such that $\bigcap\{\overline{V_k}^{\tau_{S_m}} : k > m\} = \{x_m\}$.

Given $m, n \in \omega$ such that $m < n$ observe that $\tau_{S^m} \subset \tau_{S^n}$ and hence $\overline{V_k}^{\tau_{S^n}} \subset \overline{V_k}^{\tau_{S^m}}$ for all $k \in \omega$ so it follows from (2) that

$$\{x_n\} = \bigcap\{\overline{V_k}^{\tau_{S_n}} : k > n\} \subset \bigcap\{\overline{V_k}^{\tau_{S_m}} : k > m\} = \{x_m\},$$

and hence there exists a point $x \in X$ such that $x = x_m$ for all $m \in \omega$, i.e., $\bigcap\{V_m : m \in \omega\} = \{x\}$ so (1) is proved.

The set J being upward directed, we can find a sequence $\{F^k : k \in \omega\} \subset J$ such that $F^k \ll F^{k+1}$ and $B(F^{k+1}) \neq B(F^k)$ for any $k \in \omega$. Property (1) guarantees that $\bigcap\{B(F^k) : k \in \omega\} = \{x\}$ for some $x \in X$. To show that x belongs to the set $\bigcap\{B(F) : F \in J\}$ take any $F \in J$. Since J is upward directed, there exists $H^0 \in J$ such that $F \ll H^0$ and $F^0 \ll H^0$. Since the family $\mathcal{W}$ has no minimal element, we can find $G^0 \in J$ for which $H^0 \ll G^0$ and $B(G^0) \neq B(H^0)$.

Proceeding inductively, it is easy to construct a family $\{G^k : k \in \omega\} \subset J$ such that $G^k \ll G^{k+1}$ while $F^k \ll G^k$ and $B(G^k) \neq B(G^{k+1})$ for every $k \in \omega$. Property (1) shows that $\bigcap\{B(G^k) : k \in \omega\} = \{y\}$ for some $y \in X$. Observe that the inclusion $B(G^k) \subset B(F^k)$ holds for each $k \in \omega$, and therefore

$$\{y\} = \bigcap\{B(G^k) : k \in \omega\} \subset \bigcap\{B(F^k) : k \in \omega\} = \{x\}.$$

As an immediate consequence, $y = x$ so it follows from $F \ll H^0 \ll G^0$ that $B(G^0) \subset B(F)$ and therefore $x \in B(G^0) \subset B(F)$. Since $F \in J$ was chosen arbitrarily, we have proved that $x \in \bigcap\{B(F) : F \in J\}$ which shows that $\bigcap\{B(F) : F \in J\} \neq \emptyset$ and hence the triple $(Q, \ll, B)$ satisfies (DR4), i.e., X is domain representable. ∎

4.4 Proposition. *If X is a cosmic cofinally Polish space, then for any countable family $\mathcal{U} \subset \tau^*(X)$, there exists a Polish topology $\sigma \subset \tau(X)$ on the set X such that $\mathcal{U} \subset \sigma$.*

Proof. In a cosmic space all open sets are cozero sets so we can fix, for any $U \in \mathcal{U}$, a continuous function $f_U : X \to \mathbb{R}$ such that $U = f_U^{-1}(\mathbb{R}\backslash\{0\})$. The diagonal product $f = \Delta\{f_U : U \in \mathcal{U}\}$ maps X into a second countable space $\mathbb{R}^{\mathcal{U}}$; let $Y = f(X)$. It is standard that $f(U)$ is open in Y for any $U \in \mathcal{U}$.

Apply Proposition 3.5 to find a condensation $g : X \to P$ and a continuous onto map $h : P \to Y$ such that P is a Polish space and $h \circ g = f$. Then $\sigma = \{g^{-1}(V) : V \in \tau(P)\}$ is easily seen to be a Polish topology on X such that $\mathcal{U} \subset \sigma \subset \tau(X)$. ∎

4.5 Theorem. *A cosmic space is domain representable if and only if it is cofinally Polish.*

Proof. If X is a cosmic cofinally Polish space, then it is domain representable by Proposition 4.4 and Theorem 4.3; this settles sufficiency.

Suppose that a cosmic space X is domain representable and fix a partially ordered set $(Q, \ll)$ together with a map $B : Q \to \tau^*(X)$ such that the triple $(Q, \ll, B)$ represents the space X. Recall that $C(p,q)$ is the regular refinement of the pair (p,q) for any $p, q \in Q$ (see Proposition 4.1). For every $U \in \tau^*(X)$ choose a countable set $H(U) \subset Q$ such that $U = \bigcup\{B(p) : p \in H(U)\}$.

