Common value pairs and their estimations

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Abstract

We shall give a new treatment to intersection points of two maps, named common value pairs. Given two maps $f, g\colon X \to Y$. Instead of considering intersection points on target space Y, we focus on the pairs in the domains X, the pair (u, v) with $f(u) = g(v)$. The set of all these pairs is exactly the preimage of product $f \times g$ at the diagonal in Y^2. We shall apply the idea of Nielsen root theory into such a general case: preimage of a set. Hence, some estimation for common value pairs and therefore for intersection points are obtained.

1 Introduction

The Nielsen fixed point theory (see [3] and [7]) deals with the estimation of the numbers of fixed points of self-maps. The Lefschetz number contains the existence information of fixed points, while the Nielsen number serves as a lower bound for the number of fixed points. Many generalization have been done, such as classical Nielsen coincidence theory ([6]) and root theory ([1], [8]). Recently, a type of Nielsen fixed point theoretical version of Borsuk-Ulam theorem is given [4].

The origin of intersection theory can be traced back to early stage of topology, and was formalized as cup products in cohomology. One of famous works in this direction due to H. Whitney [11], showing that any smooth n-manifold may be imbedded into $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$, where self intersection points and their indices were given. Comparing with fixed point theory, his

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treatment is an analogy of Lefschetz theory. It is a natural idea to ask for a Nielsen theory version for intersection theory. C. McCord [9, 10] initiated this study, and considered the relation between intersection and preimage, i.e. a natural generalization of root theory. Similar work can be found in [5]. Along this way, we shall give a kind of estimation of the number of intersection points by using the method from Nielsen theory, where the fundamental groups play essential roles.

Our consideration begins with a very general context: the preimage $f^{-1}(B)$ of a map $f\colon X \to Y$ for some given subset B of Y. It is obvious that the preimage is a direct generalization of roots. By a root of f at y_0, we mean a point $x \in X$ with $f(x) = y_0$. Actually, coincidence point of $f_1, f_2\colon X \to Y$ is exactly the preimage $((f_1 \times f_2)\Delta_{X^2})^{-1}(\Delta_{Y^2}(Y))$ of the composition $X \overset{\Delta_{X^2}}{\to} X^2 \overset{f_1\times f_2}{\to} Y^2$. As in classical Nielsen fixed point theory, especially root theory, we divide preimage into preimage classes, and then define kinds of "index" for each preimage classes based on homomorphism indices for roots. Such a definition is one of our main improvement on the essentiality of a class. Usually, one can say that a class is essential if it will not disappear under any homotopy. Therefore, we obtain some homotopy invariants which serve as lower bounds for the number of preimage or its components.

Our main application is the consideration of intersection points of two maps $f, g\colon X \to Y$. But, we focus on the common value pairs, i.e. the pairs (u, v) with $f(u) = g(v)$. Such a consideration has its advantage: we need not make extra assumptions, like transversality, on maps f and g. For example, a triple point of a map will be understood as three common value pairs of f and itself, and no regular requirement on f as in [11] is needed.

This paper is arranged as follow. We shall give the definition of preimage classes in Section 2. The homomorphism indices of preimage classes will introduce in Section 3. Some properties of such indices are proved there, especially lower bound property. In Section 4, we shall apply our preimage treatments into common value pairs, and explain some relations with intersections. Self common value pairs are discussed in Section 5.

Throughout this paper, we make the following convention in notions.

- X and Y: both are connected spaces with universal coverings $p_X\colon \tilde{X} \to X$ and $p_Y\colon \tilde{Y} \to Y$, respectively.
- $\mathrm{D}(\tilde{X})$ and $\mathrm{D}(\tilde{Y})$: deck transformation groups of $\tilde{X}$ and $\tilde{Y}$, respectively.
- Δ_{W^2}: the main diagonal map from W to W^2 defined by $w \mapsto (w, w)$ for any $w \in W$.

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2 Preimage classes

In this section, we consider a map $f\colon X \to Y$ and its preimage $f^{-1}(B)$ at a connected subset B of Y. In general, one can consider the preimages of components of B whenever B is not connected.

For a map $f\colon X \to Y$, a lifting $\tilde{f}$ of f (with respect to universal covering p_X and p_Y) is understood as a map $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ satisfying the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

We shall make use of liftings to study the preimage set $f^{-1}(B)$.

Proposition 2.1. *Let $\tilde{B}$ be a component of $p_Y^{-1}(B)$. Then $f^{-1}(B) = \cup_{\tilde{f}}\, p_X(\tilde{f}^{-1}(\tilde{B}))$, where $\tilde{f}$ ranges over all liftings of f.*

Proof. Let $x \in f^{-1}(B)$, i.e. $f(x) \in B$. Pick a point $\tilde{x} \in p_X^{-1}(x)$ and a lifting $\tilde{f}$ of f. Then $\tilde{f}(\tilde{x}) \in p_Y^{-1}(B)$ by definition of lifting. Since $\tilde{B}$ is a component of $p_Y^{-1}(B)$, there is an element β in deck transformation group $\mathrm{D}(\tilde{Y})$ of $\tilde{Y}$ such that $\beta\tilde{f}(\tilde{x}) \in \tilde{B}$. This implies that $\tilde{x} \in (\beta\tilde{f})^{-1}(\tilde{B})$, and hence $x \in p_X((\beta\tilde{f})^{-1}(\tilde{B}))$. It follows that $f^{-1}(B) \subset \cup_{\tilde{f}}\, p_X(\tilde{f}^{-1}(\tilde{B}))$.

Suppose that $x \in p_X(\tilde{f}^{-1}(\tilde{B}))$ for some lifting $\tilde{f}$ of f. Then there is a point $\tilde{x} \in p_X^{-1}(x)$ such that $\tilde{f}(\tilde{x}) \in \tilde{B}$. By definition of lifting, $f(x) = f(p_X(\tilde{x})) = p_Y\tilde{f}(\tilde{x}) \in p_Y(\tilde{B}) = B$, i.e. $x \in f^{-1}(B)$. We obtain that $f^{-1}(B) \supset \cup_{\tilde{f}}\, p_X(\tilde{f}^{-1}(\tilde{B}))$. ∎

Definition 2.2. *Let $\tilde{B}$ a component of $p_Y^{-1}(B)$. Two liftings $\tilde{f}'$ and $\tilde{f}''$ of f are said to equivalent with respect to $\tilde{B}$, written as $\tilde{f}'' \sim_{\tilde{B}} \tilde{f}'$, if $\tilde{f}'' = \beta\tilde{f}'\alpha$ for some element $\alpha \in \mathrm{D}(\tilde{X})$ and some element $\beta \in \mathrm{D}(\tilde{Y})$ with $\beta(\tilde{B}) = \tilde{B}$.*

The equivalent relation above leads to the following:

Proposition 2.3. *For any two liftings $\tilde{f}'$ and $\tilde{f}''$ of f, we have*
(1) If $\tilde{f}'' \sim_{\tilde{B}} \tilde{f}'$, then $p_X(\tilde{f}''^{-1}(\tilde{B})) = p_X(\tilde{f}'^{-1}(\tilde{B}))$.
(2) If $\tilde{f}'' \not\sim_{\tilde{B}} \tilde{f}'$, then $p_X(\tilde{f}''^{-1}(\tilde{B})) \cap p_X(\tilde{f}'^{-1}(\tilde{B})) = \emptyset$.

