The τ-precompact Hausdorff Group Reflection of Topological Groups*

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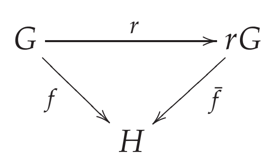
Abstract

We give three different descriptions of the τ-precompact Hausdorff group reflection of topological groups. In particular, we describe the ω-narrow Hausdorff reflection of a given topological group. We also prove that the τ-precompact Hausdorff reflection functor preserves perfect surjective homomorphisms, quotient homomorphisms and arbitrary products. As a direct application, we deduce that the compact Hausdorff reflection functor preserves arbitrary products.

1 Introduction

Let G be a topological group. A topological group rG with property $\mathcal{P}$ and a continuous homomorphism r from G to rG is called a *$\mathcal{P}$-reflection* of G if for every continuous homomorphism from G to an arbitrary topological group H with property $\mathcal{P}$, there exists a unique continuous homomorphism $\bar{f}$ from rG to H such that $\bar{f} \circ r = f$.



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It is clear from the definition that if $s\colon G \longrightarrow sG$ is another $\mathcal{P}$-reflection of G, then there exists a topological isomorphism $j\colon rG \longrightarrow sG$ such that $j \circ r = s \circ id_G$. Hence the $\mathcal{P}$-reflection of G, if exists, is essentially unique. In this case we say that r is *universal* for all continuous homomorphisms from G into a topological group with property $\mathcal{P}$. Abusing terminology we also call the topological group rG the *$\mathcal{P}$-reflection* of G.

Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$. If for each object $B \in \mathcal{B}$, there exists a $\mathcal{A}$-reflection A_B with the reflection mapping $r_B : B \longrightarrow A_B$, then there exists a unique reflection functor $R : \mathcal{B} \longrightarrow \mathcal{A}$ such that $R(B) = A_B$ for each object $B \in \mathcal{B}$ and $R(f) \circ r_B = r_{B'} \circ f$ for each $\mathcal{B}$-morphism $f : B \longrightarrow B'$.

Whenever the Hausdorff separation property is assumed, we will specify it explicitly.

Let τ be an infinite cardinal. A topological group G is called *τ-precompact* if, for every neighborhood U of the identity e in G, there exists a subset K of G with $|K| < \tau$ such that $KU = G$. An $\aleph_0$-precompact group is called *precompact* (or *totally bounded*). An $\aleph_1$-precompact group is called *ω-narrow* (see [2, Section 3.4]).

The most important compactification of a Tychonoff space is the Čech-Stone compactification. From the category theory point of view, this is the compact Hausdorff reflection of a Tychonoff space. For an arbitrary topological group G, the compact Hausdorff group reflection of G coincides with the Bohr compactification of G. Many articles have contributed to this topic. Some articles discuss the categorical properties of the Bohr compactification (see [7], [13], [15]). Many others study the relationships between the topological properties of the original group and its Bohr compactification (see [2, Section 9.9], [11]). Very recently the so-called *torsion Bohr topology* has been studied in [3], [4], [23]. The ω-narrow reflection of abelian groups has been considered in [14]. Other reflections in the class of semitopological group (such as the T_i-reflection for $i = 0, 1, 2, 3$ and the group reflection of a semitopological group) have been systematically studied by Tkachenko in [19], [20], [21], [22]. Inspired by these papers we discuss the τ-precompact Hausdorff reflection of topological groups.

Here, for an arbitrary cardinal $\tau \geq \omega$, we consider the τ-precompact Hausdorff reflection of a topological group which is a generalization of the Bohr compactification and ω-narrow Hausdorff reflection. We show that for every topological group G and every infinite cardinal τ, there exists a τ-precompact Hausdorff topological group $\varpi_\tau G$ (ω-narrow Hausdorff group ϖG) which is the τ-precompact Hausdorff group reflection (ω-narrow Hausdorff group reflection) of G. In addition, we give three different descriptions of the τ-precompact Hausdorff group reflection of a given topological group. We also show that the τ-precompact Hausdorff reflection functor preserves perfect surjective homomorphisms, quotient homomorphisms and arbitrary products. Finally, we deduce that the compact Hausdorff reflection functor preserves arbitrary products.

1.1 Notation and terminology

Throughout this paper we consider the category **TopGrp** of (*not necessarily Hausdorff*) topological groups and their continuous homomorphisms. The categories τ**PHGrp** and **CompHGrp** stand for the categories of τ-precompact Hausdorff groups and compact Hausdorff groups, respectively. The letters $G, H, K, ...$ will always denote topological groups. The formula $A \leq G$ means that A is a subgroup of a topological group G endowed with the subspace topology. The identity map of G onto itself is denoted by 1_G, while e_G denotes the identity element of G. For each $x \in G$, $\mathcal{N}_G(x)$ or simply $\mathcal{N}(x)$ denotes the family of all open neighborhoods of x in G.

For an arbitrary topological group G, we denote the 'family' of all continuous homomorphisms from G to Hausdorff groups H with $w(H) < \tau$ by the symbol $G_\tau^\sim$. [Clearly $G_\tau^\sim$ is a proper class, but it is a usual practice to identify two elements $f, g \in G_\tau^\sim$ if there exists a topological isomorphism φ of $f(G)$ onto $g(G)$ such that $\varphi \circ f = g$. This introduces an equivalence relation $\sim$ on $G_\tau^\sim$ and one can easily verify that $G_\tau^\sim/\sim$ is a set.] In fact, one can check that $[f] \in G_\tau^\sim/\sim$ and $\{g \in G_\tau^\sim : [g] = [f]\}$ induce the same topology on G, so the two families $G_\tau^\sim$ and $G_\tau^\sim/\sim$ induce the same topology. Therefore we denote the initial topology induced by the family $G_\tau^\sim$ on G by σ_τ^G and we will not distinguish the two families. If no confusion is possible, we will simply denote the topology σ_τ^G by σ_τ. We denote the group (G, σ_τ) by $\omega_\tau G$ and $\omega_\tau G/\overline{\{e_G\}}$ by $\varpi_\tau G$.

2 τ-precompact Hausdorff reflection of topological groups

In this section we define the τ-precompact Hausdorff reflection of a given topological group G and present several descriptions of the resulting Hausdorff topological group $\varpi_\tau G$.

