Inverse map and equicontinuity of power maps in locally convex algebras

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Abstract

We show that the inverse map $x \longmapsto x^{-1}$ is continuous in any unitary non commutative locally convex algebra in which the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero. As a consequence, we obtain that the inverse map is continuous in any unitary B_0-algebra not necessarily commutative in which entire functions operate.

1 Introduction.

Turpin showed that, any commutative locally convex algebra in which the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero is m-convex ([7]). Thus, the inverse map $x \longmapsto x^{-1}$ is continuous. In this paper, we obtain the same result in a more general context. We show that the inverse map $x \longmapsto x^{-1}$ is continuous in any unitary locally convex algebra not necessarily commutative in which the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero. As a consequence, we obtain that the inverse map is continuous in any unitary B_0-algebra not necessarily commutative in which entire functions operate. We give two counterexamples showing respectively that the condition of completeness and metrizability are necessary in the last result.

We show that there is a locally convex algebra in which the inverse map $x \longmapsto x^{-1}$ is continuous but the sequence of power maps $(x \longmapsto x^n)_n$ is not equicontinuous at zero. Thus the converse of our main result is not true in

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general case. But it is under additional condition, as it has already been shown by Turpin ([7]).

2 Definitions and notations.

A locally convex algebra (*l.c.a.* for short) is a Hausdorff locally convex space which is an algebra over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) with separately continuous product. If the product is continuous in two variables, it is said to be with continuous product. The topology of such algebra A can be introduced by a family $(p_i)_{i\in I}$ of seminorms such that for every index $i \in I$ there is an index $j \in I$ with

$$p_i(xy) \leq p_j(x)\, p_j(y) \qquad (1)$$

for all $x, y \in A$. If, for a locally convex algebra A, condition (1) can be replaced by

$$p_i(xy) \leq p_i(x)\, p_i(y) \qquad (2)$$

for all $x, y \in A$ and all $i \in I$, then we call A locally multiplicatively convex (shortly m-convex). A locally convex metrizable and complete algebra is called a B_0-algebra. Let $(x \longmapsto x^n)_n$ be the sequence of power maps in (A, τ). We say that $(x \longmapsto x^n)_n$ is equicontinuous at zero if, for every neighborhood U of zero in A, there exists a neighborhood V of zero in A such that

$$x^n \in U, \text{ for each } n \in \mathbb{N}^* \text{ and each } x \in V$$

An entire function $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$, $a_n \in \mathbb{K}$, operates in a *l.c.a.* (A, τ) if, for every x in A, $f(x) = \sum\limits_{n=0}^{+\infty} a_n x^n$, converges in (A, τ). A topological unital algebra is said to be Q-algebra if the set $G(A)$ of all invertible elements of A is open. The spectrum of an element x of A, denoted $Sp_A x$ (or just Spx) is a subset of $\mathbb{C}$ defined by $Sp_A x = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } A\}$. An element x of A is said to be bounded if for some nonzero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, ...\}$ is a bounded subset of (A, τ). The set of all bounded elements of (A, τ) is denoted by A_0. We say that (A, τ) is pseudo-complete if, every closed idempotent disk B is completant i.e., the space $(E_B, \|.\|_B)$ is Banach, where $E_B = \bigcup\limits_{\lambda > 0} \lambda B$ is the span of B and $\|.\|_B$ is the gauge of B. For a detailed account of basis properties of general locally m-convex algebras and B_0-algebras, we refer the reader to [8] and [12].

3 Continuity of the inverse.

It is known that the inverse map $x \longmapsto x^{-1}$ is continuous in any unitary locally convex algebra which is m-convex ([12]). We obtain the same in a more general context. But first let us start with the next useful lemma.

Lemma 1. *Let (A,τ) be a locally convex algebra not necessarily commutative in which the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero. Then, the topology τ can be given by a family of seminorms $(p_i)_{i\in I}$ such that*

1) For every $i \in I$, there exists $j \in I$ such that

$$p_i(x^n) \leq \left(p_j(x)\right)^n, \text{ for all } x \in A,\ n \in \mathbb{N}^*.$$

2) For every $i \in I$ there exists $j \in I$ such that

$$p_i(xy+yx) \leq 6p_j(x)p_j(y),\ x,y \in A.$$

Proof. **1)** Follows from ([3], proposition 2.1).