Proof. (1) Since $\tilde{f}'' \sim_{\tilde{B}} \tilde{f}'$, we have that $\tilde{f}'' = \beta\tilde{f}'\alpha$. Let $\tilde{x} \in \tilde{f}'^{-1}(\tilde{B})$, i.e. $\tilde{f}'(\tilde{x}) \in \tilde{B}$. Note that $\beta(\tilde{B}) = \tilde{B}$. We have that $\beta\tilde{f}'\alpha\alpha^{-1}(\tilde{x}) \in \tilde{B}$. This means that $\alpha^{-1}(\tilde{x}) \in \tilde{f}''^{-1}(\tilde{B})$. Thus, $\alpha^{-1}(\tilde{f}'^{-1}(\tilde{B})) \subset \tilde{f}''^{-1}(\tilde{B})$. It follows that $p_X(\tilde{f}'^{-1}(\tilde{B})) \subset p_X(\tilde{f}''^{-1}(\tilde{B}))$. In the same way, we can prove that $p_X(\tilde{f}''^{-1}(\tilde{B})) \subset p_X(\tilde{f}'^{-1}(\tilde{B}))$ because $\tilde{f}' = \beta^{-1}\tilde{f}''\alpha^{-1}$.

(2) Suppose on the contrary that $p_X(\tilde{f}''^{-1}(\tilde{B})) \cap p_X(\tilde{f}'^{-1}(\tilde{B}))$ is not empty, and therefore contains a point x. Thus, there are two points $\tilde{x}'$ and $\tilde{x}''$ in $p_X^{-1}(x)$ such that $\tilde{f}'(\tilde{x}')$ and $\tilde{f}''(\tilde{x}'')$ are both in $\tilde{B}$. It follows that $\tilde{f}''(\tilde{x}'') = \beta\tilde{f}'(\tilde{x}')$ for some $\beta \in \mathrm{D}(\tilde{Y})$. Note that $\tilde{B}$ is connected. We have that $\beta(\tilde{B}) = \tilde{B}$. Since both of $\tilde{x}'$ and $\tilde{x}''$ lie in $p_X^{-1}(x)$, we have that $\tilde{x}' = \alpha(\tilde{x}'')$ for some $\alpha \in \mathrm{D}(\tilde{X})$. Hence, $\tilde{f}''(\tilde{x}'') = \beta\tilde{f}'(\tilde{x}') = \beta\tilde{f}'(\alpha\tilde{x}'')$. We obtain that $\tilde{f}'' = \beta\tilde{f}'\alpha$, i.e. $\tilde{f}'' \sim_{\tilde{B}} \tilde{f}'$. This is a contradiction. ∎

From this Proposition, we know that the set of preimage $f^{-1}(B)$ is divided into a disjoint union of subsets $p_X(\tilde{f}^{-1}(\tilde{B}))$'s. As in fixed point theory or root theory, we have

Definition 2.4. *The subset $p_X(\tilde{f}^{-1}(\tilde{B}))$ of $f^{-1}(B)$ is said to be the preimage class of the map f at B which is determined by $\tilde{f}$ and $\tilde{B}$.*

Note that $\tilde{f}^{-1}(\tilde{B}) = \{\tilde{x} \mid \tilde{f}(\tilde{x}) \in \tilde{B}\} = \{\tilde{x} \mid \alpha\tilde{f}(\tilde{x}) \in \alpha\tilde{B}\} = (\alpha\tilde{f})^{-1}(\alpha(\tilde{B}))$ for any $\alpha \in \mathrm{D}(\tilde{X})$. Thus, the classification of preimage set is independent of the choice of $\tilde{B}$.

Similar to a fixed point class, we have

Proposition 2.5. *Each preimage class is a relatively open set of $f^{-1}(B)$.*

Proof. Let $x \in p_X(\tilde{f}^{-1}(\tilde{B}))$ for some lifting $\tilde{f}$ of f. Then $\tilde{f}(\tilde{x}) \in \tilde{B}$ for some $\tilde{x} \in p_X^{-1}(x)$. By definition of covering, $f(x)$ has a neighborhood V such that each component of $p_Y^{-1}(V)$ is homeomorphic to V by $p_Y|$. Let $\tilde{V}$ be the component of $p_Y^{-1}(V)$ containing $\tilde{f}(\tilde{x})$. Since $\tilde{B}$ is a component of $p_Y^{-1}(B)$, we have that $\tilde{V} \cap p_Y^{-1}(B) \subset \tilde{B}$. Pick a neighborhood W of x such that $f(W) \subset V$. Moreover, we can make W so that the restriction of p_X on each component of $p_X^{-1}(W)$ is a homeomorphism. If $f(x') \in B$ for some $x' \in W$, then there is a unique point $\tilde{x}' \in p_X^{-1}(x')$ such that $\tilde{x}$ and $\tilde{x}'$ lie in the same component $\tilde{W}$ of $p_X^{-1}(W)$. Note that $\tilde{f} = (p_Y|_{\tilde{V}})^{-1} f p_X$ on $\tilde{W}$. We have $\tilde{f}(\tilde{x}') \in \tilde{B}$. Thus, $x' \in p_Y(\tilde{f}^{-1}(\tilde{B}))$. It follows that $W \cap f^{-1}(B)$ is contained in the class $p_Y(\tilde{f}^{-1}(\tilde{B}))$. ∎

Corollary 2.6. *If X is compact and B is closed, then the number of preimage classes is finite for any map.*

The next two Propositions describe that the preimage classes are kinds of generalizations of the classical coincidence classes, especially root classes. The proofs are straightforward, we omit the details here.

Proposition 2.7. *Let $f, g\colon X \to Y$ be two maps. The set of preimage classes of $(f \times g)\Delta_{X^2}\colon X \to Y^2$ at the diagonal $\Delta_{Y^2}(Y)$ is the same as the set of coincidence classes of f and g.*

Proposition 2.8. *For any point $y_0 \in Y$, the set of preimage classes of a map $f\colon X \to Y$ at $\{y_0\}$ is the same as the set of root classes of the map f at y_0.*

Moreover, we can consider the influence of a homotopy on preimage classes. Let $F\colon X \times I \to Y$ be a homotopy between f_0 and f_1. A preimage class $p_X(\tilde{f}_0^{-1}(\tilde{B}))$ of f_0 and a preimage class $p_X(\tilde{f}_1^{-1}(\tilde{B}))$ of f_1 are said to *be related by the homotopy F* if $\tilde{f}_0$ and $\tilde{f}_1$ are respectively 0- and 1-slice of a lifting $\tilde{F}$ of F, i.e. $\tilde{F}(\tilde{x}, 0) = \tilde{f}_0(\tilde{x})$ and $\tilde{F}(\tilde{x}, 1) = \tilde{f}_1(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$.

It should be mentioned that "homotopy related" is not a one-to-one correspondence here, because a preimage class may be related to an empty set.

3 Indices for subsets of preimages

Let $f\colon X \to Y$ be a map, and B a subset of Y. In this section, we shall consider the indices of an isolated subset A of preimage $f^{-1}(B)$. By an isolated subset A of $f^{-1}(B)$ we mean that there is a closed neighborhood N of A such that $N \cap f^{-1}(B) = A$.

Definition 3.1. *The (homology) homomorphism index $\mathcal{L}_*(f, A, B)$ of A is defined to be the composition of following:*

$$H_*(X) \xrightarrow{j_*} H_*(X, X - A) \xrightarrow{e_*^{-1}} H_*(N, N - A) \xrightarrow{f_*} H_*(Y, Y - B),$$

where j_* is the natural homomorphism in the homology exact sequence of pair $(X, X - A)$, e_* is the exclusion, and N is a closed neighborhood of A such that $N \cap f^{-1}(B) = A$.

The definition above is obviously a generalization of homomorphism index for a root set, see [2] and [8].