Lemma 2.1. *Let (G, σ) be a topological group. Then the following assertions are equivalent:*

(a) *(G, σ) is a τ-precompact Hausdorff group;*

(b) *(G, σ) can be embedded in a product $\prod_{\alpha \in I} G_\alpha$ of Hausdorff topological groups G_α satisfying $w(G_\alpha) < \tau$ for each $\alpha \in I$;*

(c) *The topology on G induced by the family $G_\tau^\sim$ coincides with the original topology of G and this topology is Hausdorff.*

Proof. (a)$\Rightarrow$(b) follows from Theorem 5.1.10 in [2] (Guran's theorem).

(b)$\Rightarrow$(c) follows from the definition of the Tychonoff product topology.

(c)$\Rightarrow$(a). The family $\mathcal{B}$ of the sets $f^{-1}(V)$, where $f : G \longrightarrow H$ is a continuous homomorphism to a topological group H with $w(H) < \tau$ and $V \in \mathcal{N}_H(e)$, constitutes a local neighborhood base at the identity of (G, σ_τ); clearly $\mathcal{B}$ is closed under finite intersections. It follows from (c) that for an arbitrary neighborhood $U \in \mathcal{N}_G(e)$, there exists a neighborhood $f_U^{-1}(V_U) \in \mathcal{B}$ of the identity in G such that $f_U^{-1}(V_U) \subseteq U$. Then $\Delta_{U\in\mathcal{N}_G(e)} f_U : G \longrightarrow \prod_{U\in\mathcal{N}_G(e)} H_U$ is a topological isomorphism of G onto a subgroup of $\prod_{U\in\mathcal{N}_G(e)} H_U$. Hence G is τ-precompact since that each group H_U is τ-precompact and τ-precompactness is stable with respect to products and taking subgroups. ∎

Proposition 2.2. *Let G be a topological group. Every continuous homomorphism $f\colon G \longrightarrow K$ from G to a τ-precompact group K remains continuous when considered as a homomorphism from (G, σ_τ) to K, i.e. the identity mapping $1_G\colon G \longrightarrow (G, \sigma_\tau)$ is universal for all continuous homomorphisms from G to τ-precompact groups.*

Proof. Let $\pi\colon K \to K/\overline{\{e_K\}}$ be the quotient homomorphism. The quotient group $K/\overline{\{e_K\}}$ is a τ-precompact Hausdorff group. Therefore, by Lemma 2.1, the topology of the group $K/\overline{\{e_K\}}$ coincides with the topology induced by the family $(K/\overline{\{e_K\}})^{\sim}_{\tau}$. Hence the composition mapping $\pi \circ f\colon (G, \sigma_\tau) \to K/\overline{\{e_K\}}$ is a continuous homomorphism by the definition of the topology σ_τ. So the homomorphism $f\colon (G, \sigma_\tau) \to K$ is also continuous because π is a quotient mapping. ∎

Let $\pi\colon (G, \sigma_\tau) \longrightarrow G/\overline{\{e_G\}}$ be the quotient mapping. It can be easily checked that $G/\overline{\{e_G\}}$ with the quotient mapping π is the T_1-reflection of (G, σ_τ). By Proposition 2.2, the following result is clear.

Corollary 2.3. *Let G be a topological group and $\varpi_\tau G = \omega_\tau G/\overline{\{e_G\}}$. The topological group $\omega_\tau G$ is the τ-precompact reflection of G, and the Hausdorff group $\varpi_\tau G$ is the τ-precompact Hausdorff reflection of G.*

The second way to get the τ-precompact Hausdorff reflection of a topological group (G, ς) is to consider G as a discrete group, and endow G with the topology σ induced by *all* homomorphisms from G to Hausdorff groups H with $w(H) < \tau$. Then we take the group topology $\inf\{\sigma, \varsigma\}$ on G, it will be the topology of the τ-precompact reflection $\omega_\tau G$. Thus $\varpi_\tau G = (G, \inf\{\sigma, \varsigma\})/\overline{\{e_G\}}$. We explain this with more detail in Propositions 2.5 and 2.6.

First we recall the well-known Pontryagin's description of a neighborhood base at the identity of a topological group.

Theorem 2.4. *Let G be a topological group and $\mathcal{U}$ an open base at the identity e of G. Then:*

i) for every $U \in \mathcal{U}$, there is an element $V \in \mathcal{U}$ such that $V^2 \subseteq U$;

ii) for every $U \in \mathcal{U}$, there is an element $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$;

iii) for every $U \in \mathcal{U}$ and every $x \in U$, there is an element $V \in \mathcal{U}$ such that $Vx \subseteq U$;

iv) for every $U \in \mathcal{U}$ and every $x \in U$, there is an element $V \in \mathcal{U}$ such that $xVx^{-1} \subseteq U$;

v) for every $U, V \in \mathcal{U}$, there is an element $W \in \mathcal{U}$ such that $W \subseteq U \bigcap V$.

Conversely, let $\mathcal{U}$ be a family of subsets of G satisfying conditions $i) - v)$. Then the family $\mathcal{B}_{\mathcal{U}} = \{Ua : a \in G,\ U \in \mathcal{U}\}$ is a base for a topology $\tau_{\mathcal{U}}$ on G. With this topology, G is a topological group, and the family $\{aU : a \in G,\ U \in \mathcal{U}\}$ is also a base for the same topology on G.

Proposition 2.5. *Let ς and σ be group topologies on a group G. Let also $\mathcal{N}_\varsigma(e)$ and $\mathcal{N}_\sigma(e)$ be symmetric neighborhood bases at the identity e in (G,ς) and (G,σ), respectively. To functions $\varphi\colon \mathbb{N} \to \mathcal{N}_\varsigma(e)$ and $\psi\colon \mathbb{N} \to \mathcal{N}_\sigma(e)$, we assign the set*

$$O[\varphi,\psi] = \bigcup_{n\in\mathbb{N}} \bigcup_{\pi\in\mathbf{S}(2n)} U_{\pi(1)}V_{\pi(2)}\cdots U_{\pi(2n-1)}V_{\pi(2n)},$$

where $\mathbf{S}(2n)$ is the group of permutations of the set $\{1,\ldots,2n\}$, $\varphi(i) = U_i$ and $\psi(i) = V_i$ for each $i \in \mathbb{N}$. Then the family $\mathcal{N} = \{O[\varphi,\psi] : \varphi \in \mathcal{N}_\varsigma(e)^{\mathbb{N}}$ and $\psi \in \mathcal{N}_\sigma(e)^{\mathbb{N}}\}$ is a local base at the identity of the topological group $(G, \inf\{\varsigma,\sigma\})$.

