Sharp height estimate in Lorentz-Minkowski space revisited

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Abstract

In this paper, we deal with compact (necessarily with nonempty boundary) generalized linear Weingarten spacelike hypersurfaces immersed into the Lorentz-Minkowski space $\mathbb{L}^{n+1}$, which means that there exists a linear relation involving some of the corresponding higher order mean curvatures. In this setting, we obtain a sharp height estimate concerning such a hypersurfaces whose boundary is contained in a spacelike hyperplane of $\mathbb{L}^{n+1}$. Furthermore, we apply our estimate to describe the nature of the end of a complete generalized linear Weingarten spacelike hypersurface in $\mathbb{L}^{n+1}$.

1 Introduction

The last few decades have seen a steadily growing interest in the study of the geometry of spacelike hypersurfaces in Lorentzian spacetimes from both physical and mathematical points of view. From a physical point of view, such interest is motivated by their role in different problems of general relativity. For instance, Lichnerowicz [12] showed that zero mean curvature spacelike hypersurfaces are convenient as initial data for solving the Cauchy problem of the Einstein equations. We also refer to [10, 15] and references therein for other reasons justifying that interest.

From the mathematical point of view, this is mostly due to the fact that such hypersurfaces exhibit nice Bernstein-type properties, and one can truly say that

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the first remarkable results in this branch were the rigidity theorems of Calabi [6] and Cheng and Yau [9], who showed (the former for $n \leq 4$, and the latter for general n) that the only maximal complete, noncompact, spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. On the other hand, Aiyama [1] and Xin [17] simultaneous and independently characterized the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces of the Lorentz-Minkowski space having the image of its Gauss map contained in a geodesic ball of the n-dimensional hyperbolic space (see also [16] for a weaker first version of this result given by Palmer).

More recently, the study of estimates for the height function of compact spacelike hypersurfaces (necessarily with nonempty boundary; see, for instance, Section 2 of [5]) having some constant higher order mean curvature in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ has become subject of increasing research. In this setting, López [14] studied compact spacelike surfaces with constant mean curvature in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ and, when the boundary of the surface is a planar curve, he obtained an estimate for the height of the surface measured from the plane that contains its boundary. Moreover, López showed that such estimate is reached only if the surface is a planar domain or a hyperbolic cap. In [13], the second author also studied height estimates and obtained a sharp estimate for compact spacelike hypersurface with some constant higher order mean curvature in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ and with boundary contained in a spacelike hyperplane. These estimates have the important feature that they only depend on the constant higher order mean curvature of the hypersurface and on the radius of a appropriate geodesic ball in the hyperbolic space. Due to this feature, he was able to apply them to the study of topological properties of complete spacelike hypersurfaces with some positive constant higher order mean curvature in the Lorentz-Minkowski space.

Proceeding with the picture described above, in this paper our purpose is to get a height estimate for a wider class of spacelike hypersurfaces in the Lorentz-Minkowski space, which extends that one having some constant higher order mean curvature. Precisely, we consider *generalized linear Weingarten* spacelike hypersurfaces immersed in $\mathbb{L}^{n+1}$, which means that there exists a linear relation involving some of the corresponding higher order mean curvatures (for more details, see Section 3). Our aim is just to extend the technique developed in [13] in order to obtain a sharp estimate for the height function of a compact generalized linear Weingarten spacelike hypersurfaces in $\mathbb{L}^{n+1}$ (see Theorem 1 and Corollary 1). We point out that, for spacelike hypersurfaces having some constant higher order mean curvature, our results improve the estimates obtained by the second author in [13] (see Remark 1). Finally, we prove a topological result concerning the nature of the end of a complete generalized linear Weingarten spacelike hypersurface immersed in $\mathbb{L}^{n+1}$ (see Theorem 2).