Lemma 3.2. *(Independency) The homomorphism index $\mathcal{L}_*(f, A, B)$ is independent of the choice of closed neighborhoods of A.*

Proof. Let N_1 and N_2 be two closed neighborhoods of A such that $N_1 \cap f^{-1}(B) = N_2 \cap f^{-1}(B) = A$. Then $N = N_1 \cup N_2$ is also a closed neighborhood of A with $N \cap f^{-1}(B) = A$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & H_*(N_1, N_1 - A) & & \\
 & & & \nearrow^{e_{1*}^{-1}} & \downarrow e'_* & \searrow^{f|_{N_1*}} & \\
H_*(X) & \xrightarrow{j_*} & H_*(X, X - A) & \xrightarrow{e_*^{-1}} & H_*(N, N - A) & \xrightarrow{f|_{N*}} & H_*(Y, Y - B), \\
 & & & \searrow_{e_{2*}^{-1}} & \uparrow e''_* & \nearrow_{f|_{N_2*}} & \\
 & & & & H_*(N_2, N_2 - A) & &
\end{array}$$

where the e's indicate varies of exclusions. We have

$$f|_{N_1*}e_{1*}^{-1}j_* = (f|_{N*}e'_*)(e'^{-1}_* e_*^{-1})j_* = f|_{N*}e_*^{-1}j_*,$$

$$f|_{N_2*}e_{2*}^{-1}j_* = (f|_{N*}e''_*)(e''^{-1}_* e_*^{-1})j_* = f|_{N*}e_*^{-1}j_*.$$

We then obtain the conclusion. ∎

Lemma 3.3. *The homomorphism index $\mathcal{L}_*(f, A, B)$ is equal to the following composition:*

$$H_*(X) \xrightarrow{j_{Q*}} H_*(X, X - Q) \xrightarrow{e_*^{-1}} H_*(N_Q, N_Q - Q) \xrightarrow{f_*} H_*(Y, Y - B),$$

as long as Q contains A and N_Q is a closed neighborhood with $N_Q \cap f^{-1}(B) = A$.

Proof. Consider following natural diagram

$$\begin{array}{ccccccc}
H_*(X) & \xrightarrow{j_*} & H_*(X, X - A) & \xrightarrow{e_*^{-1}} & H_*(N_Q, N_Q - A) & \xrightarrow{f|_{N_Q*}} & H_*(Y, Y - B). \\
 & \searrow^{j_{Q*}} & \uparrow j'_* & & \uparrow j''_* & \nearrow_{f|_{N_Q*}} & \\
 & & H_*(X, X - Q) & \xrightarrow{e_*^{-1}} & H_*(N_Q, N_Q - Q) & &
\end{array}$$

Since $A \subset Q$, N_Q is also a closed neighborhood of A. Thus, the homomorphism index of A is the composition of homomorphisms in first row. We obtain our conclusion by the commutativity of above diagram. ∎

Lemma 3.4. *(Additivity) Let A_s be a subset of $f^{-1}(B)$ for $s = 1, \dots, m$. Suppose that there is a closed neighborhood N_s of A_s for each s with $N_s \cap f^{-1}(B) = A_s$ such that all N_s's are pairwisely disjoint. Then $\mathcal{L}_*(f, \cup_{s=1}^m A_s, B) = \sum_{s=1}^m \mathcal{L}_*(f, A_s, B)$.*

Proof. Clearly, the disjoint union $N = \cup_{s=1}^m N_s$ is a closed neighborhood of $A = \cup_{s=1}^m A_s$ with $N \cap f^{-1}(B) = A$. We then have the following commutative diagram

$$\begin{array}{ccccccc}
H_*(X) & \xrightarrow{j_*} & H_*(X, X-A) & \xrightarrow{e_*^{-1}} & H_*(N, N-A) & \xrightarrow{f|_{N*}} & H_*(Y, Y-B). \\
 & \searrow^{\sum_{s=1}^m j_{s*}} & \cong \big\downarrow \sum_{s=1}^m j'_{s*} & & \sum_{s=1}^m i_{s*} \big\uparrow \cong & \nearrow_{\sum_{s=1}^m f|_{N_s *}} & \\
 & & \oplus_{s=1}^m H_*(X, X-A_s) & \xrightarrow{e_{s*}^{-1}} & \oplus_{s=1}^m H_*(N_s, N_s - A_s) & &
\end{array}$$

where $j'_{s*}\colon H_*(X, X-A_s) \to H_*(X, X-A)$ is the natural homomorphism induced by the projection from $C_*(X, X-A)$ to $C_*(X, X-A_s)$. With the same argument as in [1, 4.10], we obtain the conclusion. ■

Lemma 3.5. *(Homotopy invariance) Let $F = \{f_t\}\colon X \times I \to Y$ be a homotopy, and C be a compact isolated subset of $F^{-1}(B)$. Then $\mathcal{L}_*(f_0, C_0, B) = \mathcal{L}_*(f_1, C_1, B)$, where $C_t = \{x \in X \mid (x,t) \in C\}$ denotes the t-slice of C for any $t \in I$.*

Proof. It is sufficient to show that the correspondence from I to $Hom(H_*(X), H_*(Y, Y-B))$, which is given by $t \mapsto \mathcal{L}_*(f_t, C_t, B)$, is locally constant for any $t \in I$.

With loss of generality, we shall prove that this correspondence is locally constant at 0. If $C_0 = \emptyset$, there is a positive number ε such that $C_t = \emptyset$ for $t \in [0, \varepsilon]$ because of the compactness of C. Thus, $H_*(X, X - C_t) = H_*(X, X) = 0$ for $t \in [0, \varepsilon]$. It follows that $\mathcal{L}_*(f_t, C_t, B)$ is a zero homomorphism for all t with $t \in [0, \varepsilon]$, and therefore locally constant at 0.

Now we consider the case $C_0 \neq \emptyset$. Since C is isolated, there is a closed neighborhood N of C in $X \times I$ such that $N \cap F^{-1}(B) = C$. For each $(x, t) \in C$, there is an open product neighborhood $V_{x,t} \times J_{x,t}$ such that its closure $\bar{V}_{x,t} \times \bar{J}_{x,t}$ is contained in the interior $Int(N)$ of N. Here, we can make a choice such that $J_{x,t}$ is in the form of either $[0, s_1)$, (s_2, s_3) or $(s_4, 1]$, where $s_1, s_2, s_3, s_4 \in (0, 1)$. Since C is compact, $\{V_{x,t} \times J_{x,t}\}_{(x,t)\in C}$ has a finite sub-cover of C, i.e. $C \subset \cup_{q=1}^m V_{x_q,t_q} \times J_{x_q,t_q}$. We reorder these subscripts so that

$$0 = \inf J_{x_1,t_1} = \cdots = \inf J_{x_k,t_k} < \inf J_{x_{k+1},t_{k+1}} \leq \cdots \leq \inf J_{x_m,t_m}.$$

Moreover, since C_0 is not empty, there is at least one J_q with $\min J_{x_q,t_q} = 0$. Thus, $1 \leq k \leq m$. Take

$$\varepsilon = \min\{\sup\{J_{x_q,t_q} \mid 1 \leq q \leq k\},\ \inf\{J_{x_q,t_q} \mid k < q \leq m\}\}.$$

Of course, the value $\inf\{J_{x_q,t_q} \mid k < q \leq m\}$ is understand to be 1 if $k = m$. (By the way, $\sup\{J_{x_q,t_q} \mid 1 \leq q \leq k\}$ can be smaller than $\inf\{J_{x_q,t_q} \mid k < q \leq m\}$. In this case, C_t will be empty for these t between these two values.)