2) Let $(p_i)_{i\in I}$ be a family of seminorms $(p_i)_{i\in I}$ verifying **1)** and $i \in I$. Then, there exists $j \in I$ such that

$$p_i(x^n) \leq \left(p_j(x)\right)^n$$

for all $x \in A$ and for all $n \in \mathbb{N}^*$. If $x_1, y_1 \in A$ such that $p_j(x_1) \leq 1$ and $p_j(y_1) \leq 1$. By using the equality $x_1y_1 + y_1x_1 = (x_1+y_1)^2 - x_1^2 - y_1^2$, one gets

$$\begin{aligned} p_i(x_1y_1+y_1x_1) &\leq p_i\left[(x_1+y_1)^2\right] + p_i(x_1^2) + p_i(y_1^2) \\ &\leq \left[p_j(x_1+y_1)\right]^2 + \left[p_j(x_1)\right]^2 + \left[p_j(y_1)\right]^2 \\ &\leq 6. \end{aligned}$$

Now let x and y be two elements of A. By applying the last inequality to $x_1 = \frac{x}{p_j(x)+\varepsilon}$ and $y_1 = \frac{y}{p_j(y)+\varepsilon}$, where $\varepsilon > 0$, we obtain

$$p_i(xy+yx) \leq 6\left(p_j(x)+\varepsilon\right)\left(p_j(y)+\varepsilon\right).$$

This being true for all $\varepsilon > 0$, it follows that

$$p_i(xy+yx) \leq 6p_j(x)p_j(y).$$

■

Remark 2. *In fact, the continuity at zero of the map $x \longmapsto x^2$ is sufficient to have a family of seminorms defining τ and satisfying **2)** of the previous lemma, but not necessarily **1)**. Indeed let V be an absolutely convex and closed neighborhood of zero. Since the map $x \longmapsto x^2$ is continuous, there is an absolutely convex and closed neighborhood U of zero such that*

$$x^2 \in V,\ x \in U.$$

Let p be the gauge associated to V and q that associated to U. We have

$$p(x^2) = \inf\left\{\lambda > 0 : x^2 \in \lambda V\right\} \text{ and } q(x) = \inf\left\{\lambda > 0 : x \in \lambda U\right\}.$$

Let $\alpha > 0$ such that $x \in \alpha U$, then $x^2 \in \alpha^2 V$. So $\alpha^2 \in \left\{\lambda > 0 : x^2 \in \lambda V\right\}$. Whence

$$p(x^2) \leq \left(q(x)\right)^2.$$

*Therefore, the family of gauges $(p_i)_{i\in I}$ associated to all absolutely convex and closed neighborhood of zero satisfies **2)**.*

Here is the main result:

Theorem 3. *Let (A, τ) be a unitary locally convex algebra not necessarily commutative in which the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero. Then the inverse map $x \longmapsto x^{-1}$ is continuous in (A, τ).*

Proof. Since the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero, the topology τ can be given, by the lemma **1**, by a family of seminorms $(p_i)_{i\in I}$ satisfying **1)** and **2)** of the same lemma. Then, for every $i \in I$, there exists h and j such that

$$p_i(xy + yx) \leq 6p_j(x)p_j(y) \text{ and } p_j(x^n) \leq [p_h(x)]^n \qquad (3)$$

for all x and y in A and for all $n \in \mathbb{N}^*$. Now let $(x_\alpha)_\alpha \in G(A)$ be such that $x_\alpha \longrightarrow e$ and $0 < \varepsilon < 1$. Then, there exists α_0 such that

$$p_h(x_\alpha - e) < \varepsilon$$

for every $\alpha \geq \alpha_0$. Let $\alpha \geq \alpha_0$. We have

$$e - x_\alpha \sum_{n=0}^{N} (e - x_\alpha)^n = (e - x_\alpha)^{N+1}.$$

Thus, by using (3), we obtain

$$\lim_{N\longrightarrow+\infty} p_j \left(e - x_\alpha \sum_{n=0}^{N} (e - x_\alpha)^n \right) = 0.$$