2 Preliminaries

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle\, ,\, \rangle = dx_1^2 + \ldots + dx_n^2 - dx_{n+1}^2,$$

where $(x_1, \ldots, x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. In this paper, for convenience, we will adopt as model for the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ the product manifold $\mathbb{R}^n \times \mathbb{R}_1$ endowed with the Lorentzian metric

$$\langle\, ,\, \rangle = \pi^*_{\mathbb{R}^n}(dx^2) - \pi^*_{\mathbb{R}}(dt^2),$$

where $\pi^*_{\mathbb{R}^n}$ and $\pi^*_{\mathbb{R}}$ denote the canonical projections from $\mathbb{R}^n \times \mathbb{R}$ on each factor, $dx^2 = dx_1^2 + \ldots + dx_n^2$ is the canonical Riemannian metric on the n-dimensional Euclidean space $\mathbb{R}^n$ and $\mathbb{R}_1$ stands for $\mathbb{R}$ furnished with the metric $-dt^2$. We observe that ∂_t is an unitary timelike vector field globally defined on $\mathbb{L}^{n+1}$, which determines a time-orientation on $\mathbb{L}^{n+1}$.

In this context, we consider a (connected) spacelike hypersurface $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ immersed in $\mathbb{L}^{n+1}$, which means that the metric induced on Σ^n via ψ is a Riemannian metric. As usual, we also denote for $\langle\, ,\, \rangle$ the metric of Σ^n induced via ψ. Since ∂_t is a globally defined timelike vector field on $\mathbb{L}^{n+1}$, there exists an unique unitary timelike normal vector field N globally defined on Σ^n which is either in the same time-orientation of ∂_t, that is, $\langle N, \partial_t\rangle \leq -1$ or in the opposite time-orientation of ∂_t, that is, $\langle N, \partial_t\rangle \geq 1$.

Let us denote by $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the shape operator of Σ^n in $\mathbb{L}^{n+1}$ with respect to a choice of orientation N of Σ^n, which is given by $AX = -\nabla^\circ_X N$, where ∇° stands for the Levi-Civita connection of $\mathbb{L}^{n+1}$. Associated to the shape operator A there are n algebraic invariants, which are the elementary symmetric functions S_r of its principal curvatures $\kappa_1, \ldots, \kappa_n$, given by

$$S_k = S_k(\kappa_1, \ldots, \kappa_n) = \sum_{i_1<\ldots<i_r} \kappa_{i_1} \cdots \kappa_{i_r}, \qquad 1 \leq k \leq n.$$

As is well known, the k-mean curvature H_k of the spacelike hypersurface Σ^n is defined by

$$\binom{n}{k} H_k = (-1)^k S_k(\kappa_1, \ldots, \kappa_n).$$

In particular, when $k = 1$,

$$H_1 = -\frac{1}{n}\sum_i \kappa_i = -\frac{1}{n}\mathrm{tr}(A) = H$$

is the mean curvature of Σ^n, which is the main extrinsic curvature of the hypersurface.

It is a classical fact that the higher order mean curvatures satisfy a very useful set of inequalities. For future reference, we collect them here. A proof can be found in [11] (see also [7], Proposition 2.3).

Lemma 1. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike hypersurface immersed into the Lorentz-Minkowski space $\mathbb{L}^{n+1}$. Suppose that there exists an elliptic point in Σ^n. If H_{k+1} is positive on Σ^n, we have that the same holds for H_j, $j = 1, \cdots, k$. Moreover,*

(a) $H_j H_{j+2} \leq H_{j+1}^2$ *for every* $j = 1, \cdots, k$;

(b) $H_1 \geq H_2^{1/2} \geq \ldots \geq H_k^{1/k}$,

and equality holds only at umbilical points.

Here, by an elliptic point in a spacelike hypersurface Σ^n we mean a point $p_0 \in \Sigma^n$ where all principal curvatures $\kappa_i(p_0)$ are negative with respect to an appropriate choice of the orientation N of Σ^n.

