Real hypersurfaces with Killing type structure Jacobi operators in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

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Abstract

In this paper, we prove that if the structure Jacobi operator of a 3-dimensional real hypersurface in a nonflat complex plane is of Killing type, then the hypersurface is either a tube of radius $\frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$ or a Hopf hypersurface with vanishing Hopf principal curvature in $\mathbb{C}H^2$. This extends the corresponding results in [6].

1 Introduction

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is said to be a *complex space form* and is denoted by $M^n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ according respectively to $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface in a complex space form $M^n(c)$, $c \neq 0$, whose Kähler metric and complex structure are denoted by $\overline{g}$ and J respectively. Then, we can define on M an *almost contact metric structure* (ϕ, ξ, η, g) induced from $\overline{g}$ and J (see Section 2), where ξ is called a *structure vector field*. We denote by D the distribution determined by tangent vectors orthogonal to ξ at each point of M. Let A be the shape operator of M in $M^n(c)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$, then M is called a *Hopf hypersurface* and α is called *Hopf principal curvature*. According to [10, 12, 15], we know that α is locally constant for any Hopf hypersurface.

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Applying some results obtained by Cecil and Ryan [3] and Takagi [27], Kimura in [9] obtained the following classification theorem.

Theorem 1.1 ([9]). *Let M be a connected Hopf hypersurface of $\mathbb{C}P^n(c)$, $n \geq 2$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following:*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$;*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k(c)$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$;*

(B) a tube of radius r over a complex quadric $\mathbb{C}Q^{n-1}$, where $0 < r < \frac{\pi}{4}$;

(C) a tube of radius r over $\mathbb{C}P^1(c) \times \mathbb{C}P^{\frac{n-1}{2}}(c)$, where $0 < r < \frac{\pi}{4}$ and $n \geq 5$ is odd;

(D) a tube of radius r over a complex Grassman $\mathbb{C}G_{2.5}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$;

(E) a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

On the other hand, Hopf real hypersurfaces in complex hyperbolic spaces were classified by Berndt [1] and Niebergall and Ryan [15] as the following.

Theorem 1.2 ([1]). *Let M be a connected Hopf hypersurface of $\mathbb{C}H^n(c)$, $n \geq 2$. Then M has constant principal curvatures if and only if M is locally congruent to one of the following:*

(A_0) *a self-tube, that is, a horosphere;*

(A_1) *a geodesic hypersphere of radius r $(0 < r < \infty)$ or a tube of radius r over a complex hyperbolic hyperplane $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}H^k(c)$ $(1 \leq k \leq n-2)$, where $0 < r < \infty$;*

(B) a tube of radius r over a totally real hyperbolic space $\mathbb{R}H^n(\frac{c}{4})$, where $0 < r < \infty$.

For simplicity, we say that a real hypersurface M in a nonflat complex space form is of type A if it is of type A_1 or A_2 in $\mathbb{C}P^n(c)$ or type A_0, A_1 or A_2 in $\mathbb{C}H^n(c)$. A well-known characterization of real hypersurfaces of type A can be expressed as the following.

Theorem 1.3 ([14, 16]). *Let M be a real hypersurface in a complex space form $M^n(c)$, $c \neq 0$, $n \geq 2$. Then M is locally congruent to one of the model spaces of type A if and only if $A\phi = \phi A$.*

Let R be the curvature tensor of a real hypersurface M in $M^n(c)$. Then we call the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector field ξ a *structure Jacobi operator*.

The problem of characterizations of real hypersurfaces under various different geometric conditions has been an important field of research for a long time.

We now recall some results regarding the structure Jacobi operator under some additional restrictions. Ortega, Pérez and Santos in [17] proved that there exist no real hypersurfaces in a nonflat complex space form $M^n(c)$, $n > 2$, whose structure Jacobi operator is parallel, i.e., $\nabla R_\xi = 0$. Later, a weaker condition named D-parallel structure Jacobi operator, i.e., $\nabla_X R_\xi = 0$ for any vector field X tangent to D, is considered by Pérez, Santos and Suh [23]. They proved that there exist no real hypersurfaces in $\mathbb{C}P^n(c)$, $n > 2$, with D-parallel structure Jacobi operators. Generalizing main results in [17] and some non-existence results of real hypersurfaces in $\mathbb{C}P^n(c)$, $n > 2$, in Pérez et al. [25], Theofanidis and Xenos in [29] proved that there exist no real hypersurfaces in a nonflat complex space form $M^n(c)$, $n > 2$, with a recurrent structure Jacobi operator, i.e., $(\nabla_X R_\xi)Y = \omega(X)R_\xi(Y)$ for any vector fields X, Y tangent to M, where ω is a 1-form. Moreover, Theofanidis and Xenos in [30] proved that there exist no real hypersurfaces in a nonflat complex plane $M^2(c)$ with D-recurrent structure Jacobi operators. Recently, this was extended to the higher dimension case, i.e., there exist no real hypersurfaces in $M^n(c)$, $n > 2$, with a D-recurrent structure Jacobi operator (see Kon et al. [11]).

