Lineability of functionals and renormings

F. J. García-Pacheco D. Puglisi

Abstract

We prove that every infinite dimensional Banach space can be equivalently renormed so that the set of norm attaining functionals contains an infinite dimensional vector subspace.

1 Introduction and background

Following the notion of a "big set" in the measure theory sense (the complementary of a measure zero set) and in the Baire theory sense (a comeager set), Gurariy coined in 1991 (see [12]) a new version of this notion in the linear sense: *lineability* and *spaceability*. However, this did not appear in the literature until the early 2000's in [3, 13]. For the last decade there has been an intensive trend to search for large algebraic and linear structures of special objects. We would like to mention the nice survey paper [5] related to this topic and the very recent monograph [2]. Let us introduce what we are meaning: A subset M of a Banach space X is said to be *lineable* (*spaceable*) if $M \cup \{0\}$ contains an infinite dimensional (closed) vector subspace. By λ-lineable (λ-spaceable) we mean that $M \cup \{0\}$ contains a (closed) vector subspace of dimension λ.

Throughout this paper, we will deal with a special friend: $\mathsf{NA}(X)$, the set of norm-attaining functionals on a Banach space X. By a classical Bishop-Phelps's theorem it is known that $\mathsf{NA}(X)$ is always "topologically generic", that is, dense in X^*, therefore it seems natural to raise the following question (originally posed by Godefroy in [11]).

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Problem 1.1 (Godefroy, [11]). *Given an infinite dimensional Banach space X, is* $\mathsf{NA}(X)$ *always lineable?*

Very recently, Rmoutil in [17] observed that the example of Read [16] of a Banach space with no proximinal subspaces of codimension 2 is also an example of a Banach space whose set of norm-attaining functionals does not contain subspaces of dimension 2. In [1] it has been shown that the above question has a positive answer for some classical Banach spaces like the $\mathcal{C}(K)$ and the $\mathsf{L}_1(\mu)$ spaces. In [9] it is observed that not all closed infinite dimensional subspaces of ℓ_∞ verify that the set of norm-attaining functionals is lineable. In the same manuscript it is also found a class of closed infinite dimensional subspaces of ℓ_∞, called filling subspaces of ℓ_∞, such that the set of norm-attaining functionals is lineable. We recall the reader that a closed infinite dimensional subspace V of ℓ_∞ is said to be filling provided that for every infinite subset A of $\operatorname{supp}(V)$ there exists $x \in \mathsf{S}_V$ with $\operatorname{supp}(x) \subseteq A$ and x attains its sup norm. In [10] the previous results are generalized in the following way.

Theorem 1.2 (García-Pacheco and Puglisi, 2017). *Let X be a Banach space. There exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\{x_i^* : i \in I\}$ is norming if and only if X is linearly isometric to a filling subspace of $\ell_\infty(\Lambda)$. In this situation,* $\mathsf{NA}(X)$ *is* $\operatorname{card}(\Lambda)$*-lineable.*

Another isometric result concerning the lineability of the norm-attaining functionals was given in [8], where it is proved that if a Banach space admits a monotonic projection basis then the set of norm-attaining functionals is lineable.

Concerning Question 1.1 in terms of spaceability, the main effort has been done by Bandyopadhyay and Godefroy in [4], where it was shown that Asplund Banach spaces with the Dunford-Pettis property cannot be equivalently renormed to make the norm-attaining functionals spaceable. In particular, if K is an infinite Hausdorff scattered compact topological space, then $\mathsf{NA}(\mathcal{C}(K))$ is lineable but not spaceable.

As far as we know, the main result obtained until now concerning the isomorphic lineability of $\mathsf{NA}(X)$ was obtained in [8], where it is shown that every Banach space admitting an infinite dimensional separable quotient can be equivalently renormed so that the set of its norm-attaining functionals is lineable. In [10] it also provided an isomorphic condition for the lineability of $\mathsf{NA}(X)$.

Theorem 1.3 (García-Pacheco and Puglisi, 2017). *Let X be a Banach space. There exists a biorthogonal system $(x_i, x_i^*)_{i\in I}$ such that $\{x_i^* : i \in I\}$ is bounded and* $\operatorname{span}\left(\overline{\operatorname{abco}}^{w^*}\{x_i^* : i \in I\}\right) = X^*$ *if and only if X is isomorphic to a filling subspace of $\ell_\infty(\Lambda)$. In this situation, X can be equivalently renormed to make* $\mathsf{NA}(X)$ *be* $\operatorname{card}(\Lambda)$*-lineable.*

In this note we solve completely the isomorphic version of Godefroy's question 1.1.