By definition of ε, we have that $C \cap (X \times [0, \varepsilon]) \subset \cup_{q=1}^k V_{x_q,t_q} \times [0, \varepsilon]$. Write $V = \cup_{q=1}^k V_{x_q,t_q}$. Then C_t is contained in V for all $t \in [0, \varepsilon]$. Note that N_t is also a closed neighborhood of V for all $t \in [0,\varepsilon]$. By Lemma 3.3, $\mathcal{L}_*(f_t, C_t, B)$ is the composition of the following

$$H_*(X) \xrightarrow{j_*} H_*(X, X-V) \xrightarrow{e_*^{-1}} H_*(N_t, N_t - V) \xrightarrow{f_*} H_*(Y, Y-B)$$

for $t \in [0,\varepsilon]$. As in the proof Lemma 3.2, above composition is actually independent of the choice of closed neighborhood N_t of V. The homomorphism index $\mathcal{L}_*(f_t, C_t, B)$ is therefore constant for $t \in [0,\varepsilon]$. Thus, we are done. ∎

From Proposition 2.5 and its corollary, we know that each preimage class of is an isolated subset of $f^{-1}(B)$, and hence has a well-defined (homology) homomorphism index. Thus, we have following definition

Definition 3.6. *We write $N(f, B; H_*)$ for the number of preimage classes of f at B with non-zero (homology) homomorphism index.*

By Lemma 3.5, $N(f, B; H_*)$ is a homotopy invariant. We obtain

Theorem 3.7. *Let $f\colon X \to Y$ be a map, and B a closed subset of Y. Suppose that X is compact. Then for any map $g\colon X \to Y$ homotopic to f, the number $N(f, B; H_*)$ is a lower bound for either (1) the number of components of $g^{-1}(B)$ or (2) the number of points in $g^{-1}(B)$.*

Proof. Suppose that $F\colon X \times I \to Y$ is a homotopy from f to g. The homotopy invariance of homomorphism index implies that F-related preimage classes have the same homomorphism indices. Moreover, if a preimage class of f does not H-related to any preimage class of f, then such a class must have zero indices. Thus, each preimage class of f with non-zero index contributes at least a component of $g^{-1}(B)$. We obtain our conclusion. ∎

We can also define similar indices by using cohomology, the coefficient can be arbitrary abelian group G, denoted $\mathcal{L}^*(f, A, B; G)$. If some dimension n is specified, we refer to $\mathcal{L}_n(f, A, B; G)$, or $\mathcal{L}^n(f, A, B; G)$. All of these indices share the same properties in the section. Thus, we may have other lower bounds for number and component number of $f^{-1}(B)$: $N(f, B; H_*(\cdot, G))$ and $N(f, B; H^*(\cdot, G))$, $N(f, B; H_n(\cdot, G))$, etc.

We would like to remark that the requirements on the space X can be weakened to normality. A topological space is said to be *normal* if any singleton is always a closed subset and any two disjoint closed subsets have disjoint neighborhoods.

4 Common value pairs

In this section, we shall apply our preimage class arguments into a special case: common value pairs, which is defined as follows.

Definition 4.1. *Let $f, g\colon X \to Y$ be two maps. The set of common value pairs* $\mathrm{CVP}(f,g)$ *of f and g is defined to be $(f \times g)^{-1}(\Delta_{Y^2}(Y)) = \{(u,v) \in X^2 \mid f(u) = g(v)\}$.*

From this definition, one can see a relation with coincidence point: $x \in \mathrm{Coin}(f,g)$ if and only if $(x,x) \in \mathrm{CVP}(f,g)$, i. e. $\mathrm{CVP}(f,g) = \Delta_{X^2}(X)(\mathrm{Coin}(f,g))$. Moreover, $\mathrm{CVP}(f,g) = \mathrm{Coin}((f \times g)p_1, (f \times g)p_2)$, where $p_1, p_2 \colon Y^2 \to Y$ are two projections. But, we consider here the behavior of the set $\mathrm{CVP}(f,g)$ under the homotopies of f and g, which is a different story about $\mathrm{Coin}((f \times g)p_1, (f \times g)p_2)$.

Note that any lifting of $f \times g$ has the form $\tilde{f} \times \tilde{g}$, where $\tilde{f}$ and $\tilde{g}$ are respectively liftings of f and g. We write p_{X^2} for the product $p_{X^2} \colon \tilde{X}^2 \to X^2$, giving the universal covering of X^2, so is p_{Y^2}. Thus,

Proposition 4.2. *Let $f, g \colon X \to Y$ be two maps, and $\tilde{f}_0$ and $\tilde{g}_0$ be respectively chosen liftings of f and g. Then*

$$\begin{aligned}\mathrm{CVP}(f,g) &= \cup_{\tilde{f}\in \mathrm{lift}(f), \tilde{g}\in \mathrm{lift}(g)} p_{X^2}(\mathrm{CVP}(\tilde{f}, \tilde{g})) \\ &= \cup_{\tilde{f}\in \mathrm{lift}(f)} p_{X^2}(\mathrm{CVP}(\tilde{f}, \tilde{g}_0)) \\ &= \cup_{\tilde{g}\in \mathrm{lift}(g)} p_{X^2}(\mathrm{CVP}(\tilde{f}_0, \tilde{g})).\end{aligned}$$

Proof. The first equality is a special case of Proposition 2.1. The second one comes from the fact that $p_{X^2}(\mathrm{CVP}(\tilde{f}, \tilde{g}_0)) = p_{X^2}(\mathrm{CVP}(\delta\tilde{f}, \delta\tilde{g}_0))$ for any $\delta \in \mathrm{D}(\tilde{Y})$, and the fact that any lifting of g has the form $\gamma\tilde{g}_0$ for some $\gamma \in \mathrm{D}(\tilde{Y})$. The third one is in the same situation. ∎

Observe that the preimage $(p_{Y^2})^{-1}(\Delta_{Y^2}(Y))$ has a natural component $\Delta_{\tilde{Y}^2}(\tilde{Y})$, because Y and hence $\tilde{Y}$ is connected. We may regard preimage classes here as those in the form $p_{X^2}((\tilde{f} \times \tilde{g})^{-1}(\Delta_{\tilde{Y}^2}(\tilde{Y})))$. This leads to the following:

Definition 4.3. *Let $f, g \colon X \to Y$ be two maps. The preimage class $p_{X^2}((\tilde{f} \times \tilde{g})^{-1}$ $(\Delta_{\tilde{Y}^2}(\tilde{Y})))$ of $f \times g$ at the subset $\Delta_{Y^2}(Y)$ of Y^2 is said to be* the common value class *of (f,g) determined by $\tilde{f} \times \tilde{g}$, or say by $\tilde{f}$ and $\tilde{g}$.*

Note that $p_{X^2}((\tilde{f} \times \tilde{g})^{-1}(\Delta_{\tilde{Y}^2}(\tilde{Y}))) = p_{X^2}(\mathrm{CVP}(\tilde{f} \times \tilde{g}))$.