Now, let us consider $P_k : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the k-th *Newton transformation* of the spacelike hypersurface $\psi : \Sigma^n \to \mathbb{L}^{n+1}$, $0 \leq k \leq n$, which are given by

$$P_k = \binom{n}{k} H_k I + \binom{n}{k-1} H_{k-1} A + \ldots + \binom{n}{1} H_1 A^{k-1} + A^k,$$

where I denotes the identity in $\mathcal{X}(\Sigma)$, or inductively, by putting $P_0 = I$ and, for $1 \leq k \leq n$,

$$P_k = \binom{n}{k} H_k I + A P_{k-1}.$$

It is easy to see that each P_k is a self-adjoint operator which commutes with the shape operator A, in the sense that if a local orthonormal frame on Σ^n diagonalizes A, then it also diagonalizes each P_k.

Let ∇ be the Levi-Civita connection of the spacelike hypersurface Σ^n. Associated to each Newton transformation P_k, one has the second order linear differential operator $L_k : C^\infty(\Sigma) \to C^\infty(\Sigma)$, defined by

$$L_k u = \operatorname{tr}(P_k \nabla^2 u).$$

where $C^\infty(\Sigma)$ stands for the ring of the smooth real functions on Σ^n and $\nabla^2 u : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of u which is given by

$$\langle \nabla^2 u(X), Y \rangle = \langle \nabla_X \nabla u, Y \rangle,$$

for all $X, Y \in \mathfrak{X}(\Sigma)$.

We observe that $L_0 = \Delta$ is just the Laplacian of Σ^n, which is always an elliptic operator in divergence form. For a general index k, it follows from equation (3.4) of [4] jointly with Corollary 3.2 of [3] that

$$L_k u = \operatorname{div}(P_k \nabla u). \tag{2.1}$$

Consequently, we conclude that the operator L_k is elliptic if and only if P_k is positive definite. For our applications, it will be useful to have some geometric conditions which guarantee the ellipticity of L_k when $k \geq 1$. For $k = 1$, the next lemma assures the ellipticity of L_1 (see Lemma 3.2 of [4]).

Lemma 2. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike hypersurface immersed into the Lorentz-Minkowski space $\mathbb{L}^{n+1}$. If $H_2 > 0$ on Σ, then L_1 is elliptic or, equivalently, P_1 is positive definite (for a appropriate choice of the Gauss map N).*

When $k \geq 2$, the following lemma give us sufficient conditions to guarantee the ellipticity of L_k. The proof follows from Proposition 3.2 of [8] (see also Lemma 3.3 of [4]).

Lemma 3. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike hypersurface immersed into the Lorentz-Minkowski space $\mathbb{L}^{n+1}$. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of the Gauss map N, and $H_{k+1} > 0$ on Σ^n, for some $2 \leq k \leq n-1$, then for all $1 \leq j \leq k$ the operator L_j is elliptic or, equivalently, P_j is positive definite (for a appropriate choice of the Gauss map N, if j is odd).*

Now, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n, namely, the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$ and the angle function $\Theta = \langle N, \partial_t \rangle$. The following formulas are, in fact, particular cases of ones obtained by Alías and Colares [4] in the context of the so-called generalized Robertson-Walker spacetimes.

Proposition 1. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike hypersurface. With the previous notation and denoting $c_k = (k+1)\binom{n}{k+1}$ for every $k = 0, \dots, n-1$, we have*

(a) $L_k h = -c_k H_{k+1} \Theta$;

(b) $L_k \Theta = \Theta \mathrm{tr}(A^2 \circ P_k) + \dfrac{c_k}{k+1} \langle \nabla H_{k+1}, \partial_t \rangle$.