Proof. We will prove that $\mathcal{N}$ constitutes a local base at e in $(G,\inf\{\varsigma,\sigma\})$. First we verify that the family $\mathcal{N}$ satisfies conditions i)-v) of Theorem 2.4.

We start with condition i). Take an element $O[\varphi,\psi] \in \mathcal{N}$ for some $\varphi \in \mathcal{N}_\varsigma(e)^{\mathbb{N}}$ and $\psi \in \mathcal{N}_\sigma(e)^{\mathbb{N}}$, and put $\varphi(i) = U_i$ and $\psi(i) = V_i$ for each $i \in \mathbb{N}$. Let $\varphi'(i) = W_i$ and $\psi'(i) = O_i$ for each $i \in \mathbb{N}$, where $W_i \in \mathcal{N}_\varsigma(e)$, $O_i \in \mathcal{N}_\sigma(e)$ and $W_i \subseteq \bigcap_{j=1}^{2i} U_j$, $O_i \subseteq \bigcap_{j=1}^{2i} V_j$. Let us verify that $O[\varphi',\psi']^2 \subseteq O[\varphi,\psi]$. By the definition of $O[\varphi,\psi]$ and $O[\varphi',\psi']$, it suffices to prove that for arbitrary $m,n \in \mathbb{N}$ and permutations $\pi \in \mathcal{S}(2n)$, $\lambda \in \mathcal{S}(2m)$, the inclusion $(W_{\pi(1)}O_{\pi(2)}\cdots W_{\pi(2n-1)}O_{\pi(2n)})(W_{\lambda(1)}O_{\lambda(2)}\cdots W_{\lambda(2m-1)}O_{\lambda(2m)}) \subseteq U_{\theta(1)}V_{\theta(2)}\cdot\cdot\cdot U_{\theta(2k-1)}V_{\theta(2k)}$ holds for an appropriately chosen integer k and a permutation $\theta \in \mathcal{S}(2k)$. In fact we can assume that $m = n$. Let $k = 2n$, $\theta(i) = 2\pi(i) - 1$ and $\theta(2n+i) = 2\lambda(i)$ for each $1 \le i \le 2n$. Then $\theta \in \mathcal{S}(4n)$ and $W_{\pi(i)} \subseteq U_{2\pi(i)-1} = U_{\theta(i)}$ for each $1 \le i \le 2n$ ($O_{\pi(i)} \subseteq V_{2\pi(i)-1} = V_{\theta(i)}$ for each $1 \le i \le 2n$) and $W_{\lambda(i)} \subseteq U_{2\lambda(i)} = U_{\theta(2n+i)}$ for each $1 \le i \le 2n$ ($O_{\lambda(i)} \subseteq V_{2\lambda(i)} = V_{\theta(2n+i)}$ for each $1 \le i \le 2n$ also holds). Then

$$\begin{aligned}&(W_{\pi(1)}O_{\pi(2)}\cdots W_{\pi(2n-1)}O_{\pi(2n)})(W_{\lambda(1)}O_{\lambda(2)}\cdots W_{\lambda(2n-1)}O_{\lambda(2n)})\\ &\subseteq (U_{\theta(1)}V_{\theta(2)}\cdots U_{\theta(2n-1)}V_{\theta(2n)})(U_{\theta(2n+1)}V_{\theta(2n+2)}\cdots U_{\theta(4n-1)}V_{\theta(4n)}).\end{aligned}$$

This implies condition i).

Now we verify condition ii). We will prove that every element $O[\varphi,\psi]$ is symmetric. By the definition of $O[\varphi,\psi]$, it suffices to show that $(U_{\pi(1)}V_{\pi(2)}\cdots U_{\pi(2n-1)}V_{\pi(2n)})^{-1}$ is contained in $O[\varphi,\psi]$ for each $\pi \in \mathcal{S}(2n)$. We define a permutation $\mu \in \mathcal{S}(2n+2)$ by $\mu(1) = 2n+1$, $\mu(2n+2) = 2n+2$ and $\mu(i) = \pi(2n+2-i)$ for each $2 \le i \le 2n+1$. Then we have

$$\begin{aligned}(U_{\pi(1)}V_{\pi(2)}\cdots U_{\pi(2n-1)}V_{\pi(2n)})^{-1} &= V_{\pi(2n)}U_{\pi(2n-1)}\cdots V_{\pi(2)}U_{\pi(1)}\\ &\subseteq U_{2n+1}V_{\pi(2n)}U_{\pi(2n-1)}\cdots V_{\pi(2)}U_{\pi(1)}V_{2n+2}\\ &= U_{\mu(1)}V_{\mu(2)}U_{\mu(3)}\cdots V_{\mu(2n))}U_{\mu(2n+1)}V_{\mu(2n+2)}.\end{aligned}$$

Now we check condition iii). Take $O[\varphi,\psi] \in \mathcal{N}$ and an element $x \in O[\varphi,\psi]$. Then there exist $n \in \mathbb{N}$ and $\pi \in \mathcal{S}(2n)$ such that $x \in U_{\pi(1)}V_{\pi(2)}\cdots U_{\pi(2n-1)}V_{\pi(2n)}$. We put $W_i = U_{2n+i}$, $O_i = V_{2n+i}$ and define $\varphi'(i) = W_i$, $\psi'(i) = O_i$ for each $i \in \mathbb{N}$. We claim that $O[\varphi',\psi']x \subseteq O[\varphi,\psi]$. It suffices to prove that for every $m \in \mathbb{N}$ and $\lambda \in \mathcal{S}(2m)$, there exists $\nu \in \mathcal{S}(2m+2n)$ such that $W_{\lambda(1)}O_{\lambda(2)}\cdots$

