On well-posedness, regularity and ill-posedness for the nonlinear fourth-order Schrödinger equation

Van Duong Dinh

Abstract

We prove the local well-posedness for the nonlinear fourth-order Schrödinger equation (NL4S) in Sobolev spaces. We also study the regularity of local solutions in the sub-critical case. A direct consequence of this regularity is the global well-posedness above mass and energy spaces under some assumptions. Finally, we show the ill-posedness for (NL4S) in some cases of the super-critical range.

1 Introduction

We consider the Cauchy problem for the fourth-order Schrödinger equation posed on $\mathbb{R}^d, d \geq 1$, namely

$$\left\{ \begin{array}{rcl} i\partial_t u(t,x) + \Delta^2 u(t,x) &=& -\mu |u|^{\nu-1} u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) &=& u_0(x), \quad x \in \mathbb{R}^d. \end{array} \right. \tag{NL4S}$$

where $\nu > 1$ and $\mu \in \{\pm 1\}$. The number $\mu = 1$ (resp. $\mu = -1$) corresponds to the defocusing case (resp. focusing case).

The fourth-order Schrödinger equation was introduced by Karpman [Kar96] and Karpman and Shagalov [KS00] taking into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium

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with Kerr nonlinearity. The study of nonlinear fourth-order Schrödinger equation has been attracted a lot of interest in a past decade (see [Pau1], [Pau2], [PS10], [HHW06], [HHW07], [HJ05], [Din2] and references cited therein).

It is worth noticing that if we set for $\lambda > 0$,

$$u_\lambda(t,x) = \lambda^{-\frac{4}{\nu-1}} u(\lambda^{-4}t, \lambda^{-1}x), \tag{1.1}$$

then (NL4S) is invariant under this scaling. An easy computation shows

$$\|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\frac{d}{2}-\frac{4}{\nu-1}-\gamma}\|u_0\|_{\dot{H}^\gamma},$$

where $\dot{H}^\gamma$ is the homogeneous Sobolev space. From this, we define the critical regularity exponent for (NL4S) by

$$\gamma_c = \frac{d}{2} - \frac{4}{\nu-1}. \tag{1.2}$$

One said that H^γ is sub-critical (critical, super-critical) if $\gamma > \gamma_c$ ($\gamma = \gamma_c$, $\gamma < \gamma_c$) respectively. Another important property of (NL4S) is that the mass and energy are formally conserved under the flow of the equation,

$$M(u(t)) = \int |u(t,x)|^2 dx, \quad E(u(t)) = \int \frac{1}{2}|\Delta u(t,x)|^2 + \frac{\mu}{\nu+1}|u(t,x)|^{\nu+1}dx.$$

The main purpose of this note is to study the well-posedness and ill-posedness results for (NL4S) in Sobolev spaces. In [Din1], the local well-posedness for the nonlinear fractional Schrödinger equation including the fourth-order Schrödinger equation in both sub-critical and critical cases are showed. We shall review the local well-posedness for the nonlinear fourth-order Schrödinger equation below. These results are very similar to the nonlinear Schrödinger equation given in [CW90]. We also give the local well-posedness in the critical Sobolev space $H^{d/2}$. The global well-posedness in L^2 is then a direct consequence of the local existence and the conservation of mass. We also recall (see e.g. [Pau1] or [Din1]) the global well-posedness in the energy space H^2 under some assumptions. We next show the regularity of local solutions in the sub-critical case. As a consequence of this regularity, we obtain the global well-posedness above the mass and energy spaces for (NL4S) under some assumptions. The second part of this note is devoted to the ill-posedness of (NL4S). It is easy to see (e.g [LS95]) that (NL4S) is ill-posed in $\dot{H}^\gamma$ for $\gamma < \gamma_c$. Indeed if u solves (NL4S) with initial data $u_0 \in \dot{H}^\gamma$, then the norm $\|u_\lambda(0)\|_{\dot{H}^\gamma}$ and the lifespan of u_λ go to zero as $\lambda \to 0$. Using the technique of Christ-Colliander-Tao given in [CCT03], we are able to prove the ill-posedness for (NL4S) in H^γ with

$$\begin{cases} \gamma \in (-\infty, -d/2] \cup [0, \gamma_c) & \text{when } \gamma_c > 0, \\ \gamma \in (-\infty, -d/2] \cap (-\infty, \gamma_c) & \text{otherwise.} \end{cases} \tag{1.3}$$

This ill-posed result is similar to the nonlinear semi-relativistic equation given in [Din3]. Note that for the nonlinear Schrödinger equation, the ill-posedness holds in H^γ for $\gamma < \max\{0, \gamma_c\}$ (see [CCT03]). The main difference is that the nonlinear Schrödinger equation has the Galilean invariance while (NL4S) does not share this property. The Galilean invariance plays a crucial role in the proof of the ill-posedness in the range $\gamma \in (-d/2, 0)$. Recently, Hong and Sire in [HS15] used the pseudo-Galilean transformation to show the ill-posedness for the nonlinear fractional Schrödinger equation in Sobolev spaces of negative exponent. Unfortunately, it seems to be difficult to control the error of the pseudo-Galilean transformation in high Sobolev norms, and their result (see [HS15, Theorem 1.5]) only holds in one dimension. We finally note that the well-posedness, regularity for (NL4S) given in this note can be applied for the nonlinear fractional Schrödinger equation of order greater than or equal to 2 without any difficulty. Moreover, the ill-posedness argument can be adapted for the nonlinear fractional Schrödinger equation of any order.

Before stating our results, let us introduce some notations (see e.g. [GV85, Appendix], [Tri83, Chapter 5] or [BL76, Chapter 6]). Given $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, the generalized Sobolev space is defined by

$$H^\gamma_q := \Big\{ u \in \mathscr{S}' \mid \|u\|_{H^\gamma_q} := \| \langle \Lambda \rangle^\gamma u\|_{L^q} < \infty \Big\}, \quad \Lambda = \sqrt{-\Delta},$$

where $\langle x \rangle = \sqrt{1+|x|^2}$ is the Japanese bracket and $\mathscr{S}'$ is the space of tempered distributions. The generalized homogeneous Sobolev space is defined by

$$\dot{H}^\gamma_q := \Big\{ u \in \mathscr{S}'_0 \mid \|u\|_{\dot{H}^\gamma_q} := \|\Lambda^\gamma u\|_{L^q} < \infty \Big\},$$

where $\mathscr{S}_0$ is a subspace of the Schwartz space $\mathscr{S}$ consisting of functions ϕ satisfying $D^\alpha \hat{\phi}(0) = 0$ for all $\alpha \in \mathbb{N}^d$ with $\hat{\cdot}$ the Fourier transform on $\mathscr{S}$, and $\mathscr{S}'_0$ is its topology dual space. One can see $\mathscr{S}'_0$ as $\mathscr{S}'/\mathscr{P}$ where $\mathscr{P}$ is the set of all polynomials on $\mathbb{R}^d$. Under these settings, H^γ_q and $\dot{H}^\gamma_q$ equipped with the norms $\|u\|_{H^\gamma_q}$ and $\|u\|_{\dot{H}^\gamma_q}$ are Banach spaces. In the sequel, we shall use $H^\gamma := H^\gamma_2$, $\dot{H}^\gamma := \dot{H}^\gamma_2$. We also have for $\gamma > 0$, $H^\gamma_q = L^q \cap \dot{H}^\gamma_q$.

Throughout this note, a pair (p, q) is said to be admissible if

$$(p,q) \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

We also denote for $(p,q) \in [1,\infty]^2$,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{4}{p}. \tag{1.4}$$

Since we are working in spaces of fractional order γ or β, we need the nonlinearity $F(z) = -\mu|z|^{\nu-1}z$ to have enough regularity. When ν is an odd integer, $F \in C^\infty(\mathbb{C},\mathbb{C})$ (in the real sense). When ν is not an odd integer, we need the following assumption

$$\lceil \gamma \rceil \text{ or } \lceil \beta \rceil \leq \nu, \tag{1.5}$$

where $\lceil \gamma \rceil$ is the smallest integer greater than or equal to γ, similarly for β. Our first result concerns the local well-posedness of (NL4S) in both sub-critical and critical cases.