Proposition 4.4. *Let $\tilde{f}' \times \tilde{g}'$ and $\tilde{f}'' \times \tilde{g}''$ be two liftings of $f \times g$. Then $\tilde{f}' \times \tilde{g}' \sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})} \tilde{f}'' \times \tilde{g}''$ if and only if there are two elements α, γ in $\mathrm{D}(\tilde{X})$ and an element $\beta \in \mathrm{D}(\tilde{Y})$ such that $\tilde{f}' = \beta\tilde{f}''\alpha$ and $\tilde{g}' = \beta\tilde{g}''\gamma$.*

Proof. By definition of the relation $\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}$ (see Definition 2.2), $\tilde{f}' \times \tilde{g}' \sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})} \tilde{f}'' \times \tilde{g}''$ means that there is an element $\alpha_1 \times \alpha_2 \in \mathrm{D}(\tilde{X}^2) = \mathrm{D}(\tilde{X}) \times \mathrm{D}(\tilde{X})$ and an element $\beta_1 \times \beta_2 \in \mathrm{D}(\tilde{Y}^2) = \mathrm{D}(\tilde{Y}) \times \mathrm{D}(\tilde{Y})$ with $(\beta_1 \times \beta_2)(\Delta_{\tilde{Y}^2}(\tilde{Y})) = \Delta_{\tilde{Y}^2}(\tilde{Y})$ such that $\tilde{f}' \times \tilde{g}' = (\beta_1 \times \beta_2) \circ (\tilde{f}'' \times \tilde{g}'') \circ (\alpha_1 \times \alpha_2)$, i.e. $\tilde{f}' = \beta_1\tilde{f}''\alpha_1$ and $\tilde{g}' = \beta_2\tilde{g}''\alpha_2$. Note that $(\beta_1 \times \beta_2)(\Delta_{\tilde{Y}^2}(\tilde{Y})) = \Delta_{\tilde{Y}^2}(\tilde{Y})$ if and only if $\beta_1 = \beta_2$. We are done. ∎

Corollary 4.5. *If X is compact and Y is a Hausdorff space, then any two maps $f, g \colon X \to Y$ has finitely many common value classes.*

By Proposition 4.2 again, if we have chosen reference liftings $\tilde{f}_0$ and $\tilde{g}_0$ of f and g. Each common value class can be written as $p_{X^2}(\mathrm{CVP}(\tilde{f}_0, \delta\tilde{g}_0)) = p_{X^2}((\tilde{f}_0 \times \delta\tilde{g}_0)^{-1}(\Delta_{\tilde{Y}^2}(\tilde{Y})))$ for some $\delta \in \mathrm{D}(\tilde{Y})$.

Proposition 4.6. *Let $\tilde{f}_0$ and $\tilde{g}_0$ be respectively liftings of f and g. Then $\tilde{f}_0 \times \delta_1 \tilde{g}_0 \sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})} \tilde{f}_0 \times \delta_2 \tilde{g}_0$ if and only if $\delta_2 = \tilde{f}_D(\alpha^{-1})\delta_1 \tilde{g}_D(\gamma)$ for some $\alpha, \gamma \in \mathrm{D}(\tilde{X})$, where $\tilde{f}_D \colon \mathrm{D}(\tilde{X}) \to \mathrm{D}(\tilde{Y})$ is the homomorphism determined by the relation $\tilde{f}_0(\eta\tilde{x}) = \tilde{f}_D(\eta)\tilde{f}_0(\tilde{x})$ for all $\eta \in \mathrm{D}(\tilde{X})$ and $\tilde{x} \in \tilde{X}$, and $\tilde{g}_D$ is defined similarly.*

Proof. By Proposition 4.4, the equivalency in the sense of $\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}$ implies that $\tilde{f}_0 = \beta\tilde{f}_0\alpha$ and $\delta_2\tilde{g}_0 = \beta\delta_1\tilde{g}_0\gamma$ for some $\alpha, \gamma \in \mathrm{D}(\tilde{X})$ and some $\beta \in \mathrm{D}(\tilde{Y})$. Thus, $\tilde{f}_0 = \beta\tilde{f}_{\mathrm{D}}(\alpha)\tilde{f}_0$ and $\delta_2\tilde{g}_0 = \beta\delta_1\tilde{g}_{\mathrm{D}}(\gamma)\tilde{g}_0$. It follows that $1 = \beta\tilde{f}_{\mathrm{D}}(\alpha)$ and $\delta_2 = \beta\delta_1\tilde{g}_{\mathrm{D}}(\gamma)$ in $\mathrm{D}(\tilde{Y})$. The former equality gives us $\beta = \tilde{f}_D(\alpha^{-1})$. From the other equality, we obtain that $\delta_2 = \tilde{f}_D(\alpha^{-1})\delta_1\tilde{g}_D(\gamma)$. ∎

Corollary 4.7. *If one of f and g induces a surjective homomorphism between fundamental group, all lifting of $f \times g$ are equivalent in the sense of $\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}$. Hence, the number of common value classes of f and g is either* 0 *or* 1.

The following Proposition gives a geometric interpretation of common value classes.

Proposition 4.8. *Two common value pairs (u_0, v_0) and (u_1, v_1) of (f, g) are in the same common value class if and only if there is a path η_U from u_0 to u_1 and a path η_V from v_0 to v_1 such that $f\eta_U$ and $g\eta_V$ are homotopic keeping end points fixed.*

Proof. "If". Assume that $(u_0, v_0) \in p_{X^2}(\mathrm{CVP}(\tilde{f}, \tilde{g}))$. Then $\tilde{f}(\tilde{u}_0) = \tilde{g}(\tilde{v}_0)$ with $p_X(\tilde{u}_0) = u_0$ and $p_X(\tilde{v}_0) = v_0$. Let $\tilde{\eta}_U$ be the lifting path of η_U with initial point $\tilde{\eta}_U(0) = \tilde{u}_0$, and $\tilde{\eta}_V$ be the lifting path of η_V with initial point $\tilde{\eta}_V(0) = \tilde{v}_0$. Thus, $\tilde{f}\tilde{\eta}_U$ is the lifting path of $f\eta_U$ with initial point $\tilde{f}(\tilde{u}_0)$, and $\tilde{g}\tilde{\eta}_V$ is the lifting path of $g\eta_V$ with initial point $\tilde{g}(\tilde{v}_0)$.

Let $H \colon I \times I \to Y$ be a homotopy from $f\eta_U$ to $g\eta_V$. We write $\tilde{H} \colon I \times I \to \tilde{Y}$ for the lifting homotopy with $\tilde{H}(0,0) = \tilde{f}(\tilde{u}_0)$. Thus, $\tilde{H}(\cdot, 0)$ is the lifting path of $f\eta_U$ with initial point $\tilde{f}(\tilde{u}_0)$. Similarly, $\tilde{H}(\cdot, 1)$ is the lifting path of $g\eta_V$ with initial point $\tilde{g}(\tilde{v}_0)$ ($= \tilde{f}(\tilde{u}_0)$). By the uniqueness of liftings, we know that $\tilde{f}\tilde{\eta}_U$ is the same as $\tilde{H}(\cdot, 0)$, and $\tilde{g}\tilde{\eta}_V$ is the same as $\tilde{H}(\cdot, 1)$. Especially, $\tilde{f}(\tilde{\eta}_U(1)) = \tilde{H}(1, 0)$ and $\tilde{g}(\tilde{\eta}_V(1)) = \tilde{H}(1, 1)$. Note that $H(1, \cdot)$ is a constant path at $f(u_1) = g(v_1)$. We have that $\tilde{H}(1, 0) = \tilde{H}(1, 1)$. Thus, $(u_1, v_1) = p_{X^2}(\eta_U(1), \eta_V(1)) \in p_{X^2}(\mathrm{CVP}(\tilde{f}, \tilde{g}))$.