3 Height estimate of generalized linear Weingarten hypersurfaces

This section is devoted to establish our results concerning estimates of the height function h of a wide class of spacelike hypersurfaces in the Lorentz-Minkowski space, which extends that ones having some constant higher order mean curvature. Specifically, let us consider $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ a spacelike hypersurface immersed into the Lorentz-Minkowski space. We say that Σ^n is (r,s)*-linear Weingarten*, for some $0 \leq r \leq s$, if there exist nonnegative real numbers $b_r, \cdots, b_s$ (at least one of them nonzero) such that the following linear relation holds on Σ^n:

$$\sum_{k=r}^{s} b_k H_{k+1} = d \in \mathbb{R}.$$

Taking into account that $\overline{R} = -H_2$, where $\overline{R}$ stands for the normalized scalar curvature of Σ^n, we observe that $(0,1)$-linear Weingarten spacelike hypersurfaces are called simply linear Weingarten spacelike hypersurfaces. Moreover, (r,r)-linear Weingarten spacelike hypersurfaces are just the spacelike hypersurfaces having H_{r+1} constant.

We also note that the Gauss map $N \in \mathfrak{X}^{\perp}(\Sigma)$ of a spacelike hypersurface Σ^n immersed into the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ can be regarded as a map $N : \Sigma^n \to \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space, that is,

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1} \, ; \, \langle x, x \rangle = -1\}.$$

In this setting, the image $N(\Sigma)$ will be called the *hyperbolic image* of Σ^n.

Now, we are in the position to state and prove our main result. More precisely, we will establish an estimate for the height function concerning compact (r, s)-linear Weingarten spacelike hypersurfaces in the Lorentz-Minkowski space.

Theorem 1. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a compact (r, s)-linear Weingarten spacelike hypersurface immersed into the Lorentz-Minkowski space such that H_{s+1} has strict sign on it and whose boundary $\partial\Sigma$ is contained in the hyperplane $\mathbb{R}^n \times \{0\}$. If the hyperbolic image of Σ^n is contained in a geodesic ball of center $e_{n+1} \in \mathbb{H}^n$ and radius $\varrho > 0$, then the height function h of Σ^n satisfies the following estimate*

$$|h| \leq \frac{\cosh \varrho - 1}{\min_{\Sigma} |H|}. \tag{3.1}$$

Moreover, the estimate (3.1) *is sharp in the sense that it is reached by the hyperbolic cap*

$$\Sigma_\lambda = \left\{ x \in \mathbb{L}^{n+1} ; \langle x, x \rangle = -\lambda^2, \, \lambda \leq x_{n+1} \leq \sqrt{1 + \lambda^2} \right\}, \tag{3.2}$$

where λ is the positive constant given by $\lambda = (\cosh \varrho - 1)^{-1/2}$.

Proof. From Lemma 1 of [5], our assumption that the boundary of Σ^n is contained into the hyperplane $\mathbb{R}^n \times \{0\}$ implies that (after an appropriate choice of orientation on Σ^n) there exists an elliptic point in Σ^n. Thus, we can suppose that $H_{s+1} > 0$ on Σ^n and apply Lemma 3 (or Lemma 2 if $s = 1$) to guarantee the ellipticity of the operators L_k for every $k = r, \ldots, s$.

Thus, we introduce the following second order linear differential operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$Lu = \sum_{k=r}^{s} (k+1) c_k^{-1} b_k L_k u.$$

It follows from (2.1) that we can write the operator L as

$$Lu = \sum_{k=r}^{s} \operatorname{div} \left((k+1) c_k^{-1} b_k P_k \nabla u \right).$$

Then, since $(k+1)c_k^{-1} b_k > 0$ for every $k = r, \ldots, s$ and each operator L_k is elliptic (equivalently, each P_k is positive definite), we have that the operator $P = \sum_{k=r}^{s} (k+1) c_k^{-1} b_k P_k$ is positive definite and, consequently, L is also elliptic.