Theorem 1.1. *Let* $\gamma \in [0, d/2)$ *be such that* $\gamma \geq \gamma_{\rm c}$, *and also, if* $\nu > 1$ *is not an odd integer,* (1.5). *Let*

$$p = \frac{8(\nu+1)}{(\nu-1)(d-2\gamma)}, \quad q = \frac{d(\nu+1)}{d+(\nu-1)\gamma}. \tag{1.6}$$

Then for all $u_0 \in H^\gamma$, *there exist* $T^* \in (0,\infty]$ *and a unique solution to* (NL4S) *satisfying*

$$u \in C([0,T^*), H^\gamma) \cap L^p_{\rm loc}([0,T^*), H^\gamma_q).$$

Moreover, the following properties hold:

(i) $u \in L^a_{\rm loc}([0,T^*), H^\gamma_b)$ *for any admissible pair* (a,b) *with* $b < \infty$ *and* $\gamma_{a,b} = 0$.

(ii) $M(u(t)) = M(u_0)$ *for any* $t \in [0,T^*)$.

(iii) If $\gamma \geq 2$, $E(u(t)) = E(u_0)$ *for any* $t \in [0,T^*)$.

(iv) If $\gamma > \gamma_{\rm c}$ *and* $T^* < \infty$, *then* $\|u(t)\|_{\dot{H}^\gamma} \to \infty$ *as* $t \to T^*$.

(v) If $\gamma = \gamma_{\rm c}$ *and* $T^* < \infty$, *then* $\|u\|_{L^p([0,T^*), H^{\gamma_{\rm c}}_q)} = \infty$.

(vi) u *depends continuously on* u_0 *in the following sense. There exists* $0 < T < T^*$ *such that if* $u_{0,n} \to u_0$ *in* H^γ *and if* u_n *denotes the solution of* (NL4S) *with initial data* $u_{0,n}$, *then* $0 < T < T^*(u_{0,n})$ *for all* n *sufficiently large and* u_n *is bounded in* $L^a([0,T], H^\gamma_b)$ *for any admissible pair* (a,b) *with* $\gamma_{a,b} = 0$ *and* $b < \infty$. *Moreover,* $u_n \to u$ *in* $L^a([0,T], L^b)$ *as* $n \to \infty$. *In particular,* $u_n \to u$ *in* $C([0,T], H^{\gamma-\epsilon})$ *for all* $0 < \epsilon < \gamma$.

(vii) If $\gamma = \gamma_{\rm c}$ *and* $\|u_0\|_{\dot{H}^{\gamma_{\rm c}}} < \varepsilon$ *for some* $\varepsilon > 0$ *small enough, then* $T^* = \infty$ *and the solution is scattering in* $H^{\gamma_{\rm c}}$, *i.e. there exists* $u_0^+ \in H^{\gamma_{\rm c}}$ *such that*

$$\lim_{t \to +\infty} \|u(t) - e^{it\Delta^2} u_0^+\|_{H^{\gamma_{\rm c}}} = 0.$$

We also have the following local well-posedness in the critical Sobolev space $H^{d/2}$.

Theorem 1.2. *Let* $\gamma = d/2$ *be such that if* $\nu > 1$ *is not an odd integer,* (1.5). *Then for all* $u_0 \in H^{d/2}$, *there exists* $T^* \in (0,\infty]$ *and a unique solution to* (NL4S) *satisfying*

$$u \in C([0,T^*), H^{d/2}) \cap L^p_{\rm loc}([0,T^*), L^\infty),$$

for some $p > \max(\nu-1, 4)$ *when* $d = 1$ *and some* $p > \max(\nu-1, 2)$ *when* $d \geq 2$. *Moreover, the following properties hold:*

(i) $u \in L^a_{\rm loc}([0,T^*), H^{d/2}_b)$ *for any admissible pair* (a,b) *with* $b < \infty$ *and* $\gamma_{a,b} = 0$.

(ii) If $T^* < \infty$, *then* $\|u(t)\|_{H^{d/2}} \to \infty$ *as* $t \to T^*$.

(iii) u *depends continuously on* u_0 *in the sense of* Theorem 1.1

The continuous dependence can be improved (see Remark 2.8) if we assume that $\nu > 1$ is an odd integer or $\lceil d/2 \rceil \leq \nu - 1$ otherwise. Concerning the well-posedness of the nonlinear Schrödinger equation in this critical space, we refer to [Kat95] and [NO98]. Note that in [NO98], the global well-posedness with small data is proved with exponential-type nonlinearity but not the local well-posedness without size restriction on the initial data.

It is well-known that (see [Caz03, Chapter 4], [Kat95] or [Tao06, Chapter 3]) that for $\gamma > d/2$, the nonlinear Schrödinger equation is locally well-posed provided the nonlinearity has enough regularity. It is not a problem to extend this result for the nonlinear fourth-order Schrödinger equation. For the sake of completeness, we state (without proof) the local well-posedness for (NL4S) in this range.

Theorem 1.3. *Let $\gamma > d/2$ be such that if $\nu > 1$ is not an odd integer,* (1.5). *Then for all $u_0 \in H^\gamma$, there exist $T^* \in (0, \infty]$ and a unique solution $u \in C([0, T^*), H^\gamma)$ to* (NL4S). *Moreover, the following properties hold:*

(i) $u \in L^a_{\text{loc}}([0, T^*), H^\gamma_b)$ *for any admissible pair* (a, b) *with* $b < \infty$ *and* $\gamma_{a,b} = 0$.

(ii) If $T^* < \infty$, *then* $\|u(t)\|_{H^\gamma} \to \infty$ *and* $\limsup \|u(t)\|_{L^\infty} \to \infty$ *as* $t \to T^*$.

(iii) u *depends continuously on* u_0 *in the following sense. There exists* $0 < T < T^*$ *such that if* $u_{0,n} \to u_0$ *in* H^γ *and if* u_n *is the solution of* (NL4S) *with the initial data* $u_{0,n}$, *then* $u_n \to u$ *in* $C([0, T], H^\gamma)$.

Corollary 1.4. *Let $\nu \in (1, 1 + 8/d)$. Then for all $u_0 \in L^2$, there exists a unique global solution to* (NL4S) *satisfying $u \in C(\mathbb{R}, L^2) \cap L^p_{\text{loc}}(\mathbb{R}, L^q)$, where (p, q) given in* (1.6).

In the energy space H^2, we have the following global well-posedness result.

Proposition 1.5 ([Paul] or [Din1]). *Let $\nu \in (1, 1 + 8/(d - 4))$ for $d \geq 5$ and $\nu > 1$ for $d \leq 4$. Then for any $u_0 \in H^2$, the solution to* (NL4S) *given in* Theorem 1.1, Theorem 1.2 and Theorem 1.3 *can be extended to the whole $\mathbb{R}$ if one of the following is satisfied:*

(i) $\mu = 1$.

(ii) $\mu = -1, \nu < 1 + 8/d$.

(iii) $\mu = -1, \nu = 1 + 8/d$ *and* $\|u_0\|_{L^2}$ *is small.*

(iv) $\mu = -1$ *and* $\|u_0\|_{H^2}$ *is small.*

Our next result concerns with the regularity of solutions of (NL4S) in the sub-critical case.

Theorem 1.6. *Let $\beta > \gamma \geq 0$ be such that $\gamma > \gamma_c$, and also, if $\nu > 1$ is not an odd integer,* (1.5). *Let $u_0 \in H^\gamma$ and u be the corresponding H^γ solution of* (NL4S) *given in* Theorem 1.1, Theorem 1.2, Theorem 1.3. *If $u_0 \in H^\beta$, then $u \in C([0, T^*), H^\beta)$.*

The following result is a direct consequence of Theorem 1.6 and the global well-posedness in Corollary 1.4 and Proposition 1.5.

Corollary 1.7. *(i) Let $\gamma \geq 0$ and $\nu \in (1, 1+8/d)$ be such that if ν is not an odd integer,* (1.5)*. Then* (NL4S) *is globally well-posed in H^γ.*

(ii) Let $\gamma \geq 2$, $\nu \in [1+8/d, 1+8/(d-4))$ for $d \geq 5$ and $\nu \in [1+8/d, \infty)$ for $d \leq 4$ be such that if ν is not an odd integer, (1.5)*. Then* (NL4S) *is globally well-posed in H^γ provided one of conditions* (i), (iii), (iv) *in* Proposition 1.5 *is satisfied.*

In [PS10], the authors proved the global existence for the L^2-critical (NL4S) in higher dimensions $d \geq 5$. More precisely, they proved that the equation is globally well-posed in L^2

- for any initial data in L^2 in the defocusing case,
- for initial data in L^2 satisfying $\|u_0\|_{L^2} < \|Q\|_{L^2}$ in the focusing case, where Q is the solution to the elliptic equation

$$\Delta^2 Q + Q = |Q|^{\frac{8}{d}} Q. \tag{1.7}$$

Moreover, in both cases, the following uniform bound holds true

$$\|u\|_{L^{2+\frac{8}{d}}(\mathbb{R}, L^{2+\frac{8}{d}})} \leq C(\|u_0\|_{L^2}).$$

With this uniform bound, we have the following global existence for the L^2-critical (NL4S) in dimensions $d \geq 5$.