"Only if". Since (u_0, v_0) and (u_1, v_1) are in the same common value class, there is a lifting $\tilde{f}$ of f and a lifting $\tilde{g}$ of g such that $\tilde{f}(\tilde{u}_0) = \tilde{g}(\tilde{v}_0)$ and $\tilde{f}(\tilde{u}_1) = \tilde{g}(\tilde{v}_1)$, where $\tilde{u}_i \in p_X^{-1}(u_i)$ and $\tilde{v}_i \in p_X^{-1}(v_i)$ for $i = 0, 1$. Let $\tilde{\eta}_U$ be a path from $\tilde{u}_0$ to $\tilde{u}_1$ in $\tilde{X}$, and $\tilde{\eta}_V$ be a path from $\tilde{v}_0$ to $\tilde{v}_1$ in $\tilde{X}$. Then $p_X\tilde{\eta}_U$ will be a path from u_0 to u_1 in X, and $p_X\tilde{\eta}_V$ will be a path from v_0 to v_1 in X. Since $\tilde{Y}$ is simply-connected, and since $\tilde{f}\tilde{\eta}_U$ has the same end points as $\tilde{g}\tilde{\eta}_V$, we have that $\tilde{f}\tilde{\eta}_U \simeq \tilde{g}\tilde{\eta}_V$. Thus, $f p_X\tilde{\eta}_U = p_Y\tilde{f}\tilde{\eta}_U \simeq p_Y\tilde{g}\tilde{\eta}_V = g p_X\tilde{\eta}_V$. ∎

Let $f, g \colon X \to Y$ be two maps. By Proposition 2.5 and its Corollary, If X is compact and Y is a Hausdorff space, then any common value class C is an isolated subset of $(f \times g)^{-1}(Y^2)$, and hence admits a well-defined homomorphism index $\mathcal{L}_*(f \times g, C, \Delta_{Y^2}(Y))$ (see Definition 3.1).

Let $F_f \colon X \times I \to Y$ be a homotopy from f_0 to f_1, and $F_g \colon X \times I \to Y$ be a homotopy from g_0 to g_1. A common value class C_0 of f_0 and g_0 and a common value class of f_1 and g_1 are said to be (F_f, F_g)-*related* if they are respectively determined by a pair of liftings $(\tilde{f}_0, \tilde{g}_0)$ of (f_0, g_0) and a pair of liftings $(\tilde{f}_1, \tilde{g}_1)$ of (f_1, g_1) such that $\tilde{f}_i(\tilde{x}) = \tilde{F}_f(\tilde{x}, i)$ and $\tilde{g}_i(\tilde{x}) = \tilde{F}_g(\tilde{x}, i)$, $i = 0, 1$, where $\tilde{F}_f\colon \tilde{X} \times I \to \tilde{Y}$ and $\tilde{F}_g\colon \tilde{X} \times I \to \tilde{Y}$ are respectively liftings of H_f and H_g.

Moreover, "(F_f, F_g)-related" is an equivalent relation preserving equivalent relation in the sense of $\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}$. By Proposition 3.5, we obtain immediately that

Lemma 4.9. *(Homotopical existence) Suppose that X is compact and Y is a Hausdorff space. Let C be a common value class of two maps f and g with non-zero index $\mathcal{L}_*(f \times g, C, \Delta_{Y^2}(Y))$. Then for any two maps f' and g' with $f \overset{F_f}{\simeq} f'$ and $g \overset{F_g}{\simeq} g'$, C is (F_f, F_g)-related to some common value class of f' and g'.*

By this Lemma and the arguments in the proof of Theorem 3.7, we obtain immediately

Theorem 4.10. *Suppose that X is compact and Y is a Hausdorff space. Let $f, g\colon X \to Y$ be two maps. Then the number $N(f \times g, \Delta_{Y^2}(Y); H_*)$ of common value classes with non-zero homology homomorphism indices is a lower bound of the number of common value pairs of f' and g' for any $f' \simeq f$ and $g' \simeq g$. So is the number $N(f \times g, \Delta_{Y^2}(Y); H^*)$ of common value classes with non-zero cohomology homomorphism indices.*

Note that a common value pair gives a consideration of an intersection coming from the domain, in other words the image of a common value pair is just an intersection. Thus, we have

Corollary 4.11. *With the same assumptions as in Theorem 4.10. If f and g have no intersection with multiple bigger than 2, then either $N(f \times g, \Delta_{Y^2}(Y); H_*)$ or $N(f \times g, \Delta_{Y^2}(Y); H^*)$ gives a lower bound for the number of intersection points of f and g.*

From intersection point of view, the usual situation is that both of X and Y are manifolds with $\dim Y = 2 \dim X$. By a standard argument in differential topology, we have

Proposition 4.12. *Let X and Y be smooth oriented closed manifolds with $\dim X = \frac{1}{2}\dim Y = m$, and Let $f, g\colon X \to Y$ be two smooth maps. Suppose that $f(u) = g(v) = y$ and that f and g are transversal at (u, v). Then (u, v) is an isolated common value pair, and its index $\mathcal{L}^*(f \times g, \{(u, v)\}) = \mathcal{L}^{2m}(f \times g, \{(u, v)\})$ in cohomologies is given by $H^{2m}(Y^2, Y^2 - \Delta_{Y^2}(Y)) \ni \tau \mapsto \epsilon\,(\omega \times \omega) \in H^{2m}(X^2)$, where τ is the Thom class, ω is the orientation class of X, and $\epsilon = \pm 1$ depending $(f \times g)_*|_{(u,v)}$ is orientation preserving or not.*

Proof. Note that $Y \cong \Delta_{Y^2}(Y)$. By Thom isomorphism theorem, we know that $H^k(Y)$ is isomorphic to $H^{k+2m}(Y^2, Y^2 - \Delta_{Y^2}(Y))$ for all k. Such an isomorphism is given by $\alpha \mapsto \alpha \cup \tau$. This means that the Thom class τ generates the infinitely cyclic group $H^{2m}(Y^2, Y^2 - \Delta_{Y^2}(Y))$. Moreover, $H^q(Y^2, Y^2 - \Delta_{Y^2}(Y)) = 0$ for $q < 2m$. On the other hand, since X has dimension m, $H^q(X^2)$ does not vanish only if $0 \leq q \leq 2m$. Thus, the possible non-trivial homomorphisms between $H^*(Y^2, Y^2 - \Delta_{Y^2}(Y))$ and $H^*(X^2)$ happen on dimension $2m$ only.

Since X is orientable, $H^{2m}(X^2) \cong \mathbb{Z}$ has a generator $\omega \times \omega$. The remaining proof is a standard argument in differential topology. ∎

Now we illustrate with an example.

Example 4.13. *Let $f,g\colon S^1 \to T^2$ be two maps. By a homotopy and re-coordinating, we may assume that they are respectively given by $f(e^{\theta i}) = (e^{m\theta i}, e^0)$ and $f(e^{\theta i}) = (e^{q\theta i}, e^{n\theta i})$ for some $m,n,q \in \mathbb{Z}$.*

Let $p_X\colon \mathbb{R}^1 \to S^1$ be the universal covering of $X = S^1$ given by $p_X(\tilde{x}) = e^{2\pi\tilde{x}i}$, and $p_Y\colon \mathbb{R}^2 \to T^2$ be the universal covering of $Y = T^2$ given by $p_Y(\tilde{y}_1,\tilde{y}_2) = (e^{2\pi\tilde{y}_1 i}, e^{2\pi\tilde{y}_2 i})$. Clearly, the deck transformation group $\mathrm{D}(\mathbb{R}^2) \cong \mathbb{Z}^2$ is generated by a and b, where $(\tilde{y}_1,\tilde{y}_2) \overset{a}{\mapsto} (\tilde{y}_1+1,\tilde{y}_2)$, $(\tilde{y}_1,\tilde{y}_2) \overset{b}{\mapsto} (\tilde{y}_1,\tilde{y}_2+1)$.