Suppose that $f : X \to M$ is a continuous onto map of X onto a second countable space M. Take a countable base $\mathcal{G}$ of the space M and let $A \subset Q$ be a countable set such that $H(f^{-1}(G)) \subset A$ for any $G \in \mathcal{G}$.

Let $\mathcal{N}$ be a countable network of the space X; there is no loss of generality to assume that all elements of $\mathcal{N}$ are closed. It is easy to find a countable set $A' \subset Q$ such that $H(X \backslash N) \subset A'$ for every $N \in \mathcal{N}$. Consider the set $R_0 = A \cup A'$.

It is easy to see that, proceeding inductively, we can construct an increasing sequence $\{R_n : n \in \omega\}$ of countable subsets of Q with the following properties:

(1) if $n \in \omega$, then $H(X \backslash \overline{B(q)}\,) \subset R_{n+1}$ for any $q \in R_n$;
(2) if $n \in \omega$ and $p, q \in R_n$, then $C(p,q) \subset R_{n+1}$.

For the set $R = \bigcup_{n\in\omega} R_n$, we will prove that the family $\mathcal{B} = \{B(q) : q \in R\}$ is a base of a Tychonoff topology on the set X. Given any $x \in X$ take any $G \in \mathcal{G}$ such that $f(x) \in G$. Then $x \in f^{-1}(G)$ and hence $x \in B(p)$ for some $p \in A \subset R$. This proves that $\bigcup \mathcal{B} = X$.

If $p, q \in R$ and $x \in B(p) \cap B(q)$, then there exists $n \in \omega$ such that $\{p, q\} \subset R_n$ and hence $C(p,q) \subset R_{n+1}$. This implies that $B(q) \cap B(q) = \bigcup\{B(r) : r \in L\}$ for some $L \subset R_{n+1} \subset R$. This shows that $x \in B(r)$ for some $r \in L$ and therefore $x \in B(r) \subset B(p) \cap B(q)$. Since $B(r) \in \mathcal{B}$, we have established that $\mathcal{B}$ is, indeed, a base for some topology σ on X; let $X' = (X, \sigma)$.

Take any point $x \in X$; the space X being Tychonoff, the set $\{x\}$ is closed in X and hence there is a family $\mathcal{N}' \subset \mathcal{N}$ such that $\{x\} = \bigcap \mathcal{N}'$. For any $N \in \mathcal{N}'$, there exists a set $A_N \subset A'$ such that $X \backslash N = \bigcup\{B(q) : q \in A_N\}$. This implies the equality $X \backslash \{x\} = \bigcup\{X \backslash N : N \in \mathcal{N}'\} = \bigcup \mathcal{C}$ for the family $\mathcal{C} = \bigcup\{B(q) : q \in \bigcup\{A_N : N \in \mathcal{N}'\}\}$. Since $\mathcal{C} \subset \mathcal{B}$, the set $X \backslash \{x\}$ is open in X' for any $x \in X$ and hence X' is a T_1-space.

Now take any $x \in X$ and $U \in \sigma$ such that $x \in U$. There exists $q \in R$ such that $x \in B(q) \subset U$. Pick $p \in C(q,q)$ such that $x \in B(p) \subset \overline{B(p)} \subset B(q)$. If $p \in R_n$, then there is a set $E \subset R_{n+1}$ such that $X \backslash \overline{B(p)} = \bigcup\{B(r) : r \in E\}$. Since $\{B(r) : r \in E\} \subset \mathcal{B}$, the set $B(p)$ is closed in X'. This proves that X' is a regular T_1-space; since it is also second countable, the space X' is Tychonoff as promised.

Fix again a point $x \in X$ and assume that $x \in B(p) \cap B(q)$ for some $p, q \in R$. Take a number $n \in \omega$ such that $\{p, q\} \subset R_n$ and observe that $C(p,q) \subset R_{n+1}$. There exists $r \in C(p,q)$ such that $x \in B(r)$ while $p \ll r$ and $q \ll r$ and hence we proved that the set $\{p \in R : x \in B(p)\}$ is upward closed for any $x \in X$. By Proposition 4.2, the space X' is domain representable and hence Polish because it is second countable (see Theorem 1.1 of [1]).

Observe that if follows from the choice of the set A that $f^{-1}(G)$ is open in X' for any $G \in \mathcal{G}$ and hence the map f is continuous on X'. If $g : X \to X'$ is the identity map and $h = f$, then $h \circ g = f$ and hence the maps g and h witness that the space X is cofinally Polish, i.e., we established necessity. ∎

4.6 Corollary. *If X is a subcompact cosmic space, then X is cofinally Polish.*

Proof. Every subcompact space is domain representable by [2, Proposition 5.1] so X is cofinally Polish by Theorem 4.5. ∎

4.7 Corollary. *For any countable space X, the following are equivalent:*

(a) X is subcompact;
(b) X is domain representable;
(c) X is cofinally Polish;
(d) X is scattered.