It follows that

$$\lim_{N\longrightarrow+\infty} p_i \left(x_\alpha^{-1} - \sum_{n=0}^{N} (e - x_\alpha)^n \right) \leq \lim_{N\longrightarrow+\infty} 3p_j \left(e - x_\alpha \sum_{n=0}^{N} (e - x_\alpha)^n \right) p_j \left(x_\alpha^{-1} \right) = 0.$$

Therefore, using (3), we have

$$\begin{aligned} p_i \left(x_\alpha^{-1} \right) &\leq p_i \left(x_\alpha^{-1} - \sum_{n=0}^{N} (e - x_\alpha)^n \right) + p_i \left(\sum_{n=0}^{N} (e - x_\alpha)^n \right) \\ &\leq p_i \left(x_\alpha^{-1} - \sum_{n=0}^{N} (e - x_\alpha)^n \right) + p_i(e) + \sum_{n=1}^{N} 3 \left[p_h (e - x_\alpha) \right]^n. \end{aligned}$$

Letting N tend to infinity and using the fact that $p_h(x_\alpha - e) < \varepsilon$, one has

$$p_i \left(x_\alpha^{-1} \right) \leq K_{i,\varepsilon}, \text{ where } K_{i,\varepsilon} = p_i(e) + \frac{3\varepsilon}{1 - \varepsilon}.$$

Consequently, for any seminorm p_l and any $0 < \varepsilon < 1$, there exists $\alpha_{l,\varepsilon}$ such that

$$p_l\left(x_\alpha^{-1}\right) \leq K_{i,\varepsilon}$$

for every $\alpha \geq \alpha_{l,\varepsilon}$. In particular, we have

$$p_j\left(x_\alpha^{-1}\right) \leq K_{j,\varepsilon}$$

for every $\alpha \geq \alpha_{j,\varepsilon}$. Whence

$$p_i(x_\alpha^{-1} - e) \leq 3p_j\left(x_\alpha^{-1}\right) p_j(x_\alpha - e) \leq 3K_{j,\varepsilon}\, p_j(x_\alpha - e)$$

for every $\alpha \geq \alpha_{j,\varepsilon}$. But we know that there exists α_1 such that

$$p_j(x_\alpha - e) \leq \frac{\varepsilon}{3K_{j,\varepsilon}}$$

for every $\alpha \geq \alpha_1$. Let β be such that $\beta \geq \alpha_{j,\varepsilon}$ and $\beta \geq \alpha_1$. So

$$p_i(x_\alpha^{-1} - e) \leq \varepsilon$$

for every $\alpha \geq \beta$. It follows that $x_\alpha^{-1} \longrightarrow e$. Now, let $x \in G(A)$ and $(x_\alpha)_\alpha \in G(A)$ be such that $x_\alpha \longrightarrow x$. We have $x_\alpha^{-1} = (x^{-1}x_\alpha)^{-1}x^{-1}$ and $x^{-1}x_\alpha \longrightarrow e$. Hence $x_\alpha^{-1} \longrightarrow x^{-1}$. Thus the inverse map is continuous. ∎

Using theorem 3 and ([1], corollary 4.2, p. 414), one can prove the following:

Corollary 4. *Let (A, τ) be a unitary and pseudo-complete l.c.a. such that the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero. Then $x \in A_0$ if and only if $Sp_A(x)$ is bounded.*

Proposition 5. *Let (A, τ) be a unitary complex locally convex algebra (not necessarily commutative) in which the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero. Then $Sp_A(x) \neq \emptyset$, for every $x \in A$.*