Except for the Levi-Civita connection, the parallelism of the structure Jacobi operator with respect to the Lie derivative and some other connections was also considered by many authors. In 2005, Pérez and Santos in [22] proved the non-existence of real hypersurfaces in $\mathbb{C}P^n(c)$, $n > 2$, whose structure Jacobi operator is Lie parallel, i.e., $\mathcal{L}_X R_\xi = 0$ for any vector field X tangent to M. Later, a real hypersurface having a Lie ξ-parallel structure Jacobi operator, i.e., $\mathcal{L}_\xi R_\xi = 0$, was studied by Pérez et al. [21] in $\mathbb{C}P^n(c)$, $n > 2$ and also by Ivey and Ryan [6] in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. A weaker condition named Lie D-parallel structure Jacobi operator, i.e., $\mathcal{L}_X R_\xi = 0$ for any $X \in D$, was introduced and studied by Pérez et al. in [26]. Extending the previous result, Panagiotidou and Xenos in [20] proved the non-existence of 3-dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with a Lie D-parallel structure Jacobi operator. Kaimakamis and Panagiotidou in [7] proved that there exist no real hypersurfaces in $M^n(c)$, $n \geq 2$ and $c \neq 0$, whose structure Jacobi operator is Lie recurrent, i.e., $(\mathcal{L}_X R_\xi)Y = \omega(X)R_\xi(Y)$ for any vector fields X, Y tangent to the hypersurface. Recently, Panagiotidou in [18] proved that there exist no real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ whose structure Jacobi operator satisfies either $\mathcal{L}_X R_\xi = \nabla_X R_\xi$ or $\mathcal{L}_X A = \nabla_X A$ for any $X \in D$. A non-existence result of real hypersurfaces in $\mathbb{C}P^n(c)$, $n > 2$, with a Codazzi type structure Jacobi operator, i.e., $(\nabla_X R_\xi)Y = (\nabla_Y R_\xi)X$ for any vector fields X, Y, was obtained by Pérez et al. in [24]. Later, this was generalized to 3-dimensional case by Theofanidis and Xenos [28, 31]. Moreover, a non-existence result of 3-dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with a Codazzi type structure Jacobi operator with respect to the generalized Tanaka-Webster connection was obtained by Kaimakamis et al. in [8]

In this paper, we investigate 3-dimensional real hypersurfaces M in a nonflat complex plane $M^2(c)$, $c \neq 0$. We prove that if the structure Jacobi operator of M is of Killing type, then the hypersurface is either a tube of radius $\frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$ or a Hopf hypersurface with vanishing Hopf principal curvature in $\mathbb{C}H^2$. Note that any parallel $(1,1)$-type tensor field must be of Killing type, however, the converse is not necessarily true. Obviously, our main result extends those in [6, Section 3] in which the authors proved that there exist no real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with a parallel structure Jacobi operator.

2 Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX \tag{2.1}$$

for any vector fields X and Y tangent to M, where A denotes the shape operator of M in $M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi. \tag{2.2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi) \tag{2.4}$$

for any vector fields X and Y on M. Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla} J = 0$) of $M^n(c)$ and using (2.1), (2.2) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any vector fields X and Y. We denote by R the Riemannian curvature tensor of M. Since $M^n(c)$ is assumed to be of constant holomorphic sectional curvature c, then the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X, Y)Z =& \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.8}$$

for any vector fields X, Y on M.

From (2.7) we see that the structure Jacobi operator R_ξ is given by

$$R_\xi X = \frac{c}{4}(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \tag{2.9}$$

for any vector field X tangent to the hypersurface.