Proposition 1.8. *Let $d \geq 5, \nu = 1+8/d$ and $\beta > 0$. Let $u_0 \in H^\beta$ be such that if $\mu = -1$, $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the solution to* (1.7)*. Then the L^2-critical* (NL4S) *is globally well-posed in H^β.*

Our final result is the following ill-posedness for (NL4S).

Theorem 1.9. *Let $\nu > 1$ be such that if ν is not an odd integer, $\nu \geq k+1$ for some integer $k > d/2$. Then* (NL4S) *is ill-posed in H^γ with γ satisfying* (1.3)*. More precisely, if $\gamma \in (-\infty, -d/2] \cup (0, \gamma_c)$ when $\gamma_c > 0$ or $\gamma \in (-\infty, -d/2] \cap (-\infty, \gamma_c)$ otherwise, then for any $t > 0$ the solution map $\mathscr{S} \ni u(0) \mapsto u(t)$ of* (NL4S) *fails to be continuous at 0 in the H^γ topology. Moreover, if $\gamma_c > 0$, the solution map fails to be uniformly continuous on L^2.*

The proof of Theorem 1.9 is based on the small dispersion analysis given in [CCT03]. Note that when $\nu = 3$ and $\mu = 1$ corresponding to the defocusing cubic nonlinearity, Pausader in [Pau2] proved the ill-posedness for (NL4S) in $H^2(\mathbb{R}^d)$ with $d \geq 9$.

This note is organized as follows. In Section 2, we recall Strichartz estimates for the inhomogeneous fourth-order Schrödinger equation and the nonlinear fractional derivatives. We end Section 2 with the proof of local well-posedness given in Theorem 1.1 and Theorem 1.2. In Section 3, we give the proofs of the regularity for solutions of (NL4S) in Theorem 1.6 and Proposition 1.8. Finally, the proof of the ill-posedness result is given in Section 4.

2 Well-posedness

In this section, we will give the proofs of local well-posedness in Theorem 1.1 and Theorem 1.2. Our proofs are based on the standard contraction mapping argument using Strichartz estimates and nonlinear fractional derivatives (see Subsection 2.2).

2.1 Strichartz estimate

In this subsection, we recall Strichartz estimates for the fourth-order Schrödinger equation.

Proposition 2.1 ([COX11]). *Let u be a (weak) solution to the inhomogeneous fourth-order Schrödinger equation, namely*

$$u(t) = e^{it\Delta^2}u_0 + \int_0^t e^{i(t-s)\Delta^2}F(s)ds,$$

for some data u_0, F. Then for all (p,q) and (a,b) admissible with $q < \infty$ and $b < \infty$,

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|u_0\|_{\dot{H}^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{2.1}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + 4. \tag{2.2}$$

Here (a,a') is a conjugate pair and similarly for (b,b').

Remark 2.2. The estimate (2.1) is exactly the one given in [Pau1] or [Pau2] where the author considered (p,q) and (a,b) are Schrödinger admissible, i.e.

$$p,q \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

We refer to [COX11] (see also [Din1]) for the proof of Proposition 2.1. Note that rather than using directly a dedicate dispersive estimate of [BKS00] for the fundamental solution of the homogeneous fourth-order Schrödinger equation, one uses the scaling technique which is similar to those of wave equation (see e.g. [KT98]).

We also have the following local Strichartz estimates (see again [Din1]).

Corollary 2.3. *Let $\gamma \geq 0$ and I be a bounded interval. If u is a weak solution to the linear fourth-order Schrödinger equation for some data u_0, F, then for all (p,q) admissible satisfying $q < \infty$,*

$$\|u\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|u_0\|_{H^\gamma} + \|F\|_{L^1(I,H^\gamma)}. \tag{2.3}$$

2.2 Nonlinear fractional derivatives

In this subsection, we recall some nonlinear fractional derivatives estimates related to our purpose. Let us start with the following fractional Leibniz rule (or Kato-Ponce inequality). We refer to [Gra14] for the proof of a more general result.

Proposition 2.4. *Let $\gamma \geq 0, 1 < r < \infty$ and $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Then there exists $C = C(d, \gamma, r, p_1, q_1, p_2, q_2) > 0$ such that for all $u, v \in \mathscr{S}$,

$$\|\Lambda^\gamma (uv)\|_{L^r} \leq C\Big(\|\Lambda^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \|\Lambda^\gamma v\|_{L^{q_2}}\Big).$$

We also have the following fractional chain rule (see [CW91] or [Sta95]).

Proposition 2.5. *Let $F \in C^1(\mathbb{C}, \mathbb{C})$ and $G \in C(\mathbb{C}, \mathbb{R}^+)$ such that $F(0) = 0$ and*

$$|F'(\theta z + (1-\theta)\zeta)| \leq \mu(\theta)(G(z) + G(\zeta)), \quad z, \zeta \in \mathbb{C}, \quad 0 \leq \theta \leq 1,$$

where $\mu \in L^1((0,1))$. Then for $\gamma \in (0,1)$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

there exists $C = C(d, \mu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\|\Lambda^\gamma F(u)\|_{L^r} \leq C\|F'(u)\|_{L^q} \|\Lambda^\gamma u\|_{L^p}.$$

Combining the fractional Leibniz rule and the fractional chain rule, one has the following result (see [Kat95, Appendix]).

Lemma 2.6. *Let $F \in C^k(\mathbb{C}, \mathbb{C}), k \in \mathbb{N}\backslash\{0\}$. Assume that there is $\nu \geq k$ such that*

$$|D^i F(z)| \leq C|z|^{\nu - i}, \quad z \in \mathbb{C}, \quad i = 1, 2, ..., k.$$

Then for $\gamma \in [0, k]$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$, there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\|\Lambda^\gamma F(u)\|_{L^r} \leq C\|u\|_{L^q}^{\nu-1} \|\Lambda^\gamma u\|_{L^p}. \tag{2.4}$$

Moreover, if F is a homogeneous polynomial in u and $\overline{u}$, then (2.4) holds true for any $\gamma \geq 0$.

Remark 2.7. By Lemma 2.6, we see that the condition (1.5) ensures the nonlinearity to have enough regularity in order to apply the fractional derivatives.

2.3 Proof of Theorem 1.1

We are now able to prove Theorem 1.1. Let us firstly comment about the choice of (p,q) given in (1.6). It is easy to see that (p,q) is admissible and $\gamma_{p,q} = 0 = \gamma_{p',q'} + 4$. This allows us to use Strichartz estimate (2.1) for (p,q). Moreover, we choose (m,n) so that

$$\frac{1}{p'} = \frac{1}{m} + \frac{\nu - 1}{p}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{\nu - 1}{n}. \tag{2.5}$$

Thanks to this choice of n, we have the Sobolev embedding $\dot{H}^\gamma_q \hookrightarrow L^n$ since

$$q \leq n = \frac{dq}{d - \gamma q}.$$

Step 1. Existence. Let us consider

$$X := \Big\{ u \in L^p(I, H^\gamma_q) \mid \|u\|_{L^p(I,\dot{H}^\gamma_q)} \leq M \Big\},$$

equipped with the distance

$$d(u,v) = \|u - v\|_{L^p(I,L^q)},$$

where $I = [0,T]$ and $M, T > 0$ to be chosen later. It is easy to verify (see e.g. [CW90] or [Caz03, Chapter 4]) that (X,d) is a complete metric space. By the Duhamel formula, it suffices to prove that the functional

$$\Phi(u)(t) = e^{it\Delta^2}u_0 + i\mu \int_0^t e^{i(t-s)\Delta^2}|u(s)|^{\nu-1}u(s)ds =: u_{\text{hom}}(t) + u_{\text{inh}}(t) \tag{2.6}$$

is a contraction on (X,d).