Proof. Let $x \in A$ and suppose that $Sp_A(x) = \emptyset$. Consider the map

$$\begin{array}{rcccc} R_x & : & \mathbb{C} & \longrightarrow & (A,\tau) \\ & & \lambda & \longmapsto & (x - \lambda e)^{-1} \end{array}$$

Applying a standard technique based on the Liouville theorem for holomorphic functions, the reader can prove that $R_x = 0$. This contradiction proves our assertion. ∎

Remark 6. *We can show that the spectrum of any element of A is non empty without resorting to the continuity of the inverse in (A, τ). Indeed, let $x \in (A, \tau)$ and $A(x)$ the maximal commutative subalgebra containing x. $A(x)$ is a unitary commutative locally convex algebra in which the sequence of power maps $(x \longrightarrow x^n)_n$ is equicontinuous at zero. Therefore, by Turpin ([7]), m-convex. Since*

$$Sp_A(x) = Sp_{A(x)}(x),$$

we deduce that $Sp_A(x) \neq \emptyset$.

As a consequence, one has

Corollary 7. *Any complex locally convex division algebra (A, τ) (commutative or not) in which the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero is isomorphic to $\mathbb{C}$.*

The following result constitutes an application to the B_0*-algebra*.

Theorem 8. *Let (A, τ) be a unitary B_0-algebra not necessarily commutative in which entire functions operate. Then the inverse map $x \longmapsto x^{-1}$ is continuous.*

Proof. Since entire functions operate in (A, τ) which is an B_0-algebra, the sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero ([2]). And we conclude by theorem **3**. ∎

Remark 9.

1. *Without completeness the previous result does not remain true. Indeed, consider the algebra $(L^\infty([0,1]), \|.\|_1)$ where*

$$\|f\|_1 = \int_0^1 |f(t)|\, dt$$

for every $f \in L^\infty([0,1])$. Since $\|.\|_1 \leq \|.\|_\infty$, entire functions operate. But the inverse map is not continuous. Indeed, let $(f_n)_{n\in\mathbb{N}^*}$ be the sequence of the $L^\infty([0,1])$ defined by:

$$f_n(t) = \begin{cases} \frac{1}{n} \text{ if } 0 \leq t \leq \frac{1}{n} \\ 1 \text{ if } \frac{1}{n} < t \leq 1 \end{cases}$$

we have $\|f_n - 1\|_1 = \frac{1}{n}\left(1 - \frac{1}{n}\right)$ which tends to zero when n tends to infinity. Moreover

$$f_n^{-1}(t) = \begin{cases} n \text{ if } 0 \leq t \leq \frac{1}{n} \\ 1 \text{ if } \frac{1}{n} < t \leq 1 \end{cases}$$

whence

$$\left\| f_n^{-1} - 1 \right\|_1 = \frac{1}{n}(n-1).$$

Therefore, $\left\| f_n^{-1} - 1 \right\|_1$ does not tend to zero when n tends to infinity.

2. *We cannot relax the condition of metrizability in theorem **8**. Indeed, consider the algebra $E = C_b(\mathbb{R}^+)$ of all continuous complex-valued functions defined in on $\mathbb{R}^+$ such that, for every $\varphi \in \Phi$, we have*

$$\|f\|_\varphi = \sup_{t\in\mathbb{R}^+} |f(t)\varphi(t)| < +\infty,$$

where Φ is the set of all complex-valued continuous functions φ defined on $\mathbb{R}^+$ satisfying the following conditions

$$0 < |\varphi(t)| \leq 1$$

for all $t \in \mathbb{R}^+$ and

$$\lim_{t\longrightarrow\infty} \varphi(t) = 0.$$

The algebra E endowed with the family of seminorms $\left(\|.\|_{\varphi}\right)_{\varphi\in\Phi}$ *is a complete non m-convex algebra on which operate all entire functions ([10]). So it is not metrizable; but the inverse map is not continuous as the following result shows.*

Proposition 10. *The inverse map* $f \longmapsto f^{-1}$ *is not continuous in E.*