In this paper, all manifolds are assumed to be connected and of class C^∞.

3 Hypersurfaces with Killing type operators in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

Let M be a real hypersurface in a complex space form $M^n(c)$. We put

$$A\xi = \alpha\xi + \beta U, \tag{3.1}$$

where $\alpha = \eta(A\xi)$, U a unit vector field orthogonal to ξ and β a smooth function. Applying (2.1) and (2.2) we see that $\beta U = -\phi\nabla_\xi\xi$. We put

$$\Omega = \{p \in M \,|\, \beta(p) \neq 0\}.$$

Then Ω is an open subset of M.

A Killing tensor field of type $(1,1)$ was first introduced by Blair [2, pp. 287]. A $(1,1)$-type tensor field T defined on a Riemannian manifold with the Riemannian connection ∇ is called a *Killing tensor field* if it satisfies

$$(\nabla_X T)X = 0 \ (\Leftrightarrow (\nabla_X T)Y + (\nabla_Y T)X = 0) \tag{3.2}$$

for any vector fields X, Y. Recently, Cho in [4] proved that there exist no real hypersurfaces in a complex space form whose structure tensor field ϕ or shape operator A is of Killing type. In this paper, we study a three-dimensional real hypersurface whose structure Jacobi operator R_ξ is of Killing type. In what follows, $M^2(c)$, $c \neq 0$, is used to denote $\mathbb{C}P^2$ or $\mathbb{C}H^2$.

Lemma 3.1 ([15, pp. 245]). *Let M be a Hopf hypersurface in a nonflat complex space form $M^n(c)$. If $AX = \lambda_1 X$ and $X \in \{\xi\}^\perp$, then we have*

$$2(2\lambda_1 - \alpha)A\phi X = (2\lambda_1\alpha + c)\phi X.$$

If in addition we define λ_2 by $A\phi X = \lambda_2\phi X$, then we have

$$\lambda_1\lambda_2 = \frac{\lambda_1 + \lambda_2}{2}\alpha + \frac{c}{4}.$$

The above lemma was, in fact, first proved by Maeda [12] for the case $\mathbb{C}P^n$ and by Montiel [13] for the case $\mathbb{C}H^n$.

Lemma 3.2 ([20, Lemma 1]). *Let M be a three-dimensional real hypersurface in a nonflat complex plane $M^2(c)$. Then the following relations hold:*

$$\begin{aligned}
&AU = \gamma U + \delta\phi U + \beta\xi,\ A\phi U = \delta U + \mu\phi U,\\
&\nabla_U\xi = -\delta U + \gamma\phi U,\ \nabla_{\phi U}\xi = -\mu U + \delta\phi U,\ \nabla_\xi\xi = \beta\phi U,\\
&\nabla_U U = \kappa_1\phi U + \delta\xi,\ \nabla_{\phi U}U = \kappa_2\phi U + \mu\xi,\ \nabla_\xi U = \kappa_3\phi U,\\
&\nabla_U\phi U = -\kappa_1 U - \gamma\xi,\ \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi,\ \nabla_\xi\phi U = -\kappa_3 U - \beta\xi,
\end{aligned} \tag{3.3}$$

where γ, δ, μ, κ_i, $i = \{1,2,3\}$, are smooth functions on M and $\{\xi, U, \phi U\}$ is an orthonormal basis of the tangent space of M at a point of M.

Since the above lemma can be seen in [19, 20, 30], then here we omit its proof. Applying this lemma, from the Codazzi equation (2.8) for $X = U$ or $X = \phi U$ and $Y = \xi$ we have

$$U(\beta) - \xi(\gamma) = \alpha\delta - 2\delta\kappa_3. \tag{3.4}$$

$$\xi(\delta) = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2. \tag{3.5}$$

$$U(\alpha) - \xi(\beta) = -3\beta\delta. \tag{3.6}$$

$$\xi(\mu) = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3. \tag{3.7}$$

$$\phi U(\alpha) = \alpha\beta + \beta\kappa_3 - 3\beta\mu. \tag{3.8}$$

$$\phi U(\beta) = \alpha\mu - \gamma\mu + \delta^2 + \xi(\delta) - \kappa_3\mu + \kappa_3\gamma + \beta^2 + \frac{c}{4}. \tag{3.9}$$