$W_{\lambda(2m-1)}O_{\lambda(2m)}x \subseteq U_{\nu(1)}V_{\nu(2)}\cdots U_{\nu(2m+2n-1)}V_{\nu(2m+2n)}$. We define a permutation ν by $\nu(i) = \lambda(i) + 2n$ for each $1 \le i \le 2m$ and $\nu(2m+i) = \pi(i)$ for each $1 \le i \le 2n$. Then $\nu \in \mathcal{S}(2m+2n)$, and

$$\begin{aligned}
&W_{\lambda(1)}O_{\lambda(2)}\cdots W_{\lambda(2m-1)}O_{\lambda(2m)}x\\
&\subseteq (W_{\lambda(1)}O_{\lambda(2)}\cdots W_{\lambda(2m-1)}O_{\lambda(2m)})(U_{\pi(1)}V_{\pi(2)}...U_{\pi(2n-1)}V_{\pi(2n)})\\
&= (U_{\lambda(1)+2n}V_{\lambda(2)+2n}\cdots U_{\lambda(2m-1)+2n}V_{\lambda(2m)+2n})(U_{\pi(1)}V_{\pi(2)}\cdots U_{\pi(2n-1)}V_{\pi(2n)})\\
&= (U_{\nu(1)}V_{\nu(2)}\cdots U_{\nu(2m-1)}V_{\nu(2m)})(U_{\nu(2m+1)}V_{\nu(2m+2)}\cdots U_{\nu(2m+2n-1)}V_{\nu(2m+2n)}).
\end{aligned}$$

This implies iii).

Let us verify condition iv). Given an arbitrary element $x \in G$ and $O[\varphi,\psi] \in \mathcal{N}$ for some $\varphi \in \mathcal{N}_\varsigma(e)^{\mathbb{N}}$, $\psi \in \mathcal{N}_\sigma(e)^{\mathbb{N}}$, let $\varphi(i) = U_i$, $\psi(i) = V_i$ for each $i \in \mathbb{N}$. We also choose $W_i \in \mathcal{N}_\varsigma(e)$, $O_i \in \mathcal{N}_\sigma(e)$ such that $xW_ix^{-1} \subseteq U_i$, $xO_ix^{-1} \subseteq V_i$ for each $i \in \mathbb{N}$. Then we define $\varphi'(i) = W_i$, $\psi'(i) = O_i$ for each $i \in \mathbb{N}$. Then

$$\begin{aligned}
&xO[\varphi',\psi']x^{-1}\\
&\quad= x\Big(\bigcup_{n\in\mathbb{N}}\bigcup_{\pi\in\mathbf{S}(2n)} W_{\pi(1)}O_{\pi(2)}\cdots W_{\pi(2n-1)}O_{\pi(2n)}\Big)x^{-1}\\
&\quad= \bigcup_{n\in\mathbb{N}}\bigcup_{\pi\in\mathbf{S}(2n)} xW_{\pi(1)}O_{\pi(2)}\cdots W_{\pi(2n-1)}O_{\pi(2n)}x^{-1}\\
&\quad= \bigcup_{n\in\mathbb{N}}\bigcup_{\pi\in\mathbf{S}(2n)} (xW_{\pi(1)}x^{-1})(xO_{\pi(2)}x^{-1})\cdots(xW_{\pi(2n-1)}x^{-1})(xO_{\pi(2n)}x^{-1})\\
&\quad\subseteq \bigcup_{n\in\mathbb{N}}\bigcup_{\pi\in\mathbf{S}(2n)} U_{\pi(1)}V_{\pi(2)}\cdots U_{\pi(2n-1)}V_{\pi(2n)}\\
&\quad= O[\varphi,\psi].
\end{aligned}$$

The verification of condition v) is easy and hence it is omitted.

Thus there exists a topological group topology ξ on G such that $\mathcal{N}$ is a local base at e in (G,ξ). It follows from the definition of the sets $O[\varphi,\psi]$ that they are all open in (G,ς) and (G,σ). Therefore $\xi \subseteq \inf\{\sigma,\varsigma\}$. It remains to prove that $\inf\{\sigma,\varsigma\} \subseteq \xi$. Given any open neighborhood W of identity e in $(G,\inf\{\sigma,\varsigma\})$, we can define a sequence $\{W_i : i \in \mathbb{N}\}$ of open neighborhoods of e in $(G,\inf\{\sigma,\varsigma\})$ such that $W_1^3 \subseteq W$, $W_{i+1}^3 \subseteq W_i$ for each $i \in \mathbb{N}$. It follows from [2, Lemma 7.2.6] that for every positive integer $n \in \mathbb{N}$ and every permutation $\pi \in \mathcal{S}(n)$, $W_{\pi(1)}W_{\pi(2)}...W_{\pi(n)} \subseteq W$. Since $\inf\{\sigma,\varsigma\}$ is coarser than ς and σ, we can take two sequences $\{U_i\}$, $\{V_i\}$, where $U_i \in \mathcal{N}_\varsigma(e)$, $V_i \in \mathcal{N}_\sigma(e)$, such that $U_i \subseteq W_i$ and $V_i \subseteq W_i$ for each $i \in \mathbb{N}$. Then we define $\varphi(i) = U_i$, $\psi(i) = V_i$ for each $i \in \mathbb{N}$. One can easily verify that $O[\varphi,\psi] \subseteq W$. Therefore $\inf\{\sigma,\varsigma\} \subseteq \xi$, which implies the equality $\inf\{\sigma,\varsigma\} = \xi$. ■

Given a topological group G, we denote by G_d the underlying group G endowed with the discrete topology.

Proposition 2.6. *Let (G,σ) be a topological group and ς be the initial topology on G induced by all homomorphisms from G_d to Hausdorff groups H with $w(H) < \tau$. Then $\inf\{\varsigma,\sigma\} = \sigma_\tau$.*

Proof. We know that σ_τ is the initial topology with respect to the family of all continuous homomorphisms G to Hausdorff groups H with $w(H) < \tau$. Therefore $\sigma_\tau \subseteq \varsigma$ and $\sigma_\tau \subseteq \sigma$, whence $\sigma_\tau \subseteq \inf\{\varsigma, \sigma\}$.