Then f has a lifting $\tilde{f}\colon \mathbb{R}^1 \to \mathbb{R}^2$ which is given by $\tilde{f}(\tilde{x}) = (m\tilde{x}, 0)$, and g has a lifting $\tilde{g}\colon \mathbb{R}^1 \to \mathbb{R}^2$ which is given by $\tilde{g}(\tilde{x}) = (q\tilde{x}, n\tilde{x})$. Hence, all liftings of g have the form $a^k b^l \tilde{g}$ for $k,l \in \mathbb{Z}$. Notice that

$$\mathrm{CVP}(\tilde{f}, a^k b^l \tilde{g}) = \{(\tilde{u},\tilde{v}) \mid m\tilde{u} = q\tilde{v} + k,\ 0 = n\tilde{v} + l\}. \tag{4.1}$$

Case 1: $m \neq 0$ and $n \neq 0$. Then $\mathrm{CVP}(\tilde{f}, a^k b^l \tilde{g}) = \{(-\frac{ql}{mn} + \frac{k}{m}, -\frac{l}{n})\}$ is singleton for all $k,l \in \mathbb{Z}$. By Proposition 2.3, two pair $(\tilde{f}, a^k b^l \tilde{g})$ and $(\tilde{f}, a^{k'} b^{l'} \tilde{g})$ determine the same common class if and only if the projections of their common value are the same. The number of common value classes of f and g is $|mn|$. These classes are determined by $(\tilde{f}, a^k b^l \tilde{g})$, $k = 0,1,\ldots,|m|-1$; $l = 0,1,\ldots,|n|-1$.

Case 2: $m = 0$ and $n \neq 0$. By (4.1), we obtain that $\mathrm{CVP}(\tilde{f}, a^k b^l \tilde{g}) = \{(\tilde{u}, -\frac{l}{n}) \mid \tilde{u} \in \mathbb{R}\}$ if $k = -\frac{qn}{n}$; $\mathrm{CVP}(\tilde{f}, a^k b^l \tilde{g}) = \emptyset$ if $k \neq -\frac{ql}{n}$. If $k = -\frac{ql}{n}$, we can make a small homotopy starting with f so that the ending map has a lifting $\tilde{f}'(\tilde{x}) = (\varepsilon, 0)$. Note that $\mathrm{CVP}(\tilde{f}', a^k b^l \tilde{g}) = \emptyset$. It follows that all common value classes have zero homomorphism indices.

Case 3: $n = 0$. By (4.1), if $\mathrm{CVP}(\tilde{f}, a^k b^l \tilde{g}) \neq \emptyset$, then l must be zero. We can make a small homotopy starting with f so that the ending map has a lifting $\tilde{f}'(\tilde{x}) = (m\tilde{x}, \varepsilon)$. Note that $\mathrm{CVP}(\tilde{f}', a^k \tilde{g}) = \emptyset$. It follows that all common value classes have zero homomorphism indices.

Such a computation shows that our theory here coincides with the method of cohomology in this special case: loops on the torus. It is not hard to see that common value classes will bring more information than cohomology even in the case of loops on surfaces. For example, non-trivial but homology vanishing loops on surfaces of genus ≥ 2.

5 Self common values, what is the special?

We shall consider in this section the number of common value pairs of a map and itself. Such kinds of common value pairs are said to be self common value pairs.

Definition 5.1. *Two self common value classes are said to be symmetric to each other if they are respectively determined by lifting pair $\tilde{f}' \times \tilde{f}''$ and $\tilde{f}'' \times \tilde{f}'$.*

A common value class is said to be self-symmetric if it is symmetric to itself. Especially, self common value classes determined by $\tilde{f} \times \tilde{f}$ are said to be trivial classes.

Note that $\tilde{f} \times \tilde{f}$ and $\tilde{f}' \times \tilde{f}'$ determine the same common value class. We have

Corollary 5.2. *Each map $f\colon X\to Y$ has a unique trivial self common value class. It is the class containing the diagonal $\Delta_{X^2}(X)$ in X^2.*

Next Proposition includes some criterions which help us to understand self-symmetric classes.

Proposition 5.3. *Let C be a self common value class of $f\colon X\to Y$. Then following four statements are equivalent:*

(1) C is a self-symmetric;

(2) for any two liftings $\tilde{f}_1$ and $\tilde{f}_2$ of f, $\tilde{f}_1\times\tilde{f}_2$ determines C if and only if $\tilde{f}_2\times\tilde{f}_1$ determines C;

(3) $(u,v)\in C$ if and only if $(v,u)\in C$;

(4) there are two points $u,v\in X$ such that (u,v) and (v,u) are both in C.

Proof. From (1) to (2): Since C is self-symmetric, we have that $C=p_{X^2}(\mathrm{CVP}(\tilde{f}',\tilde{f}''))=p_{X^2}(\mathrm{CVP}(\tilde{f}'',\tilde{f}'))$ for two liftings $\tilde{f}'$ and $\tilde{f}''$ of f. If $\tilde{f}_1\times\tilde{f}_2$ determines C, then $\tilde{f}_1\times\tilde{f}_2\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}\tilde{f}'\times\tilde{f}''$. By definition, $\tilde{f}_2\times\tilde{f}_1\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}\tilde{f}''\times\tilde{f}'$. It follows that $\tilde{f}_2\times\tilde{f}_1\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}\tilde{f}'\times\tilde{f}''$, because $\tilde{f}'\times\tilde{f}''\sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})}\tilde{f}''\times\tilde{f}'$. The converse is the same.

From (2) to (3): If $(u,v)\in C$, then there are two liftings $\tilde{f}'$ and $\tilde{f}''$ of f such that $(\tilde{u},\tilde{v})\in p_{X^2}(\mathrm{CVP}(\tilde{f}',\tilde{f}''))=C$. Since C is also determined by $\tilde{f}''\times\tilde{f}'$, (v,u) lies in C. The converse is the same.

From (3) to (4): Trivial.

From (4) to (1): If $(u,v)\in C$, then $(u,v)\in p_{X^2}(\mathrm{CVP}(\tilde{f}',\tilde{f}''))$ for two liftings $\tilde{f}'$ and $\tilde{f}''$ of f. By definition of common value pair, $(v,u)\in p_{X^2}(\mathrm{CVP}(\tilde{f}'',\tilde{f}'))$. Since $(v,u)\in C$, C is also determined by $(\tilde{f}'',\tilde{f}')$. Thus, C is self-symmetric. ∎

By using a similar argument, we have

Proposition 5.4. *Let C be a self common value class of f. The set $\{(x',x'')\mid(x'',x')\in C\}$ is the self common value class which is symmetric to C.*

In another word, the symmetric relation is an $\mathbb{Z}_2$-action on the set of all self common value classes of given map, which is induced by the natural action on X^2 defined by $(x',x'')\mapsto(x'',x')$. The self-symmetric classes are just the elements fixed by this action. Moreover, we have

Proposition 5.5. *Let C' and C'' be two self common value classes of f, which is symmetric to each other. Then C' has non-zero homomorphism index if and only if C'' has non-zero homomorphism index.*