Similarly, from the Codazzi equation for $X = U$ and $Y = \phi U$ we have

$$U(\delta) - \phi U(\gamma) = \mu\kappa_1 - \gamma\kappa_1 - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu. \tag{3.10}$$

$$U(\mu) - \phi U(\delta) = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1. \tag{3.11}$$

Moreover, applying again Lemma 3.2, from the Gauss equation (2.7) and the definition of the Riemannian curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ we have

$$U(\kappa_2) - \phi U(\kappa_1) = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c. \tag{3.12}$$

$$\phi U(\kappa_3) - \xi(\kappa_2) = 2\beta\mu - \mu\kappa_1 + \delta\kappa_2 + \kappa_3\kappa_1 + \beta\kappa_3. \tag{3.13}$$

Applying Lemma 3.2 and replacing X by U and ϕU, respectively, in (2.9), we get

$$\begin{aligned} R_\xi U &= \left(\frac{c}{4} + \alpha\gamma - \beta^2\right) U + \alpha\delta\phi U, \\ R_\xi \phi U &= \alpha\delta U + \left(\frac{c}{4} + \alpha\mu\right)\phi U. \end{aligned} \tag{3.14}$$

Lemma 3.3. *Let M be a three-dimensional real hypersurface in a nonflat complex plane $M^2(c)$, $c \neq 0$, whose structure Jacobi operator is of Killing type. Then M is a Hopf hypersurface.*

Proof. Suppose that a real hypersurface M in $M^2(c)$, $c \neq 0$, is non-Hopf, then $\beta \neq 0$ and Ω is a non-empty subset. If the structure Jacobi operator is of Killing type, replacing both X and Y by ξ in (3.2) we have $R_\xi \nabla_\xi \xi = 0$. Using this and (3.1) in relation (2.9) we acquire

$$A\phi U = -\frac{c}{4\alpha}\phi U, \tag{3.15}$$

where $\alpha \neq 0$. Here we remark that if $\alpha = 0$, from $R_\xi \nabla_\xi \xi = 0$ we obtain $c = 0$, a contradiction. Comparing (3.15) with the first term of (3.3) we obtain

$$\delta = 0, \ \alpha\mu = -\frac{c}{4}. \tag{3.16}$$

Since R_ξ is of Killing type, we also have $(\nabla_U R_\xi)U = 0$. The last equation is further analyzed with the aid of (2.9), the first of (3.14) and Lemma 3.2, giving

$$U(\alpha\gamma - \beta^2) = 0,\ \kappa_1\left(\alpha\gamma - \beta^2 + \frac{c}{4}\right) = 0, \tag{3.17}$$

where (3.16) has been used.

Similarly, because R_ξ is of Killing type we have $(\nabla_{\phi U} R_\xi)\phi U = 0$. Hence, by applying (2.9), the second term of (3.14), (3.16) and Lemma 3.2 in this relation we have

$$\phi U(\alpha\mu) = 0,\ \kappa_2\left(\alpha\gamma - \beta^2 + \frac{c}{4}\right) = 0. \tag{3.18}$$

In view of second term of relation (3.16), we see that the first term of (3.18) holds trivially. If $\alpha\gamma - \beta^2 + \frac{c}{4} = 0$ holds, then it is combined with (3.16) and (3.14) in order to yield $R_\xi = 0$, that is R_ξ vanishes. However, Ivey and Ryan in [6, Proposition 7] proved that this cannot occur.

Next, we consider the only possible case $\alpha\gamma - \beta^2 + \frac{c}{4} \neq 0$ which holds on some open subset. Applying this in (3.17) and (3.18) we have

$$\kappa_1 = 0,\ \kappa_2 = 0. \tag{3.19}$$

Applying (3.19) in relations (3.5) and (3.12), respectively, we have

$$\begin{aligned} &\alpha\gamma + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2 = 0, \\ &2\gamma\mu + \gamma\kappa_3 + \mu\kappa_3 + c = 0. \end{aligned} \tag{3.20}$$

Using (3.16) and (3.19) in (3.7) and (3.11), respectively, we obtain

$$\xi(\mu) = 0,\ U(\mu) = 0. \tag{3.21}$$

By virtue of Lemma 3.2 and relation (3.16) we obtain $[U, \xi] = \nabla_U \xi - \nabla_\xi U = (\gamma - \kappa_3)\phi U$. Thus, using (3.21), the action of the above relation on μ gives

$$(\gamma - \kappa_3)\phi U(\mu) = 0. \tag{3.22}$$

Because of (3.22), we separate our discussions into two cases.