Let us suppose that the Gauss map N is in the same time-orientation as ∂_t, that is, $\Theta \leq -1$. In this case, from Proposition 1 and from the definition of the operator L we see that

$$Lh = \sum_{k=r}^{s} (k+1) c_k^{-1} b_k L_k h = -\sum_{k=r}^{s} (k+1) b_k H_{k+1} \Theta.$$

Thus, since H_{k+1} is positive for every $k = r, \dots, s$, we get $Lh \geq 0$ on Σ^n. Then, by the weak maximum principle, we must to have $h \leq 0$ on Σ^n. Now, we consider on Σ^n the smooth function $\varphi = ch - \Theta$, where $c \in \mathbb{R}$ is a negative constant to be chosen in an appropriated way. Observe that our hypothesis on the hyperbolic image of Σ^n implies

$$1 \leq -\Theta \leq \cosh \varrho,$$

which gives $\varphi \leq \cosh \varrho$ on $\partial\Sigma^n$.

On the other hand, using once more Proposition 1 and the fact that Σ^n is (r, s)-linear Weingarten, we obtain

$$\begin{aligned} L\varphi &= \sum_{k=r}^{s} (k+1) c_k^{-1} b_k L_k \varphi \\ &= -\sum_{k=r}^{s} (k+1) b_k \Theta \left(c H_{k+1} + c_k^{-1} \mathrm{tr}(A^2 \circ P_k) \right). \end{aligned}$$

With a straightforward computation, we can to show that

$$\mathrm{tr}(A^2 \circ P_k) = \frac{c_k}{k+1} (nHH_{k+1} - (n-k-1)H_{k+2}),$$

which implies

$$\begin{aligned} L\varphi &= -\sum_{k=r}^{s} b_k \Theta \left((k+1) c H_{k+1} + nHH_{k+1} - (n-k-1)H_{k+2} \right) \\ &= -\sum_{k=r}^{s} b_k \Theta \left((k+1) H_{k+1}(H+c) + (n-k-1)(HH_{k+1} - H_{k+2}) \right). \end{aligned}$$

But, from Lemma 1 we have that

$$HH_{k+1} - H_{k+2} \geq HH_{k+1} - H_{k+1}^2 H_k^{-1} = \frac{H_{k+1}}{H_k} \left(HH_k - H_{k+1} \right).$$

Thus, using once more Lemma 1, we get

$$HH_{k+1} - H_{k+2} \geq \frac{H_{k+1}}{H_k} \left(HH_k - H_k^{(k+1)/k} \right) = H_{k+1}(H - H_k^{1/k}) \geq 0.$$

Consequently, we obtain that

$$L\varphi \geq -\sum_{k=r}^{s} (k+1) b_k H_{k+1} \Theta \left(H + c \right).$$

So, choosing $c = -\min_\Sigma H$ in the definition of the function φ, it follows from the last inequality that $L\varphi \geq 0$ on Σ^n and, using once more the weak maximum principle, we conclude that $\varphi \leq \cosh \varrho$ on Σ^n. Therefore,

$$h \geq \frac{\cosh \varrho - 1}{c}.$$

Now, suppose that the Gauss map N is in the opposite time-orientation of ∂_t, that is, $\Theta \geq 1$. Then, from Proposition 1, $Lh \leq 0$ on Σ^n and, in this case, $h \geq 0$ on Σ^n. Hence, at this point we can reason in a similar way as before using the smooth function $\varphi = ch + \Theta$, with $c = \min_\Sigma H$, and conclude that

$$h \leq \frac{\cosh \varrho - 1}{c}.$$

Finally, it is not difficult to verify that the hyperbolic cap Σ_λ defined in (3.2) is a spacelike hypersurface of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ which has constant $(r+1)$-mean curvature given by

$$H_{r+1} = \frac{1}{\lambda^{r+1}} > 0,$$

for every $0 \leq r \leq n-1$ (if we choose the Gauss map N in the same time-orientation of e_{n+1}, for the case r even). Moreover, the hyperbolic image of Σ_λ is contained in the geodesic ball of center $e_{n+1} \in \mathbb{H}^{n+1}$ and radius

$$\varrho = \cosh^{-1} \sqrt{1 + \frac{1}{\lambda^2}}.$$

Thus, the height function of Σ_λ is given by

$$h = \frac{\cosh \varrho - 1}{\min_{\Sigma_\lambda} H},$$

showing that the estimate (3.1) is sharp. ■

Remark 1. We point out that, for a spacelike hypersurface with constant $(r+1)$-mean curvature H_{r+1}, Theorem 1 improves the estimate obtained by the second author in Theorem 4.2 of [13]. Indeed, it follows from item (b) of Lemma 1 that

$$\frac{\cosh \varrho - 1}{\min_\Sigma H} \leq \frac{\cosh \varrho - 1}{H_{r+1}^{1/(r+1)}}$$

for every $r = 0, \ldots, n-1$.