Proof. It follows from [7, Theorem 2.1] that (a) $\iff$ (d) and (b) $\iff$ (c) by Theorem 4.5. The implication (a)$\Longrightarrow$(b) is also known (see Proposition 5.1 of the paper [2]) so assume that X is a cofinally Polish countable space. There exists a condensation $f : X \to M$ for some second countable space M. We can find continuous maps $g : X \to P$ and $h : P \to M$ such that $f = h \circ g$, and the space P is Polish. It follows from bijectivity of f that g is also a bijection so P is scattered being a Polish countable space. Therefore X is also scattered and hence subcompact by [7, Theorem 2.1]. ∎

4.8 Corollary. *If X is a cosmic cofinally Polish space, then every closed subset of X is cofinally Polish.*

Proof. The space X must be domain representable by Theorem 4.5. If $F \subset X$ is closed, then it is a G_δ-subset of X so F is also domain representable by Theorem 3.2 of [1]. Applying Theorem 4.5 once again we conclude that F is cofinally Polish. ∎

4.9 Observation. For a general cofinally Polish space, Corollary 4.8 does not necessarily hold. Indeed, consider a Mrowka–Isbell Ψ-space X, (see e.g., [12, Problem 142]) and observe that X is pseudocompact but there exists a closed subset $F \subset X$ that is discrete and uncountable. The space X is cofinally Polish by Proposition 3.18 while F is not cofinally Polish by Corollary 3.14.

4.10 Corollary. *If X is a cosmic cofinally Polish space, then every closed subspace of X has the Baire property.*

Proof. If a set $F \subset X$ is closed, then it is cofinally Polish by Corollary 4.8. Theorem 4.5 shows that F is domain representable and hence has the Baire property. ∎

5 G_δ-subspaces of Eberlein compacta

The unique result of this section gives a positive answer to Problem 3.11 from the paper [7].

5.1 Theorem. *If X is an Eberlein compact space, then every G_δ-subspace of X is subcompact.*

Proof. Take any G_δ-subspace Y of the space X. It was proved in [11] that there exists a subcompact set $P \subset Y$ such that P is open in Y and $F = Y \setminus P$ has no points of local compactness. The set $K = \overline{F}^X$ is Eberlein compact and F is a dense G_δ-subspace of K.

There exists a metrizable dense G_δ-subspace M of the space K; it is standard that $E = M \cap F$ is dense in F. Since E is metrizable, we can choose pairwise disjoint families $\mathcal{B}_n$ of open subsets of E such that $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ is a base in E. Choose a point $x_U \in U$ for each $U \in \mathcal{B}_n$ and let $D_n = \{x_U : U \in \mathcal{B}_n\}$. It is immediate that every set D_n is discrete and $\bigcup_{n \in \omega} D_n$ is dense in E and hence in F.

For every $U \in \mathcal{B}_n$ take a set $O(U) \in \tau(F)$ such that $O(U) \cap E = U$; it is easy to deduce from density of E in F that the family $\{O(U) : U \in \mathcal{B}_n\}$ is disjoint for every $n \in \omega$. Since the space F is Čech-complete and has no points of local compactness, we can apply Theorem 4.3 of [11] to conclude that F is subcompact. Therefore $Y = P \cup F$ is subcompact by [7, Theorem 2.5]. ∎

6 Open questions

The first thing that has to be discovered about a completeness property is how it interacts with classical properties. The list of open questions given below shows that our paper has only scratched the surface in this direction.

6.1 Question. *Is it true that every cosmic cofinally Polish space is subcompact?*

6.2 Question. *Suppose that X is a Čech-complete Lindelöf space. Must X be cofinally Polish?*

6.3 Question. *Suppose that X is a subcompact Lindelöf space. Must X be cofinally Polish?*

6.4 Question. *Suppose that X is a scattered Lindelöf space. Must X be cofinally Polish?*

6.5 Question. *Suppose that X is a Čech-complete hereditarily Lindelöf space. Must X be cofinally Polish?*

6.6 Question. *Suppose that X is a subcompact hereditarily Lindelöf space. Must X be cofinally Polish?*

6.7 Question. *Suppose that X is a perfect cofinally Polish space. Must X have countable extent?*

6.8 Question. *Suppose that X is a space such that $C_p(X)$ is cofinally Polish. Then $C_p(X)$ has the Baire property but must it be pseudocomplete?*

6.9 Question. *Suppose that $C_p(X)$ is Lindelöf and cofinally Polish. Must X be discrete?*

6.10 Question. *Suppose that X is a space such that $C_p(X)$ has the Baire property. Must $C_p(X)$ be cofinally Polish?*

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