Case i. We assume that $\phi U(\mu) = 0$. The last relation and (3.21) mean that μ is a constant. Hence, by the second term of (3.16), we see that α is a non-zero constant. Applying this in (3.8) and in view of $\beta \neq 0$ we have

$$\kappa_3 = 3\mu - \alpha. \tag{3.23}$$

Thus, κ_3 is also a constant. Since κ_3 is a constant, we make use of (3.16), (3.19) and (3.13) in order to obtain

$$2\mu + \kappa_3 = 0, \tag{3.24}$$

where we have used $\beta \neq 0$. Combining (3.24) with (3.23) we may write

$$\mu = \frac{1}{5}\alpha,\ \kappa_3 = -\frac{2}{5}\alpha. \tag{3.25}$$

Moreover, replacing with (3.24) in the second term of (3.20), we acquire $\mu\kappa_3 + c = 0$ which is combined with the second term of (3.25) giving $-\frac{2}{5}\alpha\mu + c = 0$. The combination of the last equation with the second term of (3.16) gives $c = 0$, which is a contradiction.

Case ii. We assume that $\phi U(\mu) \neq 0$ holds on certain open subset. It follows from (3.22) that $\gamma = \kappa_3$. In this case, using the first term of (3.16) and relation (3.19) in (3.7) and (3.11), respectively, we obtain

$$\xi(\alpha) = U(\alpha) = 0,\ \xi(\mu) = U(\mu) = 0, \tag{3.26}$$

where we have used the second term of (3.16), $c \neq 0$ and $\alpha \neq 0$. Using $U(\alpha) = 0$ and $\delta = 0$ in (3.6) we have $\xi(\beta) = 0$. We also observe that under the assumption $\phi U(\mu) \neq 0$ ($\Rightarrow \gamma = \kappa_3$), relation (3.20) becomes

$$\alpha\gamma + \frac{c}{4} - \gamma^2 - \beta^2 = 0,\ 3\gamma\mu + \gamma^2 + c = 0. \tag{3.27}$$

Moreover, from relations (3.8) and (3.16) we have

$$\phi U(\mu) = \frac{4\beta\mu^2}{c}(\alpha + \gamma - 3\mu).$$

Also, using the first term of (3.16) and relation (3.19) in (3.10) we have

$$\phi U(\gamma) = \beta(\gamma + 2\mu).$$

Finally, applying the above two relations, $\beta \neq 0$, the action of ϕU on the second term of (3.27) gives

$$c(\gamma + 2\mu)(2\gamma + 3\mu) + 12\gamma\mu^2(\alpha + \gamma - 3\mu) = 0. \tag{3.28}$$

From the second terms of (3.16) and (3.27) we see that both α and γ depend only on μ and c. Consequently, from (3.28) we conclude that either there exists no solution for μ or μ is a constant. In view of $\phi U(\mu) \neq 0$, in both cases we arrive at a contradiction. This completes the proof. ∎

Theorem 3.1. *Let M be a 3-dimensional real hypersurface in a nonflat complex plane $M^2(c)$. If the structure Jacobi operator of M is of Killing type, then M is either a tube of radius $\frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$ or a Hopf hypersurface with vanishing Hopf principal curvature in $\mathbb{C}H^2$.*

Proof. According to Lemma 3.3, a real hypersurface in a nonflat complex plane $M^2(c)$ is Hopf. Then, because of $\beta = 0$, (3.14) becomes

$$\begin{aligned} R_\xi U &= \left(\frac{c}{4} + \alpha\gamma\right) U + \alpha\delta\phi U, \\ R_\xi \phi U &= \alpha\delta U + \left(\frac{c}{4} + \alpha\mu\right)\phi U. \end{aligned} \tag{3.29}$$

Since R_ξ is assumed to be a Killing tensor field, we obtain $(\nabla_U R_\xi)U = 0$. Using (3.29) in this relation and applying Lemma 3.2, we acquire

$$\alpha U(\gamma) = 0,\ \kappa_1\alpha(\gamma - \mu) = 0,\ \delta = 0. \tag{3.30}$$

Similarly, by virtue of (3.29) and Lemma 3.2, relation $(\nabla_{\phi U} R_\xi)\phi U = 0$ is analyzed to give

$$\alpha\phi U(\mu) = 0,\ \kappa_2\alpha(\gamma-\mu) = 0,\ \delta = 0. \tag{3.31}$$

We continue our discussions by the following three cases.