The topology $\lambda = \inf\{\varsigma, \sigma\}$ is coarser than ς, so it is τ-precompact. Clearly λ is coarser than σ. Hence the identity mapping $1_G\colon (G, \sigma_\tau) \to (G, \lambda)$ is continuous by Proposition 2.2. This implies that $\lambda \subseteq \sigma_\tau$, so $\inf\{\varsigma, \sigma\} = \sigma_\tau$. ■

Now we describe the third method to construct the τ-precompact Hausdorff group reflection of an arbitrary topological group (G, σ). Let $N_G = \bigcap_{f \in G^{\sim}_{\tau}} ker(f)$. Then N_G is a closed normal subgroup of G. Let $(G/N_G, \overline{\sigma})$ be the quotient topological group and $r_{N_G}\colon G \longrightarrow G/N_G$ be the canonical homomorphism. Let $(G/N_G, \overline{\sigma}_\tau)$ denote the group G/N_G with the topology $\overline{\sigma}_\tau$ induced by the family $(G/N_G)^{\sim}_{\tau}$.

Theorem 2.7. *Let (G, σ) be a topological group. Then $(G/N_G, \overline{\sigma}_\tau)$ is a τ-precompact Hausdorff group, and $r_{N_G}\colon G \longrightarrow (G/N_G, \overline{\sigma}_\tau)$ is universal for all continuous homomorphisms from G to τ-precompact Hausdorff groups, i.e. $(G/N_G, \overline{\sigma}_\tau)$ is the τ-precompact Hausdorff reflection of G.*

Proof. By Lemma 2.1, $(G/N_G, \overline{\sigma}_\tau)$ is a τ-precompact Hausdorff group. Let $f\colon G \longrightarrow H$ be a continuous homomorphism, where H is a τ-precompact Hausdorff group. It follows from the definition of N_G that $N_G \subseteq kerf$. Hence there exists a homomorphism $\bar{f}\colon G/N_G \longrightarrow H$ satisfying $\bar{f} \circ r_{N_G} = f$, by Proposition 1.5.10 in [2]. Furthermore the homomorphism $\bar{f}\colon (G/N_G, \overline{\sigma}) \longrightarrow H$ is continuous since $\bar{f} \circ r_{N_G} = f$ and r_{N_G} is a quotient homomorphism. Let us show that $\bar{f}\colon (G/N_G, \overline{\sigma}_\tau) \longrightarrow H$ is continuous as well.

Take an open neighborhood O of e_H in H. By Lemma 2.1, we can assume that $O = g^{-1}(W)$, where $g\colon H \longrightarrow K$ is a continuous homomorphism from H to a Hausdorff group K with $w(K) < \tau$ and $W \in \mathcal{N}_K(e)$. We have $\bar{f}^{-1}(g^{-1}(W)) = (g \circ \bar{f})^{-1}(W) \in \overline{\sigma}_\tau$. This implies that $\bar{f}\colon (G/N_G, \overline{\sigma}_\tau) \longrightarrow H$ is a continuous homomorphism. Hence $(G/N_G, \overline{\sigma}_\tau)$ is the τ-precompact Hausdorff group reflection of G. ■

Remark 2.8. Since the $\mathcal{P}$-reflection is unique (see also Proposition 4.19 in [1]), the three different descriptions of the τ-precompact reflection of a topological group are equivalent.

3 Some properties of canonical homomorphisms and reflection functors

In this section, we mainly discuss the properties preserved by the τ-precompact Hausdorff reflection functor. In Proposition 3.1 we characterize the topological groups G such that the homomorphism $r_{N_G}\colon G \longrightarrow \varpi_\tau G$ is open. We also prove that the τ-precompact Hausdorff reflection functor preserves perfect surjective homomorphisms, quotient homomorphisms, and arbitrary products.

Let G be a topological group and U be an open set in G. We say that U is *τ-open* in G if there exists a continuous homomorphism f from G to a Hausdorff topological group H with $w(H) < \tau$ such that $U = f^{-1}(V)$, for some open set V in H.

Proposition 3.1. *Let $\varpi_\tau G$ be the τ-precompact Hausdorff reflection of a topological group G, and $r_{N_G}\colon G \longrightarrow \varpi_\tau G$ the canonical homomorphism. Then the homomorphism r_{N_G} is open if and only if for each $U \in \mathcal{N}(e_G)$, there exists a τ-open set V in G with $e_G \in V$ such that $VN_G \subset UN_G$.*

Proof. Assume that the homomorphism r_{N_G} is open. Let $U \in \mathcal{N}(e_G)$ be an open neighborhood of e_G. Since the τ-open sets constitute a local base at the identity of $\varpi_\tau G$, there exists a τ-open set W in $\varpi_\tau G$ such that $W \subseteq r_{N_G}(U)$. Then $V = r_{N_G}^{-1}(W)$ is a τ-open neighborhood of the identity in G satisfying $VN_G = r_{N_G}^{-1}(r_{N_G}(V)) \subset r_{N_G}^{-1}(r_{N_G}(U)) = UN_G$.

Conversely, assume that for each $U \in \mathcal{N}(e_G)$, there exists a τ-open set V in G such that $e_G \in V$ and $VN_G \subset UN_G$. Since $r_{N_G}(V)$ is a τ-open set in the quotient group $(G/N_G, \overline{\sigma})$, the set $r_{N_G}(V)$ is open in $\varpi_\tau G$. Clearly $r_{N_G}(V) \subset r_{N_G}(U)$, hence r_{N_G} is open by [2, Proposition 1.5.15]. ∎

Theorem 3.2. *The functor $\varpi_\tau\colon$ **TopGrp** $\longrightarrow \tau$**PHGrp** preserves perfect surjective homomorphisms and open surjective homomorphisms (i.e. quotient homomorphisms).*

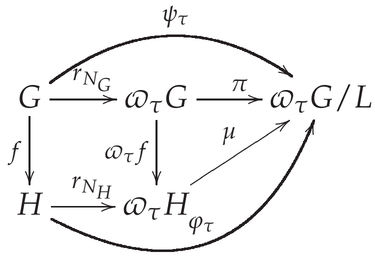
Proof. Assume that $f\colon G \longrightarrow H$ is a perfect surjective homomorphism of topological groups. By [1, Propositon 4.22], there exists a unique functor $\varpi_\tau\colon$ **TopGrp** $\longrightarrow \tau$**PHGrp** such that $\varpi_\tau(G) = (G/N_G, \overline{\sigma}_\tau)$ and $\varpi_\tau(f) \circ r_{N_G} = r_{N_H} \circ f$ for the homomorphism $f : G \longrightarrow H$ where $\varpi_\tau(f)(gN_G) = f(g)N_H$ for each $gN_G \in \varpi_\tau G$. We have to show that $\varpi_\tau f\colon \varpi_\tau G \longrightarrow \varpi_\tau H$ is also perfect and surjective. In the sequel we use notation introduced immediately before Theorem 2.7.