2 Main results

Let $(X, \|\cdot\|)$ be a Banach space. A closed subspace M of X^* is said to be *total* if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$. For a total subspace $M \subseteq X^*$ one can define a norm on X

$$\|x\|_M = \sup\{|f(x)| : f \in M, \|f\| \leq 1\}.$$

It is clear that $\|\cdot\|_M \leq \|\cdot\|$. If $\|\cdot\|_M$ is equivalent to $\|\cdot\|$, then M is said to be *norming*. A first example of a total non-norming subspace goes back to S. Mazurkiewicz [14]. Observe that if M is a total non-norming subspace of X^*, then B_X is not a neighborhood of 0 in $(X, \|\cdot\|_M)$ and, since B_X is absolutely convex, we deduce that B_X has empty interior in $(X, \|\cdot\|_M)$ as well as in its completion. In [7], W.J. Davis and J. Lindenstrauss proved that a Banach space X has a total non-norming subspace in X^* if and only if X has infinite codimension in its second dual, i.e. $\dim X^{**}/X = \infty$ (see also [15]).

Lemma 2.1. *Let X be a Banach space and A a closed absolutely convex subset of X with empty interior. Then for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathsf{S}_{X^*}$ such that $|f_\varepsilon(a)| \leq \varepsilon$ for all $a \in A$.*

Proof. Consider the polar set $A^0 := \{f \in X^* : |f(a)| \leq 1 \ \ \forall a \in A\}$. We will show that A^0 is unbounded. Otherwise, there exists $\alpha > 0$ such that $A^0 \subseteq \alpha\mathsf{B}_{X^*}$. Then $\alpha\mathsf{B}_X = (\alpha\mathsf{B}_{X^*})_0 \subseteq \left(A^0\right)_0 = \overline{\text{abco}}(A) = A$. This contradicts the fact that A has empty interior, therefore A^0 is unbounded. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A^0 such that $(\|f_n\|)_{n\in\mathbb{N}}$ diverges to ∞. Let $n_0 \in \mathbb{N}$ such that $\|f_{n_0}\| > \frac{1}{\varepsilon}$. Finally, it suffices to take $f_\varepsilon := f_{n_0}/\|f_{n_0}\|$. ∎

Lemma 2.2. *Let X be a topological vector space, A and B non-empty subsets of X, and Y a proper subspace of X. If $A + B \subseteq Y$, then both A and B have empty interior.*

Proof. We will show that A has empty interior. In a similar way it can be shown that B has empty interior. Fix an arbitrary $b \in B$. Then $A + b \subseteq A + B \subseteq Y$ and since Y is proper we have that Y has empty interior in X, therefore $A + b$ has empty interior in X. Since translations are homeomorphisms, we deduce that A has empty interior in X. ∎

We are now in the right position to state and prove the main result in this manuscript. The argument used in the proof resembles the one in [15].

Theorem 2.3. *Every infinite dimensional Banach space X admits an equivalent norm such that $\mathsf{NA}\,(X)$ is lineable.*

Proof. In case $\dim X^{**}/X < \infty$, X is a quasi–reflexive space and hence by [18] X is a direct sum of a reflexive subspace Y and a separable subspace Z. Therefore it has a separable infinite–dimensional quotient space and the thesis follows directly by [8, Corollary 3.3].

Let us suppose that $\dim X^{**}/X = \infty$. By the Davis-Lindenstrauss's theorem [7], there exists a closed subspace $M \subseteq X^*$ which is total non-norming. Let us define X_M to be the completion of $(X, \|\cdot\|_M)$ and let

$$E_0 : X \hookrightarrow X_M$$

be the natural embedding.

We have that $E_0(\mathsf{B}_X)$ does not have interior point in X_M. Therefore, by the Lemma 2.1 there exists $f_1 \in \mathsf{S}_{X_M^*}$ such that

$$|f_1(E_0(x))| \leq \frac{1}{2 \cdot 3}, \quad \forall x \in \mathsf{B}_X.$$

Let $v_0 \in X$ such that $\|E_0(v_0)\|_M \leq 2$ and $f_1(E_0(v_0)) = 1$ and define

$$E_1 : X \longrightarrow X_M$$

by

$$E_1(x) = E_0(x) - f_1(E_0(x))E_0(v_0).$$

Therefore we have

(i$_0$) $(E_0 - E_1)(X) = \operatorname{span}\{E_0(v_0)\}$,

(ii$_0$) $\|E_0^*(f_1)\|_{X^*} \leq \frac{1}{2 \cdot 3}$,

(iii$_0$) $\|E_0 - E_1\| \leq \frac{1}{3}$,

(iv$_0$) $E_1(X) \subseteq \ker(f_1) \cap E_0(X)$.

According to Lemma 2.2, $E_1(\mathsf{B}_X)$ does not have interior points in X_M. Hence, we can proceed exactly as before with E_1 instead of E_0, to create an operator $E_2 : X \longrightarrow X_M$ and $f_2 \in \mathsf{S}_{X_M^*}$ satisfying suitable conditions. Iterating this process, for each $n \in \mathbb{N} \cup \{0\}$, we obtain a sequence of operators

$$E_n : X \longrightarrow X_M,$$

a sequence of functionals $(f_n)_n \subseteq \mathsf{S}_{X_M^*}$ and $(v_n)_n \subseteq X$, such that

(i$_n$) $f_{n+1}(E_n(v_n)) = 1$,

(ii$_n$) $\|E_n^*(f_{n+1})\|_{X^*} \leq \frac{1}{2 \cdot 3^{n+1}}$,

(iii$_n$) $(E_n - E_{n+1})(X) = \operatorname{span}\{E_n(v_n)\}$,

(iv$_n$) $\|E_n - E_{n+1}\| \leq \frac{1}{3^{n+1}}$,

(v$_n$) $E_{n+1}(X) \subseteq \ker(f_{n+1}) \cap E_n(X) \subseteq \left(\cap_{i=1}^{n+1} \ker(f_i)\right) \cap E_0(X)$..