Proof. We write $\tau : X^2\to X^2$ for the natural involution given by $\tau(x',x'')=(x'',x')$. Pick a closed neighborhood N' of C' with $N'\cap\mathrm{CVP}(f,f)=C'$, it follows that $N''=\tau(N')$ is a closed neighborhood of C'' with $N''\cap\mathrm{CVP}(f,f)=C''$. We have following commutative diagram:

$$\begin{array}{ccccccc}
H_*(X^2) & \xrightarrow{j_*} & H_*(X^2,X^2-C') & \xrightarrow{e_*^{-1}} & H_*(N',N'-C') & \xrightarrow{(f\times f)_*} & H_*(Y^2,Y^2-\Delta_{Y^2}(Y)).\\
\downarrow{\scriptstyle\tau_*} & & \downarrow{\scriptstyle\tau_*} & & \downarrow{\scriptstyle\tau_*} & & \downarrow{\scriptstyle\tau'_*}\\
H_*(X^2) & \xrightarrow{j_*} & H_*(X^2,X^2-C'') & \xrightarrow{e_*^{-1}} & H_*(N'',N''-C'') & \xrightarrow{(f\times f)_*} & H_*(Y^2,Y^2-\Delta_{Y^2}(Y)),
\end{array}$$

where $\tau' \colon Y^2 \to Y^2$ is the natural involution. It is obvious that the first row gives the homomorphism index of C', and that the second row gives that of C''. Thus, we prove this proposition because τ_* and τ'_* are both isomorphisms. ■

Similar to Theorem 4.10, we may have an estimation of self common value pairs for given map. But, the estimation of self-intersections is a little different because symmetric pairs are mapped into the same intersection point. Thus, we have

Theorem 5.6. *Suppose that X is compact and Y is a Hausdorff space. If f no self-intersection with multiple bigger than 2, then*

$$\sharp\{\textit{non-trival self-symmetric self common value classe with non-zero indices }\}$$
$$+\frac{1}{2}\sharp\{\textit{non-self-symmetric self common value classe with non-zero indices }\}$$

gives a lower bound for the number of self-intersection points of f.

We can also explain the self-symmetric classes in sense of deck transformation groups.

Proposition 5.7. *Let $\tilde{f}_0$ be a lifting of f, and δ an element in $\mathrm{D}(\tilde{Y})$. Then $\tilde{f}_0 \times \delta\tilde{f}_0 \sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})} \delta\tilde{f}_0 \times \tilde{f}_0$ if and only if $\delta = \tilde{f}_{\mathrm{D}}(\alpha_1)\delta^{-1}\tilde{f}_{\mathrm{D}}(\alpha_2)$ for some $\alpha_1, \alpha_2 \in \mathrm{D}(\tilde{X})$.*

Proof. Suppose that $\tilde{f}_0 \times \delta\tilde{f}_0 \sim_{\Delta_{\tilde{Y}^2}(\tilde{Y})} \delta\tilde{f}_0 \times \tilde{f}_0$. By Proposition 4.4, $\delta\tilde{f}_0 = \beta\tilde{f}_0\alpha$ and $\tilde{f}_0 = \beta\delta\tilde{f}_0\gamma$. It follows that $\delta = \beta\tilde{f}_{\mathrm{D}}(\alpha)$ and $1 = \beta\delta\tilde{f}_{\mathrm{D}}(\gamma)$. By canceling β, we obtain that $\delta = \tilde{f}_{\mathrm{D}}(\gamma^{-1})\delta^{-1}\tilde{f}_{\mathrm{D}}(\alpha)$.

Conversely, if $\delta = \tilde{f}_{\mathrm{D}}(\alpha_1)\delta^{-1}\tilde{f}_{\mathrm{D}}(\alpha_2)$ for some $\alpha_1, \alpha_2 \in \mathrm{D}(\tilde{X})$, then $\delta\tilde{f}_0 = \tilde{f}_{\mathrm{D}}(\alpha_1)\delta^{-1}\tilde{f}_{\mathrm{D}}(\alpha_2)\tilde{f}_0 = \tilde{f}_{\mathrm{D}}(\alpha_1)\delta^{-1}\tilde{f}_0\alpha_2$ and $\tilde{f}_0 = \tilde{f}_{\mathrm{D}}(\alpha_1)\tilde{f}_0\alpha_1^{-1} = \tilde{f}_{\mathrm{D}}(\alpha_1)\delta^{-1}(\delta\tilde{f}_0)\alpha_1^{-1}$. This implies that $\tilde{f}_0 \times \delta\tilde{f}_0$ and $\delta\tilde{f}_0 \times \tilde{f}_0$ are equivalent. ■

Now, we give a simple example showing the self common value classes.

Example 5.8. *Let $f \colon S^1 \to T^2 = S^1 \times S^1$ be a map given by $f(e^{\theta i}) = (e^{m\theta i}, e^0)$.*

We can homotope f into a map f', defined by $f'(e^{\theta i}) = (e^{m\theta i}, e^{(\sin\frac{\theta}{2})^2 i})$. We shall using the same notations as in Example 4.13 for universal coverings and their deck transformation groups. Then the map f' has a lifting $\tilde{f}'$, defined by $\tilde{f}'(\tilde{x}) = (m\tilde{x}, \frac{1}{2\pi}\sin^2\pi\tilde{x})$. A direct computation show that

$$\begin{aligned}
\mathrm{CVP}(\tilde{f}', a^k\tilde{f}') &= \{(\tilde{u},\tilde{v}) \mid m\tilde{u} = m\tilde{v} + k,\ \sin^2\pi\tilde{u} = \sin^2\pi\tilde{v}\} \\
&= \{(\tilde{u},\tilde{v}) \mid \tilde{u} = \tilde{v} + \tfrac{k}{m},\ \tilde{u} = \pm\tilde{v} + s, s \in \mathbb{Z}\} \\
&= \begin{cases} \{(\tilde{v} + \frac{k}{m}, \tilde{v}) \mid \tilde{v} \in \mathbb{R}\} & \text{if } m|k, \\ \{(\frac{s}{2} + \frac{k}{2m}, \frac{s}{2} - \frac{k}{2m}) \mid s \in \mathbb{Z}\} & \text{if } m \not| k. \end{cases}
\end{aligned}$$

In fact, there is no more self common value class, because $\mathrm{CVP}(\tilde{f}', a^k b^l \tilde{f}') = \emptyset$ if $l \neq 0$. Thus, there are m self common value classes: $p_{X^2}(\mathrm{CVP}(\tilde{f}', a^k\tilde{f}'))$, where $k = 0, 1, \ldots, m-1$. For $k = 0$, the class $p_{X^2}(\mathrm{CVP}(\tilde{f}', \tilde{f}')) = \{(e^{\theta i}, e^{\theta i}) = \Delta_{(S^1)^2}(S^1)$ is the trivial class. If $k = 1, \ldots, m-1$, then

$$p_{X^2}(\mathrm{CVP}(\tilde{f}', a^k\tilde{f}')) = \{(e^{\frac{k\pi i}{m}}, e^{-\frac{k\pi i}{m}}),\ (e^{\pi i + \frac{k\pi i}{m}}, e^{\pi i - \frac{k\pi i}{m}})\}.$$

The class $p_{X^2}(\mathrm{CVP}(\tilde{f}', a^k \tilde{f}'))$ and $p_{X^2}(\mathrm{CVP}(\tilde{f}', a^{m-k} \tilde{f}'))$ are symmetric to each other. Clearly, if m is even, then the $p_{X^2}(\mathrm{CVP}(\tilde{f}', a^{m/2} \tilde{f}'))$ is self-symmetric.

Note that the homomorphism index of a self common value class comes from the constructed map $f \times f$. It is easy to see that we can homotope f into f'' so that the image of f and f'' have no intersection. The homotopy invariance of the homomorphism index implies that all non-trivial self common value classes have zero homomorphism indices. From a geometric argument, we know that that there must be self-intersection points and hence non-trivial self common value pairs if $|m| > 1$, because f is not a simple closed curve. This phenomenon shows that vanishing of index does not mean intersection free in general.

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