Let K be the kernel of f and $L = r_{N_G}(K)$. Then L is a compact invariant subgroup of the Hausdorff group $\varpi_\tau G$. Denote by π the canonical homomorphism of $\varpi_\tau G$ onto the quotient topological group $\varpi_\tau G/L$ and let $\psi_\tau = \pi \circ r_{N_G}$. The homomorphism π is perfect by Theorem 1.5.7 in [2]. Since $ker(\psi_\tau) = r_{N_G}^{-1}(L) = r_{N_G}^{-1}(r_{N_G}(K)) = K \cdot ker(r_{N_G})$, we see that $ker(f) \subseteq ker(\psi_\tau)$. This means, by [2, Proposition 1.5.10], that there exists a homomorphism $\varphi_\tau\colon H \longrightarrow \varpi_\tau(G)/L$ such that $\varphi_\tau \circ f = \psi_\tau$. The equality $f^{-1}(\varphi_\tau^{-1}(F)) = \psi_\tau^{-1}(F)$ holds for an arbitrary closed subset F of $\varpi_\tau(G)/L$, which implies that $\varphi_\tau^{-1}(F) = f(\psi_\tau^{-1}(F))$. We conclude that $\varphi_\tau^{-1}(F)$ is closed in H since f is closed and ψ_τ is continuous. Therefore the homomorphism φ_τ is continuous. The group $\varpi_\tau G$ and its quotient $\varpi_\tau G/L$ are τ-precompact Hausdorff groups. Since $\varpi_\tau H$ is the τ-precompact Hausdorff reflection of H, there exists a continuous homomorphism $\mu\colon \varpi_\tau H \longrightarrow \varpi_\tau G/L$ such that $\varphi_\tau = \mu \circ r_{N_H}$. We have

$$\mu \circ \varpi_\tau f \circ r_{N_G} = \mu \circ r_{N_H} \circ f = \varphi_\tau \circ f = \psi_\tau = \pi \circ r_{N_G},$$

which implies that $\mu \circ \varpi_\tau f = \pi$ because r_{N_G} is a surjective homomorphism. Since π is a perfect homomorphism and $\varpi_\tau f$ is a surjective homomorphism onto the Hausdorff group $\varpi_\tau H$, we conclude that both μ and $\varpi_\tau f$ are perfect by Proposition 3.7.5 in [8]. Since $\varpi_\tau f \circ r_{N_G} = r_{N_H} \circ f$ is surjective, so is $\varpi_\tau f$. Therefore $\varpi_\tau f$

is a perfect surjective homomorphism.



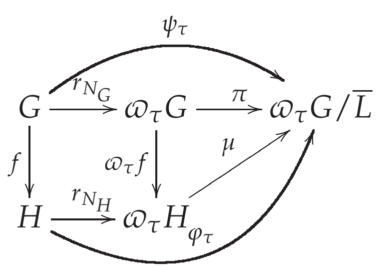
Assume that $f\colon G \longrightarrow H$ is an open surjective homomorphism. We will show that $\varpi_\tau f\colon \varpi_\tau G \longrightarrow \varpi_\tau H$ is an open surjective homomorphism as well.

Let K be the kernel of f and $L = r_{N_G}(K)$. Denote by $\overline{L}$ the closure of L in $\varpi_\tau G$. Then $\overline{L}$ is a closed invariant subgroup of $\varpi_\tau G$. Let $\pi\colon \varpi_\tau G \longrightarrow \varpi_\tau G/\overline{L}$ be the quotient homomorphism and $\psi_\tau = \pi \circ r_{N_G}$. It follows from

$$ker(\psi_\tau) = r_{N_G}^{-1}(\overline{L}) \supset \overline{r_G^{-1}(r_{N_G}(K))} \supset K \cdot \overline{ker(r_{N_G})}$$

that $ker(f) \subset ker(\psi_\tau)$. As above we have continuous homomorphisms $\varphi_\tau\colon H \longrightarrow \varpi_\tau(G)/\overline{L}$ and $\mu\colon \varpi_\tau H \longrightarrow \varpi_\tau G/\overline{L}$ such that $\varphi_\tau = \mu \circ r_{N_H}$. We claim that μ is a topological isomorphism between the groups $\varpi_\tau H$ and $\varpi_\tau G/\overline{L}$.

As in the above argument, we have $\mu \circ \varpi_\tau f = \pi$, where π is open and surjective, so μ is an open surjective homomorphism by Proposition 2.1.3 in [8]. Hence it suffices to verify that μ is injective, i.e. the kernel of μ is trivial. It follows from the equality $\mu \circ \varpi_\tau f = \pi$ that $ker(\varpi_\tau f) \subseteq ker(\pi) = \overline{L}$. On the other hand, $\varpi_\tau f(L) = \varpi_\tau f(r_{N_G}(K)) = r_{N_H}(f(K)) = e_{\varpi_\tau H}$. Hence $L \subseteq ker(\varpi_\tau f)$, which implies that $\overline{L} \subseteq ker(\varpi_\tau f)$ since $ker(\varpi_\tau f)$ is closed in $\varpi_\tau G$. Therefore $\overline{L} = ker(\pi) = ker(\varpi_\tau f)$. This implies that $ker(\mu)$ is trivial. Hence μ is a topological isomorphism. It follows that $\varpi_\tau f = \mu^{-1} \circ \pi$ is an open surjective homomorphism.



■

Remark 3.3. Let $\mathcal{P}$ be a property and assume that there exists the $\mathcal{P}$-reflection in the class of topological groups. Assume also that the reflection homomorphism $r\colon G \longrightarrow rG$ is surjective, for each topological group G, and that quotient homomorphisms preserve property $\mathcal{P}$. Then the $\mathcal{P}$-reflection functor preserves perfect surjective homomorphisms and quotient homomorphisms.