Let us firstly consider the case $\gamma > \gamma_c$. In this case, we have $1 < m < p$ and

$$\frac{1}{m} - \frac{1}{p} = 1 - \frac{(\nu - 1)(d - 2\gamma)}{8} =: \theta > 0. \tag{2.7}$$

Using Strichartz estimate (2.1), we obtain

$$\|\Phi(u)\|_{L^p(I,\dot{H}^\gamma_q)} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F(u)\|_{L^{p'}(I,\dot{H}^\gamma_{q'})},$$
$$\|\Phi(u) - \Phi(v)\|_{L^p(I,L^q)} \lesssim \|F(u) - F(v)\|_{L^{p'}(I,L^{q'})},$$

where $F(u) = |u|^{\nu-1}u$ and similarly for $F(v)$. It then follows from Lemma 2.6, (2.5), Sobolev embedding and (2.7) that

$$\|F(u)\|_{L^{p'}(I,\dot{H}^\gamma_{q'})} \lesssim T^\theta \|u\|^\nu_{L^p(I,\dot{H}^\gamma_q)}, \tag{2.8}$$

$$\|F(u) - F(v)\|_{L^{p'}(I,L^{q'})} \lesssim T^\theta \Big(\|u\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)} + \|v\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)} \Big) \|u - v\|_{L^p(I,L^q)}. \tag{2.9}$$

This shows that for all $u, v \in X$, there exists $C > 0$ independent of T and $u_0 \in H^\gamma$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^p(I,\dot{H}^\gamma_q)} &\leq C\|u_0\|_{\dot{H}^\gamma} + CT^\theta M^\nu,\\ d(\Phi(u),\Phi(v)) &\leq CT^\theta M^{\nu-1} d(u,v).\end{aligned}$$

If we set $M = 2C\|u_0\|_{\dot{H}^\gamma}$ and choose $T > 0$ so that

$$CT^\theta M^{\nu-1} \leq \frac{1}{2},$$

then Φ is a strict contraction on (X, d).

We now turn to the case $\gamma = \gamma_c$. We have from Strichartz estimate (2.1) that

$$\|u_{\rm hom}\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \lesssim \|u_0\|_{\dot{H}^{\gamma_c}}.$$

This shows that $\|u_{\rm hom}\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \leq \varepsilon$ for some $\varepsilon > 0$ small enough provided that T is small or $\|u_0\|_{\dot{H}^{\gamma_c}}$ is small. We also have from (2.1) that

$$\|u_{\rm inh}\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \lesssim \|F(u)\|_{L^{p'}(I,\dot{H}^{\gamma_c}_{q'})}.$$

Lemma (2.6), (2.5) and Sobolev embedding (note that in this case $m = p$) then yield that

$$\|F(u)\|_{L^{p'}(I,\dot{H}^{\gamma_c}_{q'})} \lesssim \|u\|^\nu_{L^p(I,\dot{H}^{\gamma_c}_q)}, \tag{2.10}$$

$$\|F(u) - F(v)\|_{L^{p'}(I,L^{q'})} \lesssim \left(\|u\|^{\nu-1}_{L^p(I,\dot{H}^{\gamma_c}_q)} + \|v\|^{\nu-1}_{L^p(I,\dot{H}^{\gamma_c}_q)}\right)\|u - v\|_{L^p(I,L^q)}. \tag{2.11}$$

This implies that for all $u, v \in X$, there exists $C > 0$ independent of T and $u_0 \in H^{\gamma_c}$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^p(I,\dot{H}^{\gamma_c}_q)} &\leq \varepsilon + CM^\nu,\\ d(\Phi(u),\Phi(v)) &\leq CM^{\nu-1} d(u,v).\end{aligned}$$

If we choose ε and M small so that

$$CM^{\nu-1} \leq \frac{1}{2}, \quad \varepsilon + \frac{M}{2} \leq M,$$

then Φ is a contraction on (X, d).

Therefore, in both sub-critical and critical cases, Φ has a unique fixed point in X. Moreover, since $u_0 \in H^\gamma$ and $u \in L^p(I, H^\gamma_q)$, the Strichartz estimate shows that $u \in C(I, H^\gamma)$ (see e.g. [CW90] or [Caz03, Chapter 4]). This shows the existence of solution $u \in C(I, H^\gamma) \cap L^p(I, H^\gamma_q)$ to (NL4S). Note that in the case $\gamma = \gamma_c$, if $\|u_0\|_{\dot{H}^{\gamma_c}}$ is small enough, then we can take $T = \infty$.

Step 2. Uniqueness. It follows easily from (2.9) and (2.11) using the fact that $\|u\|_{L^p(I,\dot{H}^\gamma_q)}$ can be small if T is small.

Step 3. Item (i). Let $u \in C(I, H^\gamma) \cap L^p(I, H^\gamma_q)$ be a solution to (NLFS) where

$I=[0,T]$ and (a,b) an admissible pair with $b<\infty$ and $\gamma_{a,b}=0$. Then Strichartz estimate (2.1) implies

$$\|u\|_{L^a(I,L^b)} \lesssim \|u_0\|_{L^2} + \|F(u)\|_{L^{p'}(I,L^{q'})}, \tag{2.12}$$

$$\|u\|_{L^a(I,\dot{H}^\gamma_b)} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F(u)\|_{L^{p'}(I,\dot{H}^\gamma_{q'})}. \tag{2.13}$$

It then follows from (2.8) and (2.10) that $u \in L^a(I,H^\gamma_b)$.

Step 4. Item (ii) and (iii). The conservation of mass and energy follows similarly as for the Schrödinger equation (see e.g. [CW90], [Caz03, Chapter 4] or [Gin98, Chapter 5]).

Step 5. Item (iv). The blowup alternative in sub-critical case is easy since the time of existence depends only on $\|u_0\|_{\dot{H}^\gamma}$.

Step 6. Item (v). It also follows from a standard argument (see e.g. [CW90]). Indeed, if $T^*<\infty$ and $\|u\|_{L^p([0,T^*),H^{\gamma_c}_q)}<\infty$, then Strichartz estimate (2.1) implies that $u\in C([0,T^*],H^{\gamma_c})$. Thus, one can extend the solution to (NL4S) beyond T^*. It leads to a contradiction with the maximality of T^*.

Step 7. Item (vi). We use the argument given in [CW90]. From Step 1, in the sub-critical case, we can choose T and M so that the fixed point argument can be carried out on X for any initial data with $\dot{H}^\gamma$ norm less than $2\|u_0\|_{\dot{H}^\gamma}$. In the critical case, there exist T, M and an $\dot{H}^{\gamma_c}$ neighborhood U of u_0 such that the fixed point argument can be carried out on X for all initial data in U. Now let $u_{0,n}\to u_0$ in H^γ. In both sub-critical and critical cases, we see that $T<T^*(u_0)$, $\|u\|_{L^p([0,T],\dot{H}^\gamma_q)}\leq M$, and that for sufficiently large n, $T<T^*(u_{0,n})$ and $\|u_n\|_{L^p([0,T],\dot{H}^\gamma_q)}\leq M$. Thus, (2.12) and (2.13) together with (2.8) and (2.10) yield that u_n is bounded in $L^a([0,T],H^\gamma_b)$ for any admissible pair (a,b) with $b<\infty$ and $\gamma_{a,b}=0$. We also have from (2.9), (2.11) and the choice of T that

$$d(u_n,u)\leq C\|u_{0,n}-u_0\|_{L^2}+\frac{1}{2}d(u_n,u) \text{ or } d(u_n,u)\leq 2C\|u_{0,n}-u_0\|_{L^2}.$$

This shows that $u_n\to u$ in $L^p([0,T],L^q)$. Again (2.13) together with (2.9) and (2.11) implies that $u_n\to u$ in $L^a([0,T],L^b)$ for any admissible pair (a,b) with $b<\infty$ and $\gamma_{a,b}=0$. The convergence in $C(I,H^{\gamma-\epsilon})$ follows from the boundedness in $L^\infty(I,H^\gamma)$ and the convergence in $L^\infty(I,L^2)$ and that $\|u\|_{H^{\gamma-\epsilon}}\leq \|u\|^{1-\frac{\epsilon}{\gamma}}_{H^\gamma}\|u\|^{\frac{\epsilon}{\gamma}}_{L^2}$.