Proof. Let V be a neighborhood of the unit 1. There exists $\varphi \in \Phi$ and $\varepsilon > 0$ such that

$$B_{\|.\|_{\varphi}}(1,\varepsilon) \subset V,$$

where

$$B_{\|.\|_{\varphi}}(1,\varepsilon) = \left\{ f \in E : \|f-1\|_{\varphi} \leq \varepsilon \right\}.$$

Let $a \in \mathbb{R}$ such that $\varphi(a) < \varepsilon$. Then there exists $\delta > 0$ such that

$$\varphi(x) < \varepsilon$$

for every $x \in]a-\delta, a+\delta[$. Let f be a continuous function such that

$$0 \leq f \leq 1,\ f(a) = 0$$

and

$$f = 1 \text{ besides }]a-\delta, a+\delta[.$$

Now consider $\psi \in \Phi$ and the function

$$g = \frac{1+\psi(a)}{1+2\psi(a)} f + \frac{\psi(a)}{1+2\psi(a)}.$$

Clearly g is continuous,

$$g = 1 \text{ besides }]a-\delta, a+\delta[,$$

and

$$1 \leq g^{-1} \leq \frac{1+2\psi(a)}{1+\psi(a)}$$

Furthermore $g \in V$ for

$$\|g-e\|_{\varphi} = \sup_{|x-a|<\delta} |\varphi(x)(g(x)-1)| \leq \varepsilon.$$

But,

$$\begin{aligned}
\left\|g^{-1}-e\right\|_{\psi} &= \sup_{x\in\mathbb{R}} \left|\psi(x)(g^{-1}(x)-1\right| \\
&\geq \psi(a)(g^{-1}(a)-1) \\
&= 1+\psi(a) > 1
\end{aligned}$$

thereby $g \in V$ and $g^{-1} \notin B_{\|.\|_{\psi}}\left(1,\frac{1}{2}\right)$. Thus the inverse map $f \longmapsto f^{-1}$ is not *continuous* in E. ∎

The following example shows that, in general case, the converse of theorem **3** is not true.

Example 11. *Let A be the algebra of the polynomial functions, with complex coefficients, on $\mathbb{C}$ endowed with the following norm*

$$\|f\|_1 = \int_0^1 |f(t)|\, dt$$

for every $f \in A$. Since $G(A)$ consists of constants, the inverse is continuous. But, by Turpin ([7]), the sequence of power maps $(x \longmapsto x^n)_n$ is not equicontinuous at zero for the algebra $(A, \|.\|_1)$ is not m-convex. Indeed, suppose conversely that there exists an equivalent system of submultiplicative seminorms $(|.|_i)_{i\in I}$ defining the topology of $(E, \|.\|_1)$. Then, for a given submultiplicative seminorm $|.|_i$, we can find a positive constants p and q such that

$$p\,\|f\|_1 \leq |f|_i \leq q\,\|f\|_1\,, \text{ for every } f \in A.$$

Thus, for every $k, p \in \mathbb{N}^*$, we have

$$\begin{aligned} p\left\|t^k t^p\right\|_1 &\leq \left|t^k t^p\right|_i \\ &\leq \left|t^k\right|_i |t^p|_i \\ &\leq q^2\left\|t^k\right\|_1 \|t^p\|_1\,. \end{aligned}$$

Which give $\frac{(k+1)(p+1)}{k+p+1} \leq \frac{q^2}{p}$, for all $k, p \in \mathbb{N}^*$. This contradiction proves that the algebra $(A, \|.\|_1)$ is not m-convex.

In general case of locally convex algebra, the continuity of the inverse map is not equivalent to the equicontinuity at zero of the power maps. But it can be obtained under additional condition.

Proposition 12. *Let (A, τ) be a locally convex unital algebra which is a Q-algebra. Then the following assertions are equivalent:*

1. *The sequence of power maps $(x \longmapsto x^n)_n$ is equicontinuous at zero.*
2. The *inverse map is continuous.*

Proof. **1**$\Longrightarrow$**2** By theorem **3**.
2$\Longrightarrow$**1** By Turpin ([7]). ∎

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