Now we show that the τ-precompact reflection functor $\omega_\tau\colon$ **TopGrp** $\longrightarrow \tau$**PGrp** preserves finite products, i.e. $\omega_\tau(G \times H)$ is topologically isomorphic to $\omega_\tau G \times \omega_\tau H$ for any topological groups G and H. Afterwards, in Theorem 3.7, we extend this result to arbitrary products.

Lemma 3.4. *Let G and H be topological groups and $\omega_\tau G$, $\omega_\tau H$ the τ-precompact group reflections of G and H, respectively. Then the τ-precompact group $\omega_\tau G \times \omega_\tau H$ is the τ-precompact reflection of $G \times H$, so the τ-precompact reflection functor commutes with finite products.*

Proof. Let $i_G \colon G \longrightarrow \omega_\tau G$, $i_H \colon H \longrightarrow \omega_\tau H$ and $i_{G\times H} \colon G \times H \longrightarrow \omega_\tau(G \times H)$ be the τ-precompact reflections of the groups G, H, and $G \times H$, respectively. By the universality of $i_{G\times H}$, there exists a continuous homomorphism $j \colon \omega_\tau(G \times H) \longrightarrow \omega_\tau G \times \omega_\tau H$ such that $i_G \times i_H = j \circ i_{G\times H}$. It suffices to show that j is a topological isomorphism. It is clear that j is one-to-one and onto. Let us verify that j is open.

Take an arbitrary open neighborhood O of (e_G, e_H) in $\omega_\tau(G \times H)$. There exists a continuous homomorphism $f \colon G \times H \longrightarrow K$ onto a topological group K with $w(K) < \tau$ such that $f^{-1}(W) \subset O$, for some open neighborhood W of the identity e_K in the group K. Let V be an open neighborhood of e_K satisfying $V^2 \subset W$. We define homomorphisms $f_1 \colon G \longrightarrow K$ and $f_2 \colon H \longrightarrow K$ by letting $f_1(x) = f(x, e_H)$ and $f_2(y) = f(e_G, y)$, respectively. Then $f_1^{-1}(V) \times f_2^{-1}(V) \in \sigma_\tau^G \times \sigma_\tau^H$ and $f_1^{-1}(V) \times f_2^{-1}(V) \subset f^{-1}(V^2) \subset f^{-1}(W) \subset O$. Hence the homomorphism j is open since the τ-open sets form a local base at the identity of the group $\omega_\tau(G \times H)$. ■

The following result shows that the functor ω_τ preserves quotient groups.

Lemma 3.5. *Let N be a (not necessarily closed) normal subgroup of a topological group G and $j \colon G \longrightarrow \omega_\tau G$ the identity homomorphism. Then the identity mapping of $\omega_\tau G/j(N)$ onto $\omega_\tau(G/N)$ is a topological isomorphism.*

Proof. The groups $\omega_\tau G/j(N)$ and $\omega_\tau(G/N)$ are algebraically the same, hence it suffices to prove that the identity mapping $Id \colon \omega_\tau G/j(N) \longrightarrow \omega_\tau(G/N)$ is a homeomorphism.

Let π be the quotient homomorphism of G to G/N. It is clear that the homomorphism $\omega_\tau \pi \colon \omega_\tau G \longrightarrow \omega_\tau(G/N)$ is continuous. Let also $q \colon \omega_\tau G \longrightarrow \omega_\tau G/j(N)$ be the quotient homomorphism. Since $\omega_\tau \pi = Id \circ q$, where $\omega_\tau \pi$ is continuous and q is open, we see that Id is continuous. Finally, the group $\omega_\tau G/j(N)$ is evidently τ-precompact, so the continuity of the identity mapping $Id \colon \omega_\tau(G/N) \longrightarrow \omega_\tau G/j(N)$ is immediate. Therefore Id is a homeomorphism, as claimed. ■

Remark 3.6. In general, the τ-precompact reflection functor does not preserve subgroups, even for $\tau = \aleph_0$. Let $G = SL(2, \mathbb{C})$ be the special linear group of 2×2 matrices with complex entries and H be the diagonal subgroup of G. Then $H \cong \mathbb{T}$ is a compact abelian subgroup of G, hence $\omega_\tau H \cong \mathbb{T}$, for each $\tau \geq \aleph_0$. Further, every precompact group is a subgroup of a compact group and every continuous homomorphism of $SL(2, \mathbb{C})$ to a compact Hausdorff topological group is constant (i.e. the group $SL(2, \mathbb{C})$ is *minimally almost periodic*, see [16]), so for $\tau = \aleph_0$, $\omega_\tau G$ is the group G which carries the trivial antidiscrete topology. In particular, the subgroup H of $\omega_\tau G$ is antidiscrete as well. Therefore the $\aleph_0$-precompact reflection functor does not preserve subgroups of *locally compact* non-Abelian groups.

Theorem 3.7. *Let $\{G_\alpha : \alpha \in I\}$ be a family of topological groups and for each $\alpha \in I$, $i_\alpha \colon G_\alpha \longrightarrow \omega_\tau G_\alpha$ the τ-precompact reflection of G_α. Then the group $G^* = \prod_{\alpha \in I} \omega_\tau G_\alpha$ is the τ-precompact reflection of $G = \prod_{\alpha \in I} G_\alpha$, so the τ-precompact reflection functor commutes with arbitrary products.*

Proof. The identity isomorphism $\varphi \colon \omega_\tau G \to G^*$ is continuous since $\omega_\tau G$ is the τ-precompact reflection of the group G and the product group G^* is τ-precompact. Hence it suffices to prove that φ is an open mapping. Take an arbitrary element $f^{-1}(W)$ of a neighborhood base at the identity of $\omega_\tau G$, where $f \colon G \to H$ is a continuous homomorphism to a topological group H with $w(H) < \tau$ and W is a neighborhood of the identity in H. Let also V be a neighborhood of the identity in H such that $V^2 \subseteq W$. Then the set $f^{-1}(V)$ contains a canonical open neighborhood $O = \prod_{\alpha \in I_0} U_\alpha \times \prod_{\alpha \in I \setminus I_0} G_\alpha$ of the identity in G, where I_0 is a finite subset of I. Let $N = \prod_{\alpha \in I_0} \{e_\alpha\} \times \prod_{\alpha \in I \setminus I_0} G_\alpha$ and $M = \prod_{\alpha \in I_0} G_\alpha \times \prod_{\alpha \in I \setminus I_0} \{e_\alpha\}$. Then $G \cong M \times N$ and $\omega_\tau G \cong \omega_\tau M \times \omega_\tau N$, by Lemma 3.4. Notice that $N \subset O \subset f^{-1}(V)$.