Step 8. Item (vii). As mentioned in Step 1, when $\|u_0\|_{\dot{H}^{\gamma_c}}$ is small, we can take $T^*=\infty$. It remains to prove the scattering property. To do so, we make use of the adjoint estimate to the homogeneous Strichartz estimate, namely $L^2\ni u_0\mapsto e^{it\Delta^2}u_0\in L^p(\mathbb{R},L^q)$ to obtain

$$\begin{aligned}\|e^{-it_2\Delta^2}u(t_2)-e^{-it_1\Delta^2}u(t_1)\|_{\dot{H}^{\gamma_c}} &= \Big\|i\mu\int_{t_1}^{t_2}e^{-is\Delta^2}F(u)(s)ds\Big\|_{\dot{H}^{\gamma_c}}\\ &= \Big\|i\mu\int_{t_1}^{t_2}\Lambda^{\gamma_c}e^{-is\Delta^2}(\mathbb{1}_{[t_1,t_2]}F(u))(s)ds\Big\|_{L^2}\\ &\lesssim \|F(u)\|_{L^{p'}([t_1,t_2],\dot{H}^{\gamma_c}_{q'})}.\end{aligned} \tag{2.14}$$

Similarly,

$$\|e^{-it_2\Delta^2}u(t_2) - e^{-it_1\Delta^2}u(t_1)\|_{L^2} \lesssim \|F(u)\|_{L^{p'}([t_1,t_2],L^{q'})}. \tag{2.15}$$

Thanks to (2.10) and (2.11), we get

$$\|e^{-it_2\Delta^2}u(t_2) - e^{-it_1\Delta^2}u(t_1)\|_{H^{\gamma_c}} \to 0,$$

as $t_1, t_2 \to +\infty$. This implies that the limit

$$u_0^+ := \lim_{t\to+\infty} e^{-it\Delta^2}u(t)$$

exists in H^{γ_c}. Moreover,

$$u(t) - e^{it\Delta^2}u_0^+ = -i\mu \int_t^{+\infty} e^{i(t-s)\Delta^2}F(u(s))ds.$$

Using again (2.14) and (2.15) together with (2.10) and (2.11), we have

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta^2}u_0^+\|_{H^{\gamma_c}} = 0.$$

This completes the proof of Theorem 1.1. ■

2.4 Proof of Theorem 1.2

We now turn to the proof of the local well-posedness in $H^{d/2}$. To do so, we firstly choose $p > \max(\nu-1, 4)$ when $d = 1$ and $p > \max(\nu-1, 2)$ when $d \geq 2$ and then choose $q \in [2,\infty)$ such that

$$\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

Step 1. Existence. We will show that Φ defined in (2.6) is a contraction on

$$X := \left\{u \in L^\infty(I, H^{d/2}) \cap L^p(I, H_q^{d/2-\gamma_{p,q}}) \mid \|u\|_{L^\infty(I,H^{d/2})} + \|u\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} \leq M\right\},$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^\infty(I,L^2)} + \|u-v\|_{L^p(I,H^{-\gamma_{p,q}})},$$

where $I = [0,T]$ and $M, T > 0$ to be determined. The local Strichartz estimate (2.3) gives

$$\|\Phi(u)\|_{L^\infty(I,H^{d/2})} + \|\Phi(u)\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} \lesssim \|u_0\|_{H^{d/2}} + \|F(u)\|_{L^1(I,H^{d/2})},$$

$$\|\Phi(u)-\Phi(v)\|_{L^\infty(I,L^2)} + \|\Phi(u)-\Phi(v)\|_{L^p(I,H_q^{-\gamma_{p,q}})} \lesssim \|F(u)-F(v)\|_{L^1(I,L^2)}.$$

Thanks to the assumptions on ν, Lemma 2.6 implies

$$\|F(u)\|_{L^1(I,H^{d/2})} \lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\|u\|_{L^\infty(I,H^{d/2})} \lesssim T^\theta \|u\|^{\nu-1}_{L^p(I,L^\infty)}\|u\|_{L^\infty(I,H^{d/2})}, \tag{2.16}$$

$$\begin{aligned}\|F(u)-F(v)\|_{L^1(I,L^2)} &\lesssim \left(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\right)\|u-v\|_{L^\infty(I,L^2)} \\ &\lesssim T^\theta\left(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)}\right)\|u-v\|_{L^\infty(I,L^2)}, \end{aligned} \tag{2.17}$$

where $\theta = 1 - \frac{\nu-1}{p} > 0$. Using the fact that $d/2 - \gamma_{p,q} > d/q$, the Sobolev embedding implies $H^{d/2-\gamma_{p,q}}_q \hookrightarrow L^\infty$. Thus,

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(I,H^{d/2})} + \|\Phi(v)\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} &\\ \lesssim \|u_0\|_{H^{d/2}} + T^\theta \|u\|^{\nu-1}_{L^p(I,H_q^{d/2-\gamma_{p,q}})}\|u\|_{L^\infty(I,H^{d/2})}&, \\ d(\Phi(u),\Phi(v)) \lesssim T^\theta\left(\|u\|^{\nu-1}_{L^p(I,H_q^{d/2-\gamma_{p,q}})} + \|v\|^{\nu-1}_{L^p(I,H_q^{d/2-\gamma_{p,q}})}\right)d(u,v).&\end{aligned}$$

Thus for all $u, v \in X$, there exists $C > 0$ independent of $u_0 \in H^{d/2}$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(I,H^{d/2})} + \|\Phi(v)\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} &\leq C\|u_0\|_{H^{d/2}} + CT^\theta M^\nu, \\ d(\Phi(u),\Phi(v)) &\leq CT^\theta M^{\nu-1} d(u,v).\end{aligned}$$

If we set $M = 2C\|u_0\|_{H^{d/2}}$ and choose $T > 0$ small enough so that $CT^\theta M^{\nu-1} \leq \frac{1}{2}$, then Φ is a contraction on X.

Step 2. Uniqueness. It is easy using (2.17) since $\|u\|_{L^p(I,L^\infty)}$ is small if T is small.

Step 3. Item (i). It follows easily from Step 1 and Strichartz estimate (2.3) that for any admissible pair (a, b) with $b < \infty$ and $\gamma_{a,b} = 0$,

$$\|u\|_{L^a(I,H_b^{d/2})} \lesssim \|u_0\|_{H^{d/2}} + \|F(u)\|_{L^1(I,H^{d/2})}.$$

Step 4. Item (ii). The blowup alternative is obvious since the time of existence depends only on $\|u_0\|_{H^{d/2}}$.

Step 5. Item (iii). The continuous dependence is similar to Step 7 of the proof of Theorem 1.1 using (2.17). ∎

Remark 2.8. If we assume that $\nu > 1$ is an odd integer or $\lceil d/2 \rceil \leq \nu - 1$ otherwise, then the continuous dependence holds in $C(I, H^{d/2})$. Indeed, we consider X as above equipped with the following metric

$$d(u,v) := \|u-v\|_{L^\infty(I,H^{d/2})} + \|u-v\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})}.$$

Thanks to the assumptions on ν, we are able to apply the fractional derivatives estimates (see e.g. [Kat95, Appendix] or [Din1, Corollary 3.5]) to have

$$\begin{aligned}&\|F(u)-F(v)\|_{L^1(I,H^{d/2})} \lesssim (\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)})\|u-v\|_{L^\infty(I,H^{d/2})} \\ &+ (\|u\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)})(\|u\|_{L^\infty(I,H^{d/2})} + \|v\|_{L^\infty(I,H^{d/2})})\|u-v\|_{L^{\nu-1}(I,L^\infty)}.\end{aligned}$$

The Sobolev embedding then implies that for all $u, v \in X$,

$$d(\Phi(u), \Phi(v)) \lesssim T^{\theta} M^{\nu-1} d(u,v).$$

The continuous dependence in $C(I, H^{d/2})$ follows as Step 7 of the proof of Theorem 1.1.

3 Regularity

The main purpose of this section is to prove the regularity of solutions of (NL4S) given in Theorem 1.6 and Proposition 1.8.

3.1 Proof of Theorem 1.6.

We follow the argument given in [Caz03, Chapter 5]. To do so, we will split γ into three cases $\gamma \in [0, d/2)$, $\gamma = d/2$ and $\gamma > d/2$.