Case i. If $\alpha = 0$, we conclude that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$ (see [3]) or a Hopf hypersurface with vanishing Hopf principal curvature in $\mathbb{C}H^2$ (see [6]).

Case ii. If $\alpha \neq 0$ and $\gamma \neq \mu$ hold on certain open subset, then it follows from (3.30) and (3.31) that $\kappa_1 = \kappa_2 = 0$. In this case, from (3.5), (3.9) and (3.12) we have

$$\begin{aligned}
&\alpha\gamma + \kappa_3(\mu-\gamma) + \frac{c}{4} - \gamma\mu = 0,\\
&\alpha\gamma + \frac{c}{2} - 2\gamma\mu + \alpha\mu = 0,\\
&2\gamma\mu + \kappa_3(\gamma+\mu) + c = 0.
\end{aligned} \tag{3.32}$$

The subtraction of the second term of (3.32) from the first one of (3.32) gives $\kappa_3(\mu-\gamma) - \frac{c}{4} + \gamma\mu - \alpha\mu = 0$. Combining the last equation with the second term of (3.32) we obtain $\kappa_3 = \frac{\alpha}{2}$, where we have used the assumption $\gamma \neq \mu$. In view of $\beta = 0$ and $\delta = 0$, applying Lemma 3.2 we have $AU = \gamma U$ and $A\phi U = \mu\phi U$. Then, from Lemma 3.1 we have

$$\gamma\mu = \frac{\alpha}{2}(\gamma+\mu) + \frac{c}{4}. \tag{3.33}$$

The replacement of $\kappa_3 = \frac{\alpha}{2}$ in the third term of (3.32) gives $2\gamma\mu + \frac{\alpha}{2}(\gamma+\mu) + c = 0$. Comparing this with (3.33) we have

$$\gamma\mu = -\frac{c}{4},\ \alpha(\gamma+\mu) = -c. \tag{3.34}$$

From this we see that both γ and μ are constants. Since R_ξ is of Killing type, then we have $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$ for vector fields X and Y tangent to M. In particular, replacing X and Y by U and ϕU, respectively, in the above relation, we obtain $\nabla_U R_\xi \phi U - R_\xi \nabla_U \phi U + \nabla_{\phi U} R_\xi U - R_\xi \nabla_{\phi U} U = 0$. The application of Lemma 3.2 in the last relation, gives

$$\gamma\left(\frac{c}{4} + \alpha\mu\right) = \mu\left(\frac{c}{4} + \alpha\gamma\right),$$

where we have used (3.29) and $\kappa_1 = \kappa_2 = \delta = 0$. Obviously, in view of $c \neq 0$, using the first term of (3.34) in the previous relation, we obtain $\gamma = \mu$ which is a contradiction.

Case iii. If $\alpha \neq 0$ and $\gamma = \mu$, from (3.29) we have

$$R_\xi U = \left(\frac{c}{4} + \alpha\gamma\right)U,\ R_\xi\phi U = \left(\frac{c}{4} + \alpha\gamma\right)\phi U. \tag{3.35}$$

Since R_ξ is assumed to be of Killing type, we obtain $(\nabla_U R_\xi)\xi + (\nabla_\xi R_\xi)U = 0$, or equivalently, $-R_\xi\nabla_U\xi + \nabla_\xi R_\xi U - R_\xi\nabla_\xi U = 0$. In the last relation, by making use of $\beta = \delta = 0$, (3.35), (3.3) and (3.4), we obtain

$$\gamma\left(\frac{c}{4} + \alpha\gamma\right) = 0. \tag{3.36}$$

With regard to (3.36), if $\gamma = 0$, from the first two terms of (3.3) together with $\beta = \delta = 0$ we have $AU = 0$ and $A\phi U = 0$. The application of this in Lemma 3.1 gives $c = 0$, a contradiction. Otherwise, if $\frac{c}{4} + \alpha\gamma = 0$, from (3.35) we see that the structure Jacobi operator R_ξ vanishes. As pointed out before, it was proved in [6, section 3] that there exist no real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with parallel structure Jacobi operator. Thus, we arrive again at a contradiction. This completes the proof. ∎

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