# Nielsen-Reidemeister indices for multivalued maps

M. C. Crabb

**Abstract**

A Nielsen-Reidemeister index is constructed for multivalued maps defined by fractions $f/p$ where $p : \tilde{X} \to X$ is a fibrewise manifold with closed fibres over a compact ENR and $f : \tilde{X} \to X$ is a continuous map. In the case that $p$ is a finite $n$-fold cover, this index is shown to agree with the index of the $n$-valued map $\tilde{X} \multimap \tilde{X}$ associated with $f/p$ by a construction of Brown [4].

## 1 Introduction

Let $p : \tilde{X} \to X$ be a finite (not necessarily connected) $n$-fold cover of a compact ENR (Euclidean Neighbourhood Retract) $X$ and suppose that $f : \tilde{X} \to X$ is a continuous map. Then we can define a multivalued map $F : X \multimap X$ by

$$F(x) = \{f(\tilde{x}) \mid \tilde{x} \in \tilde{X},\, p(\tilde{x}) = x\} \qquad (x \in X).$$

Thus $F(x)$ is a finite non-empty set with cardinality at most $n$: $\#F(x) \leqslant n$. If $\#F(x) = n$ for all $x$, the multivalued map $F$ is an *$n$-valued map* in the sense of Schirmer [16]. In any case, following Brown [4] we can associate with the pair $(f, p)$ an $n$-valued map $\tilde{F} : \tilde{X} \multimap \tilde{X}$ by setting

$$\tilde{F}(\tilde{x}) = \{\tilde{y} \in \tilde{X} \mid p(\tilde{y}) = f(\tilde{x})\} \qquad (\tilde{x} \in \tilde{X}).$$

Received by the editors in October 2016 - In revised form in December 2016.
Communicated by K. Dekimpe, D.L. Gonçalves and P. Wong.
2010 *Mathematics Subject Classification :* Primary 54H25, 55M20, Secondary 55P42, 55R70.
*Key words and phrases :* Lefschetz index, Reidemeister trace, Nielsen number, multivalued map, coincidence index.

(To be accurate, Brown considered only the case of a map $f$ that factors through $p$ as a composition $\tilde{X} \to X \to X$.)

A systematic study of the fixed point theory of the pair $(f, p)$, by analogy with the theory of Vietoris fractions (see [13, Section 19]), was undertaken in [8], using the notation '$f/p$' for the pair considered as a fraction. The primary object of study, which we shall call the fixed-point set of $f/p$, is the coincidence set of the pair.

**Definition 1.1.** The *fixed-point set* of $f/p$ is the closed subspace

$$\mathrm{Fix}(f/p) = \{\tilde{x} \in \tilde{X} \mid f(\tilde{x}) = p(\tilde{x})\}$$

of $\tilde{X}$. It projects by $p$ onto the fixed subspace $\mathrm{Fix}(F) = \{x \in X \mid x \in F(x)\}$ of the multivalued map $F$.

The basic fixed-point index of $f/p$ was constructed in [8], and called there the *homotopy Lefschetz index*, as an element

$$h\text{-}L(f/p) \in \omega_0(\text{h-Fix}(f/p))$$

of the stable homotopy group of the homotopy fixed-point set of $f/p$.

**Definition 1.2.** The *homotopy fixed-point set* of $f/p$ is the subspace

$$\text{h-Fix}(f/p) = \{(\tilde{x}, \alpha) \mid \tilde{x} \in \tilde{X},\ \alpha : [0,1] \to X,\ \alpha(0) = p(\tilde{x}),\ \alpha(1) = f(\tilde{x})\},$$

of $\tilde{X} \times \mathrm{map}([0,1], X)$. Thus, each element is given by a point $\tilde{x}$ of $\tilde{X}$ and a (continuous) path from $p(\tilde{x})$ to $f(\tilde{x})$ in $X$. We write $\pi : \text{h-Fix}(f/p) \to \tilde{X}$ for the projection to the first factor. The fixed-point set is included as a subspace of the homotopy fixed-point set, $\mathrm{Fix}(f/p) \hookrightarrow \text{h-Fix}(f/p)$, by mapping to $\tilde{x}$ to $(\tilde{x}, \alpha)$ where $\alpha$ is the constant path at $p(\tilde{x}) = f(\tilde{x})$.