The case $\gamma \in [0, d/2)$ Let $\beta > \gamma$. If $u_0 \in H^{\beta}$, then Theorem 1.1 or Theorem 1.2 or Theorem 1.3 shows that there exists a maximal solution to (NL4S) satisfying $u \in C([0,T), H^{\beta}) \cap L^a_{\text{loc}}([0,T), H^{\beta}_b)$ for any admissible pair (a,b) with $b < \infty$ and $\gamma_{a,b} = 0$. Since H^{β}-solution is in particular an H^{γ}-solution, the uniqueness implies that $T \leq T^*$. We will show that T is actually equal to T^*. Suppose that $T < T^*$, then the blowup alternative implies

$$\|u(t)\|_{H^{\beta}} \to \infty \text{ as } t \to T. \tag{3.1}$$

Moreover, since $T < T^*$, we have

$$\|u\|_{L^p((0,T), H^{\gamma}_q)} + \sup_{0 \leq t \leq T} \|u(t)\|_{H^{\gamma}} < \infty,$$

where (p,q) given in (1.6). Using Strichartz estimate (2.1), we have for any interval $I \subset (0,T)$,

$$\begin{aligned} \|u\|_{L^{\infty}(I,L^2)} + \|u\|_{L^p(I,L^q)} &\lesssim \|u_0\|_{L^2} + \|F(u)\|_{L^{p'}(I,L^{q'})}, \\ \|u\|_{L^{\infty}(I,\dot{H}^{\beta})} + \|u\|_{L^p(I,\dot{H}^{\beta}_q)} &\lesssim \|u_0\|_{\dot{H}^{\beta}} + \|F(u)\|_{L^{p'}(I,\dot{H}^{\beta}_{q'})}. \end{aligned}$$

Now, let (m,n) be as in (2.5). Lemma 2.6, (2.5) and Sobolev embedding then give

$$\begin{aligned} \|F(u)\|_{L^{p'}(I,L^{q'})} &\lesssim \|u\|^{\nu-1}_{L^p(I,L^n)} \|u\|_{L^m(I,L^q)} \lesssim \|u\|^{\nu-1}_{L^p(I,\dot{H}^{\gamma}_q)} \|u\|_{L^m(I,L^q)} \lesssim \|u\|_{L^m(I,L^q)}, \\ \|F(u)\|_{L^{p'}(I,\dot{H}^{\beta}_{q'})} &\lesssim \|u\|^{\nu-1}_{L^p(I,L^n)} \|u\|_{L^m(I,\dot{H}^{\beta}_q)} \lesssim \|u\|^{\nu-1}_{L^p(I,\dot{H}^{\gamma}_q)} \|u\|_{L^m(I,\dot{H}^{\beta}_q)} \lesssim \|u\|_{L^m(I,\dot{H}^{\beta}_q)}. \end{aligned}$$

Here we use the fact that $\|u\|_{L^p((0,T),H^{\gamma}_q)}$ is bounded. This shows that

$$\|u\|_{L^{\infty}(I,H^{\beta})} + \|u\|_{L^p(I,H^{\beta}_q)} \lesssim \|u_0\|_{H^{\beta}} + \|u\|_{L^m(I,H^{\beta}_q)},$$

for every interval $I \subset (0,T)$. Now let $0<\epsilon<T$ and consider $I=(0,\tau)$ with $\epsilon<\tau<T$. We have

$$\|u\|_{L^m(I,H^\beta_q)} \leq \|u\|_{L^m((0,\tau-\epsilon),H^\beta_q)} + \|u\|_{L^m((\tau-\epsilon,\tau),H^\beta_q)} \leq C_\epsilon + \epsilon^\theta \|u\|_{L^p(I,H^\beta_q)},$$

where θ given in (2.7). Here we also use the fact that $u \in L^p_{\mathrm{loc}}([0,T),H^\beta_q)$ since $\gamma_{p,q}=0$. Thus,

$$\|u\|_{L^\infty(I,H^\beta)} + \|u\|_{L^p(I,H^\beta_q)} \leq C + C_\epsilon + \epsilon^\theta C\|u\|_{L^p(I,H^\beta_q)},$$

where the various constants are independent of $\tau<T$. By choosing ϵ small enough, we have

$$\|u\|_{L^\infty(I,H^\beta)} + \|u\|_{L^p(I,H^\beta_q)} \leq C,$$

where C is independent of $\tau<T$. Let $\tau \to T$, we get a contradiction with (3.1).

The case $\gamma = d/2$ Since $u_0 \in H^{d/2}$, Theorem 1.2 shows that there exists a unique, maximal solution to (NL4S) satisfying $u \in C([0,T^*),H^{d/2}) \cap L^p_{\mathrm{loc}}([0,T^*),L^\infty)$ for some $p>\max(\nu-1,4)$ when $d=1$ and $p>\max(\nu-1,2)$ when $d\geq 2$. This implies in particular that

$$u \in L^{\nu-1}_{\mathrm{loc}}([0,T^*),L^\infty). \tag{3.2}$$

Now let $\beta>\gamma$. If $u_0 \in H^\beta$, then we know that u is an H^β solution defined on some maximal interval $[0,T)$ with $T \leq T^*$. Suppose that $T<T^*$. Then the unitary property of $e^{it\Delta^2}$ and Lemma 2.6 imply that

$$\|u(t)\|_{H^\beta} \leq \|u_0\|_{H^\beta} + \int_0^t \|F(u)(s)\|_{H^\beta} ds \leq \|u_0\|_{H^\beta} + C\int_0^t \|u(s)\|^{\nu-1}_{L^\infty}\|u(s)\|_{H^\beta} ds,$$

for all $0 \leq t < T$. The Gronwall's inequality then yields

$$\|u(t)\|_{H^\beta} \leq \|u_0\|_{H^\beta} \exp\Big(C\int_0^t \|u(s)\|^{\nu-1}_{L^\infty} ds\Big)$$

for all $0\leq t<T$. Using (3.2), we see that $\limsup \|u(t)\|_{H^\beta} < \infty$ as $t \to T$. This is a contradiction with the blowup alternative in H^β.

The case $\gamma > d/2$ Let $\beta>\gamma$. If $u_0 \in H^\beta$, then Theorem 1.3 shows that there is a unique maximal solution $u \in C([0,T),H^\beta)$ to (NL4S). By the uniqueness, we have $T \leq T^*$. Suppose $T<T^*$. Then

$$\sup_{0\leq t\leq T} \|u(t)\|_{H^\beta} < \infty,$$

and hence

$$\sup_{0\leq t\leq T} \|u(t)\|_{L^\infty} < \infty.$$

This is a contradiction with the fact that $\limsup \|u(t)\|_{L^\infty} = \infty$ as $t \to T$. The proof of Theorem 1.6 is now complete. ■

3.2 Proof of Proposition 1.8

Let us now give the proof of Proposition 1.8. Let $\beta > 0$ and $u_0 \in H^\beta$ be such that if $\mu = -1$, $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the solution to (1.7). We learn from the result of Pausader-Shao [PS10] that the L^2-critical (NL4S) is globally well-posed in L^2. Moreover, the unique solution enjoys the uniform bound

$$\|u\|_{L^{2+\frac{8}{d}}(\mathbb{R},L^{2+\frac{8}{d}})} \leq C(\|u_0\|_{L^2}).$$

Since $u_0 \in H^\beta$, we have from Theorem 1.1, Theorem 1.2 and Theorem 1.3 that there exists a maximal solution to the L^2-critical (NL4S) satisfying $C([0,T),H^\beta) \cap L^a_{\text{loc}}([0,T),H^\beta_b)$ for any admissible pair (a,b) with $b < \infty$ and $\gamma_{a,b} = 0$. By the blowup alternative, it suffices to show that $\|u\|_{L^\infty((0,T),H^\beta)} < \infty$. Let $p = 2 + 8/d$. It is easy to see that (p,p) is a admissible pair with $\gamma_{p,p} = 0$. Since $\|u\|_{L^p((0,T),L^p)} < \infty$, we decompose $(0,T)$ into a finite number of subintervals I_k so that $\|u\|_{L^p(I_k,L^p)} < \epsilon$ for some $\epsilon > 0$ to be chosen later. By Strichartz estimates,

$$\begin{aligned}\|u\|_{L^\infty(I_k,H^\beta)} + \|u\|_{L^p(I_k,H^\beta_p)} &\lesssim \|u_0\|_{H^\beta} + \|F(u)\|_{L^{p'}(I_k,H^\beta_{p'})} \\ &\lesssim \|u_0\|_{H^\beta} + \|u\|^{\frac{8}{d}}_{L^p(I_k,L^p)}\|u\|_{L^p(I_k,H^\beta_p)} \\ &\lesssim \|u_0\|_{H^\beta} + \epsilon^{\frac{8}{d}}\|u\|_{L^p(I_k,H^\beta_p)}.\end{aligned}$$

By choosing $\epsilon > 0$ small enough, we get $\|u\|_{L^\infty(I_k,H^\beta)} \leq C$ for some constant C independent of I_k. By summing over all subintervals I_k, we obtain $\|u\|_{L^\infty((0,T),H^\beta)} < \infty$. The proof is complete. ■

4 Ill-posedness

In this section, we will give the proof of Theorem 1.9 using the technique of [CCT03]. We follow closely the argument of [Din3]. Let us start with the small dispersion analysis.