For a fixed real number $t_0 \in \mathbb{R}$, we recall that the translation by t_0, $\Phi_{t_0} : \mathbb{L}^{n+1} \to \mathbb{L}^{n+1}$, defined by

$$\Phi_{t_0}(p_1, \ldots, p_{n+1}) = (p_1, \ldots, p_{n+1} - t_0),$$

is an isometry of $\mathbb{L}^{n+1}$. In particular, for any spacelike hyperplane $\mathbb{R}^n \times \{t\}$ of $\mathbb{L}^{n+1}$ we have that $\Phi_{t_0}(\mathbb{R}^n \times \{t\}) = \mathbb{R}^n \times \{t - t_0\}$. So, from Theorem 1 we obtain the following result.

Corollary 1. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a compact (r,s)-linear Weingarten spacelike hypersurface immersed into the Lorentz-Minkowski space such that $(s+1)$-mean curvature H_{s+1} has strict sign and boundary contained into the spacelike hyperplane $\mathbb{R}^n \times \{t\}$. If the hyperbolic image of Σ^n is contained in a geodesic ball of center $e_{n+1} \in \mathbb{H}^n$ and radius $\varrho > 0$, then*

$$\Sigma \subset \mathbb{R}^n \times [t, t + C], \quad \text{or} \quad \Sigma \subset \mathbb{R}^n \times [t - C, t],$$

where $C = \dfrac{\cosh \varrho - 1}{\min_\Sigma |H|}$.

According to Section 5 of [13], a complete spacelike hypersurface $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ immersed into the Lorentz-Minkowski space with one end can be regarded as

$$\Sigma^n = \Sigma^n_t \cup \mathcal{C}^n,$$

where Σ^n_t is a compact hypersurface in $\mathbb{L}^{n+1}$ with boundary contained into a spacelike slice $\mathbb{R}^n \times \{t\}$ and $\mathcal{C}^n$ is a hypersurface diffeomorphic to the cylinder $\mathbb{S}^{n-1} \times [t, \infty)$. In this setting, we say that the end of Σ^n is *divergent* when, considering $\mathcal{C}^n$ with coordinates $p = (q, s) \in \mathbb{S}^{n-1} \times [t, \infty)$, we have that

$$\lim_{s \to \infty} h(p) = \infty,$$

where h denotes the height function of Σ^n.

Taking into account Theorem 1, we can reason as in the proof of Theorem 5.1 of [13] to get the following

Theorem 2. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a complete (r, s)-linear Weingarten spacelike hypersurface immersed into the Lorentz-Minkowski space with one end. Suppose that $(s+1)$-mean curvature H_{s+1} satisfies $\inf_\Sigma |H_{s+1}| > 0$. If the hyperbolic image of Σ^n is contained in a geodesic ball of $\mathbb{H}^n$, then its end cannot be divergent.*

Remark 2. Finally, we also note that our assumption on the hyperbolic image of the spacelike hypersurface in Theorem 2 is necessary. Indeed, given a positive constant λ,

$$\Sigma^n = \left\{ x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -\lambda^2,\ x_{n+1} \geq \lambda \right\}$$

is a complete spacelike hypersurface with positive constant higher order mean curvatures and, consequently, it is a (r, s)-linear Weingarten for any $0 \leq r \leq s$. Moreover, Σ^n has one end which is divergent, but its hyperbolic image is just the hyperbolic space $\mathbb{H}^n$.

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