Let $j \colon G \longrightarrow \omega_\tau G$ be the identity mapping. It is clear that $j(N)$ is a normal (not necessarily closed) subgroup of $\omega_\tau G$. By Lemma 3.5, $\omega_\tau G / j(N)$ is topologically isomorphic to $\omega_\tau(G/N)$. We denote by q the quotient homomorphism of $\omega_\tau G$ onto $\omega_\tau G / j(N)$ and by p the projection of $\prod_{\alpha \in I} \omega_\tau(G_\alpha)$ onto $\prod_{\alpha \in I_0} \omega_\tau(G_\alpha)$. Let also ψ be the natural topological isomorphism of $\omega_\tau\big(\prod_{\alpha \in I} G_\alpha\big)/N$ onto $\omega_\tau\big(\prod_{\alpha \in I_0} G_\alpha\big)$. With this notation we have the following commutative diagram.

$$\begin{array}{ccccc}
\prod_{\alpha \in I} G_\alpha \xrightarrow{\ j\ } & \omega_\tau\big(\prod_{\alpha \in I} G_\alpha\big) & \xrightarrow{\ q\ } & \omega_\tau\big(\prod_{\alpha \in I} G_\alpha\big)/j(N) \cong & \omega_\tau\big(\big(\prod_{\alpha \in I} G_\alpha\big)/N\big) \\
& \Big\downarrow \varphi & & & \Big\downarrow \psi \\
& \prod_{\alpha \in I} \omega_\tau(G_\alpha) & \xrightarrow{\ \ \ p\ \ \ } & \prod_{\alpha \in I_0} \omega_\tau(G_\alpha) \ \cong & \omega_\tau\big(\prod_{\alpha \in I_0} G_\alpha\big)
\end{array}$$

It follows from the definition of the homomorphism q that

$$q^{-1}q(f^{-1}(V)) = f^{-1}(V)N \subseteq f^{-1}(V)^2 \subseteq f^{-1}(V^2) \subseteq f^{-1}(W).$$

Since q is open, the set $U = q(f^{-1}(V))$ is a neighborhood of the identity in $\omega_\tau G / j(N) \cong \omega_\tau(G/N)$. Making use of the commutativity of the above diagram and applying the fact that the mappings in the diagram, except for p and q, are bijections, we see that

$$p^{-1}(\psi(U)) = \varphi(q^{-1}(U)) \subseteq \varphi(f^{-1}(W)).$$

Hence $\varphi(f^{-1}(W))$ is a neighborhood of the identity in G^*. This proves that φ is an open mapping. Thus the τ-precompact reflection functor commutes with arbitrary products. ∎

Let G be a topological group and $r_G \colon G \longrightarrow G/\overline{\{e\}}$ the T_1-reflection mapping of G. We denote the quotient group $G/\overline{\{e\}}$ by $T_1(G)$.

Lemma 3.8. *The T_1-reflection functor commutes with arbitrary products.*

Proof. Given an arbitrary family $\{G_\alpha : \alpha \in I\}$ of topological groups, we have to prove that $T_1\big(\prod_{\alpha\in I} G_\alpha\big) \cong \prod_{\alpha\in I} T_1(G_\alpha)$, i.e. $\big(\prod_{\alpha\in I} G_\alpha\big)/\overline{\{e\}} \cong \prod_{\alpha\in I}(G_\alpha/\overline{\{e_\alpha\}})$, where e is the identity element of $G = \prod_{\alpha\in I} G_\alpha$. This follows from the facts that $\overline{\{e\}} = \prod_{\alpha\in I} \overline{\{e_\alpha\}}$ (see Proposition 2.3.3 in [8]) and that the groups $\big(\prod_{\alpha\in I} G_\alpha\big)/\prod_{\alpha\in I} \overline{\{e_\alpha\}}$ and $\prod_{\alpha\in I}(G_\alpha/\overline{\{e_\alpha\}})$ are topologically isomorphic. Hence the T_1-reflection functor commutes with products. ■

Theorem 3.9. *The τ-precompact Hausdorff group reflection functor commutes with products.*

Proof. This follows directly from Theorem 3.7 and Lemma 3.8. ■

Corollary 3.10. *The compact Hausdorff reflection functor c:* **TopGrp** $\longrightarrow$ **CompHGrp** *commutes with topological products.*

Proof. The composition of the precompact Hausdorff reflection and Raĭkov completion is the compact Hausdorff reflection. Since the Raĭkov completion functor commutes with products by Corollary 3.6.23 in [2], the compact Hausdorff reflection functor also commutes with products by Theorem 3.9. ■

Remark 3.11. As a direct generalization of precompactness, I. Guran [9] introduced the concept of ω-narrowness. He also gave a characterization of ω-narrow groups. By a theorem in [9], a Hausdorff topological group G is ω-narrow if and only if G is topologically isomorphic to a subgroup of the product of some family of second-countable Hausdorff groups.

Let $Q = [0,1]^\omega$ be the countable power of the unit interval and $Aut(Q)$ be the group of homeomorphisms of Q onto itself endowed with compact-open topology. Then $Aut(Q)$ is a Hausdorff topological group with a countable base. In [25], V.V. Uspenskij proved that the group $Aut(Q)$ is universal for all second countable Hausdorff groups, i.e. every second countable topological group can be topologically embedded in $Aut(Q)$. Using the universality of $Aut(Q)$ we can give a simpler construction of the ω-narrow Hausdorff reflection of a topological group.

By Corollary 2.3, the ω-narrow Hausdorff group reflection of a topological group G will be $\varpi G = (G,\sigma)/\overline{e_{(G,\sigma)}}$, where σ is the initial topology on G induced by all continuous homomorphisms from G to $Aut(Q)$.

Question 3.12. *Does there exist a universal group (w.r.t. embeddings) for all groups of weight less than or equal to τ, where τ is an uncountable cardinal?*

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