4.1 Small dispersion analysis

Let us consider for $0 < \delta \ll 1$ the following equation

$$\left\{\begin{aligned} i\partial_t\phi(t,x) + \delta^4\Delta^2\phi(t,x) &= -\mu|\phi|^{\nu-1}\phi(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ \phi(0,x) &= \phi_0(x), \quad x \in \mathbb{R}^d. \end{aligned}\right. \tag{4.1}$$

Note that (4.1) can be transformed back to (NL4S) by using

$$u(t,x) := \phi(t,\delta x). \tag{4.2}$$

Lemma 4.1. *Let $k > d/2$ be an integer. If ν is not an odd integer, then we assume also the additional regularity condition $\nu \geq k+1$. Let ϕ_0 be a Schwartz function. Then there* *exists* $C, c > 0$ *such that if* $0 < \delta \leq c$ *sufficiently small, then there exists a unique solution* $\phi^{(\delta)} \in C([-T,T], H^k)$ *of* (4.1) *with* $T = c|\log\delta|^c$ *satisfying*

$$\|\phi^{(\delta)}(t) - \phi^{(0)}(t)\|_{H^k} \leq C\delta^3, \tag{4.3}$$

for all $|t| \leq c|\log\delta|^c$, *where*

$$\phi^{(0)}(t,x) := \phi_0(x)\exp(-i\mu t|\phi_0(x)|^{\nu-1}) \tag{4.4}$$

is the solution of (4.1) *with* $\delta = 0$.

Proof. The proof of Lemma 4.1 is essentially given in [Pau2, Lemma 4.1] where the author treated the cubic fourth-order Schrödinger equation. The extension to the general power-type nonlinearity here is completely similar. Note that H^k with $k > d/2$ is an algebra. ■

Remark 4.2. By the same argument as in [CCT03], we can get the following better estimate

$$\|\phi^{(\delta)}(t) - \phi^{(0)}(t)\|_{H^{k,k}} \leq C\delta^3, \tag{4.5}$$

for all $|t| \leq c|\log\delta|^c$, where $H^{k,k}$ is the weighted Sobolev space

$$\|\phi\|_{H^{k,k}} := \sum_{|\alpha|=0}^{k} \| \langle x\rangle^{k-|\alpha|} D^\alpha \phi\|_{L^2}.$$

Now let $\lambda > 0$ and set

$$u^{(\delta,\lambda)}(t,x) := \lambda^{-\frac{4}{\nu-1}}\phi^{(\delta)}(\lambda^{-4}t, \lambda^{-1}\delta x). \tag{4.6}$$

It is easy to see that $u^{(\delta,\lambda)}$ is a solution of (NL4S) with initial data $u^{(\delta,\lambda)}(0) = \lambda^{-\frac{4}{\nu-1}}\phi_0(\lambda^{-1}\delta x)$. We have the following estimate of the initial data $u^{(\delta,\lambda)}(0)$.

Lemma 4.3. *Let* $\gamma \in \mathbb{R}$ *and* $0 < \lambda \leq \delta \ll 1$. *Let* $\phi_0 \in \mathscr{S}$ *be such that if* $\gamma \leq -d/2$,

$$\hat{\phi}_0(\xi) = O(|\xi|^\kappa) \text{ as } \xi \to 0,$$

for some $\kappa > -\gamma - d/2$, *where* $\hat{\cdot}$ *is the Fourier transform. Then there exists* $C > 0$ *such that*

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} \leq C\lambda^{\gamma_c - \gamma}\delta^{\gamma - d/2}. \tag{4.7}$$

The proof of this result follows the same lines as in [CCT03, Section 4] for the nonlinear Schrödinger equation. We also refer to [Din3, Lemma 3.3] for the nonlinear half-wave context.

4.2 Proof of Theorem 1.9

We now give the proof of Theorem 1.9. We only consider the case $t \geq 0$, the one for $t < 0$ is similar. Let $\epsilon \in (0, 1]$ be fixed and set

$$\lambda^{\gamma_c - \gamma}\delta^{\gamma - d/2} =: \epsilon, \tag{4.8}$$

equivalently

$$\lambda \sim \delta^{\theta}, \text{ where } \theta = \frac{d/2-\gamma}{\gamma_c - \gamma} > 1,$$

hence $0 < \lambda \leq \delta \ll 1$. We note that we are considering the super-critical range, i.e. $\gamma < \gamma_c$. We will split the proof into several cases.

The case $0 < \gamma < \gamma_c$. We firstly have from Lemma 4.3 and (4.8) that

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} \leq C\epsilon.$$

Since the support of $\phi^{(0)}(t, x)$ is independent of t (see (4.4)), we see that for t large enough, depending on γ,

$$\|\phi^{(0)}(t)\|_{H^\gamma} \sim t^\gamma,$$

whenever $\gamma \geq 0$ provided either $\nu > 1$ is an odd integer or $\gamma \leq \nu - 1$ otherwise. Thus for $\delta \ll 1$ and $1 \ll t \leq c|\log\delta|^c$, (4.3) implies

$$\|\phi^{(\delta)}(t)\|_{H^\gamma} \sim t^\gamma. \tag{4.9}$$

Next, using

$$[u^{(\delta,\lambda)}(\lambda^4 t)]\hat{}(\xi) = \lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d[\phi^{(\delta)}(t)]\hat{}(\lambda\delta^{-1}\xi),$$

we have

$$\begin{aligned}
\|u^{(\delta,\lambda)}(\lambda^4 t)\|^2_{H^\gamma} &= \int (1+|\xi|^2)^\gamma |[u^{(\delta,\lambda)}(\lambda^4 t)]\hat{}(\xi)|^2 d\xi \\
&= \lambda^{-\frac{8}{\nu-1}}(\lambda\delta^{-1})^d \int (1+|\lambda^{-1}\delta\xi|^2)^\gamma |[\phi^{(\delta)}(t)]\hat{}(\xi)|^2 d\xi \\
&\geq \lambda^{-\frac{8}{\nu-1}}(\lambda\delta^{-1})^{d-2\gamma} \int_{|\xi|\geq 1} |\xi|^{2\gamma} |[\phi^{(\delta)}(t)]\hat{}(\xi)|^2 d\xi \\
&\geq \lambda^{-\frac{8}{\nu-1}}(\lambda\delta^{-1})^{d-2\gamma}\Big(c\|\phi^{(\delta)}(t)\|^2_{H^\gamma} - C\|\phi^{(\delta)}(t)\|^2_{L^2}\Big).
\end{aligned}$$

We also have from (4.9) that $\|\phi^{(\delta)}(t)\|_{L^2} \ll \|\phi^{(\delta)}(t)\|_{H^\gamma}$ for $t \gg 1$. This yields that

$$\|u^{(\delta,\lambda)}(\lambda^4 t)\|_{H^\gamma} \geq c\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^{d/2-\gamma}\|\phi^{(\delta)}(t)\|_{H^\gamma} \geq c\epsilon t^\gamma,$$

for $1 \ll t \leq c|\log\delta|^c$. We now choose $t = c|\log\delta|^c$ and pick $\delta > 0$ small enough so that

$$\epsilon t^\gamma > \epsilon^{-1}, \quad \lambda^4 t < \epsilon.$$

Therefore, for any $\varepsilon > 0$, there exists a solution of (NL4S) satisfying

$$\|u(0)\|_{H^\gamma} < \varepsilon, \quad \|u(t)\|_{H^\gamma} > \varepsilon^{-1}$$

for some $t \in (0, \varepsilon)$. Thus for any $t > 0$, the solution map $\mathscr{S} \ni u(0) \mapsto u(t)$ for the Cauchy problem (NL4S) fails to be continuous at 0 in the H^γ-topology.