Directly from the construction it follows that

- The sequence $(f_n)_n$ is linearly independent. Indeed, by (i$_n$) f_{n+1} does not vanish on $E_n(X)$ and then by (v$_n$), it does not vanish on $\cap_{i=0}^{n} \ker(f_i)$.
- For all $n \in \mathbb{N}$, $E_0(v_0), \ldots, E_n(v_n) \in \operatorname{span}\{E_0(v_i) : 0 \leq i \leq n\}$.

By ($\mathrm{iv_n}$), the sequence $(E_n)_n$ converges in the norm-operator topology to some operator

$$D : X \longrightarrow X_M,$$

which by ($\mathrm{v_n}$)

$$D(X) \subseteq \left(\bigcap_{n=0}^{\infty} \ker(f_n) \right) \cap E_0(X). \tag{2.1}$$

We obtain that

$$E_0 = \sum_{n=0}^{\infty} (E_n - E_{n+1}) + D.$$

From this equality, since $E_0(X)$ is dense in X_M, we easily obtain that

$$\overline{\operatorname{span}}\{E_n(v_n) :\ n \in \mathbb{N}\} \oplus \overline{D(X)} \ \text{ is dense in } X_M. \tag{2.2}$$

By ($\mathrm{ii_n}$) above, we have that $\sum_{n\geq 0} |E_n^*(f_{n+1})(x)| < \infty$ for all $x \in X$. Thus

$$\sum_{n\geq 1} |f_n(x)| < \infty \quad \text{for all } x \in X_M/\overline{D(X)}. \tag{2.3}$$

Next, we will use a classical basic sequence construction. Let $(\varepsilon_n)_n$ be a sequence such that $0 < \varepsilon_n < 1$ and $\sum_n \varepsilon_n < \infty$. Using (2.3) inductively one can find a strictly increasing sequence $(p_n)_n$ in $\mathbb{N}$, and an increasing sequence of finite sets $A_n \subseteq \mathsf{B}_{X_M/\overline{D(X)}}$ such that

- For each $u \in \left(\operatorname{span}\{f_{p_1}, \ldots, f_{p_n}\}\right)^*$ with $\|u\| \leq 1$ there is an $x \in A_n$ such that
$$|u(f) - f(x)| \leq \frac{\varepsilon_n}{3}\|f\| \quad \text{for every } f \in \operatorname{span}\{f_{p_1}, \ldots, f_{p_n}\}$$
- $|f_{p_n+1}(x)| \leq \frac{\varepsilon_n}{3}$ for every $x \in A_n$.

Therefore, it is easy to check that

$$\|f + \lambda f_{p_n+1}\| \geq (1 - \varepsilon_n)\|f\| \quad \text{for all } f \in \operatorname{span}\{f_{p_1}, \ldots, f_{p_n}\},\ \lambda \in \mathbb{R}.$$

By the classical Mazur lemma, the sequence $(f_{p_n})_n$ is basic in $\left(X_M/\overline{D(X)}\right)^*$, and passing to a quotient if necessary we can assume without any loss of generality that the corresponding biorthogonal sequence of coordinates $(z_n)_n$ is a Schauder basis in $X_M/\overline{D(X)}$. Moreover we can consider an equivalent norm on $X_M/\overline{D(X)}$ such that such that $(z_n)_n$ is a monotone Schauder basis.

Now we apply [4, Lemma 2.4] to find an equivalent norm $|\cdot|$ on X_M which coincides with the original norm on $\overline{D(X)}$ and makes $\overline{D(X)}$ proximinal.

At this point, we may assume that $(z_n)_{n\in\mathbb{N}} \subseteq \overline{\operatorname{span}}\{E_0(v_n) : n \in \mathbb{N}\}$. Thus there exists a bounded subset A of X such that every f_n attains its norm at an element of A.

Therefore, the norm whose unit ball is $\overline{\operatorname{abco}}\,(\mathsf{B}_X \cup A)$ defines an equivalent renorming on X that makes

$$\operatorname{span}\{f_n :\ n \in \mathbb{N}\} \subseteq \mathsf{NA}\,(X)\,. \qquad \blacksquare$$

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Department of Mathematics, University of Cadiz,
Puerto Real, 11519, Spain (EU)
email: garcia.pacheco@uca.es

Department of Mathematics and Computer Sciences,
University of Catania,
Catania, 95125, Italy (EU)
emaildpuglisi@dmi.unict.it