The case $\gamma \leq -d/2$ and $\gamma < \gamma_c$. Using again Lemma 4.3 and (4.8), we have

$$\|u^{(\delta,\lambda)}(0)\|_{H^\gamma} \leq C\epsilon,$$

provided $0 < \lambda \leq \delta \ll 1$ and $\phi_0 \in \mathscr{S}$ satisfying

$$\hat{\phi}_0(\xi) = O(|\xi|^\kappa) \text{ as } \xi \to 0,$$

for some $\kappa > -\gamma - d/2$. We recall that

$$\phi^{(0)}(t,x) = \phi_0(x)\exp(-i\mu t|\phi_0(x)|^{\nu-1}).$$

It is clear that we can choose ϕ_0 so that

$$\Big|\int \phi^{(0)}(1,x)dx\Big| \geq c \text{ or } |[\phi^{(0)}(1)]\hat{}(0)| \geq c,$$

for some constant $c > 0$. Since $\phi^{(0)}(1)$ is rapidly decreasing, the continuity implies that

$$|[\phi^{(0)}(1)]\hat{}(\xi)| \geq c,$$

for $|\xi| \leq c$ with $0 < c \ll 1$. Since $H^{k,k}$ controls L^1 when $k > d/2$, (4.5) implies

$$|[\phi^{(\delta)}(1)]\hat{}(\xi) - [\phi^{(0)}(1)]\hat{}(\xi)| \leq C\delta^3,$$

and then

$$|[\phi^{(\delta)}(1)]\hat{}(\xi)| \geq c, \tag{4.10}$$

for $|\xi| \leq c$ provided δ is taken small enough. We now have from (4.6) that

$$u^{(\delta,\lambda)}(\lambda^4,x) = \lambda^{-\frac{4}{\nu-1}}\phi^{(\delta)}(1,\lambda^{-1}\delta x),$$

and

$$[u^{(\delta,\lambda)}(\lambda^4)]\hat{}(\xi) = \lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d[\phi^{(\delta)}(1)]\hat{}(\lambda\delta^{-1}\xi).$$

The estimate (4.10) then yields

$$[u^{(\delta,\lambda)}(\lambda^4)]\hat{}(\xi) \geq c\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d,$$

for $|\xi| \leq c\lambda^{-1}\delta$.

In the case $\gamma < -d/2$, we have from (4.8) that

$$\|u^{(\delta,\lambda)}(\lambda^4)\|_{H^\gamma} \geq c\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d = c\epsilon(\lambda\delta^{-1})^{\gamma+d/2}.$$

Here $0 < \lambda \leq \delta \ll 1$, thus $(\lambda\delta^{-1})^{\gamma+d/2} \to +\infty$. We can choose δ small enough so that $\lambda \to 0$ and $(\lambda\delta^{-1})^{\gamma+d/2} \geq \epsilon^{-2}$ or

$$\|u^{(\delta,\lambda)}(\lambda^4)\|_{H^\gamma} \geq \epsilon^{-1}.$$

In the case $\gamma = -d/2$, we have

$$\begin{aligned}\|u^{(\delta,\lambda)}(\lambda^4)\|_{H^{-d/2}} &\geq c\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d\Big(\int_{|\xi|\leq c\lambda^{-1}\delta}(1+|\xi|)^{-d}d\xi\Big)^{1/2}\\ &= c\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d(\log(c\lambda^{-1}\delta))^{1/2}\\ &= c\epsilon(\log(c\lambda^{-1}\delta))^{1/2}.\end{aligned}$$

By choosing δ small enough so that $\lambda \to 0$ and $\log(c\lambda^{-1}\delta) \geq \epsilon^{-4}$, we see that

$$\|u^{(\delta,\lambda)}(\lambda^4)\|_{H^{-d/2}} \geq \epsilon^{-1}.$$

Combining both cases, we see that the solution map fails to be continuous at 0 in H^γ-topology.

The case $\gamma = 0 < \gamma_c$. Let $a, a' \in [1/2, 2]$. Let $\phi^{(a,\delta)}$ be the solution to (4.1) with initial data

$$\phi^{(a,\delta)}(0) = a\phi_0.$$

Then, Lemma 4.1 gives

$$\|\phi^{(a,\delta)}(t) - \phi^{(a,0)}(t)\|_{H^k} \leq C\delta^3, \tag{4.11}$$

for all $|t| \leq c|\log\delta|^c$, where

$$\phi^{(a,0)}(t,x) = a\phi_0(x)\exp(-i\mu a^{\nu-1}t|\phi_0(x)|^{\nu-1}) \tag{4.12}$$

is the solution of (4.1) with $\delta = 0$ and the same initial data as $\phi^{(a,\delta)}$. Note that the constant C, c above can be taken to be independent of a since a belongs to a compact set. We next define

$$u^{(a,\delta,\lambda)}(t,x) := \lambda^{-\frac{4}{\nu-1}}\phi^{(a,\delta)}(\lambda^{-4}t, \lambda^{-1}\delta x). \tag{4.13}$$

Thanks to (4.2) and the scaling (1.1), we see that $u^{(a,\delta,\lambda)}$ is also a solution of (NL4S). On the other hand, using (4.12), a direct computation shows that

$$\|\phi^{(a,0)}(t) - \phi^{(a',0)}(t)\|_{L^2} \geq c > 0,$$

for some time t satisfying $|a-a'|^{-1} \leq t \leq c|\log\delta|^c$ provided that δ is small enough so that $c|\log\delta|^c \geq |a-a'|^{-1}$. This estimate and (4.11) yield

$$\|\phi^{(a,\delta)}(t) - \phi^{(a',\delta)}(t)\|_{L^2} \geq c,$$

for all $|a-a'|^{-1} \leq t \leq c|\log\delta|^c$. Now, let ϵ be as in (4.8), i.e.

$$\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^{d/2} =: \epsilon, \tag{4.14}$$

or $\lambda = \delta^\theta$ with $\theta = \frac{d/2}{\gamma_c} > 1$. Moreover, using the fact

$$[u^{(a,\delta,\lambda)}(\lambda^4 t)]\hat{\ }(\xi) = \lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d[\phi^{(a,\delta)}(t)]\hat{\ }(\lambda\delta^{-1}\xi),$$

we obtain

$$\|u^{(a,\delta,\lambda)}(\lambda^4 t) - u^{(a',\delta,\lambda)}(\lambda^4 t)\|_{L^2} = \lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^{d/2}\|\phi^{(a,\delta)}(t) - \phi^{(a',\delta)}(t)\|_{L^2} \geq c\epsilon.$$

Similarly, using

$$[u^{(a,\delta,\lambda)}(0)]\hat{}(\xi) = a\lambda^{-\frac{4}{\nu-1}}(\lambda\delta^{-1})^d\hat{\phi}_0(\lambda\delta^{-1}\xi),$$

the choice of ϵ in (4.14) gives

$$\|u^{(a,\delta,\lambda)}(0)\|_{L^2}, \|u^{(a',\delta,\lambda)}(0)\|_{L^2} \leq C\epsilon,$$

and

$$\|u^{(a,\delta,\lambda)}(0) - u^{(a',\delta,\lambda)}(0)\|_{L^2} \leq C\epsilon|a - a'|.$$

Since $|a - a'|$ can be arbitrarily small, this shows that for any $0 < \epsilon, \sigma < 1$ and for any $t > 0$, there exist u_1, u_2 solutions of (NL4S) with initial data $u_1(0), u_2(0) \in \mathscr{S}$ such that

$$\|u_1(0)\|_{L^2}, \|u_2(0)\|_{L^2} \leq C\epsilon, \quad \|u_1(0) - u_2(0)\|_{L^2} \leq C\sigma, \quad \|u_1(t) - u_2(t)\|_{L^2} \geq c\epsilon.$$

This shows that the solution map fails to be uniformly continuous on L^2. This completes the proof of Theorem 1.9.

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Institut de Mathématiques de Toulouse UMR5219,
Université Toulouse CNRS,
31062 Toulouse Cedex 9, France
email: dinhvan.duong@math.univ-toulouse.fr