Now $\omega_0(\text{h-Fix}(f/p))$ is just the direct sum $\bigoplus_\gamma \mathbb{Z}[\gamma]$, of copies of $\mathbb{Z}$ indexed by the components $\gamma$ of $\text{h-Fix}(f/p)$, and we can decompose $h\text{-}L(f/p)$ as

$$h\text{-}L(f/p) = \sum_\gamma l_\gamma[\gamma],$$

where $l_\gamma \in \mathbb{Z}$ is zero for all but finitely many components $\gamma$. Classically, when $p$ is the identity $\tilde{X} = X \to X$, this is the Reidemeister trace of $f$, and the Nielsen number counts the components $\gamma$ such that $l_\gamma \neq 0$. We make the same definition for the fraction $f/p$.

**Definition 1.3.** The *Nielsen number* $N(f/p)$ of $f/p$ is the number of components $\gamma$ of $\text{h-Fix}(f/p)$ such that $l_\gamma$ is non-zero.

It is a consequence of [8, Proposition 3.5] that, when $F$ is an $n$-valued map, $N(f/p)$ is equal to Schirmer's Nielsen index $N(F)$. The main purpose of this note is to show that, in general, $N(f/p)$ is equal to the Nielsen number $N(\tilde{F})$ of the $n$-valued map $\tilde{F}$ studied by Brown. This will be deduced in Section 3 from the commutativity property of the fixed-point index.

In Section 2, expanding a brief account in [7, Section 5], we give a definition of the fixed-point index following the pattern of Dold's construction [10, 11, 12] of the index for single-valued maps on ENRs, treating the finite cover $p : \tilde{X} \to X$ as the 0-dimensional special case of a fibrewise smooth manifold $p : \tilde{X} \to X$ over a compact ENR $X$ with each fibre a closed manifold of some fixed dimension $m$.

Section 4 computes the index in a particular example involving projective spaces.

**Notation.** Given a real vector bundle $\xi$ over a space $X$ and a subspace $U$ of $X$, we shall use the superscript notation $U^{\xi}$ for the *Thom space of the restriction* $\xi|U$ of the vector bundle $\xi$ to the subspace $U$. Similar notation is used for virtual vector bundles. In particular, the Thom space $U^{-\xi}$ of the negative $-\xi|U$ is realized, by a trivialization $\xi \oplus \eta \cong X \times \mathbb{R}^k$ for some vector bundle $\eta$ over $X$, as the desuspension $\Sigma^{-k}(U^{\eta})$ of the Thom space of $\eta|U$.

## 2 Construction of the fixed-point index

We now fix a compact ENR $X$ and a fibrewise smooth manifold $p : \tilde{X} \to X$ with each fibre a closed smooth manifold of dimension $m$. Its fibrewise tangent bundle, which is an $m$-dimensional real vector bundle over $\tilde{X}$, will be denoted by $\tau(p)$.

An account of general fibrewise manifolds (or manifolds over a base) can be found in [1, Section 1] or [9, Part II, Section 11]. Many examples arise as follows. Suppose that $G$ is a compact Lie group, $M$ is a closed $G$-manifold of dimension $m$ and $P \to X$ is a principal $G$-bundle. Then $\tilde{X} = P \times_G M \to X$ is a fibrewise manifold. Its fibrewise tangent bundle is $P \times_G \tau M \to P \times_G M$, where $\tau M \to M$ is the tangent bundle of $M$. Every finite $n$-fold covering space arises in this way, with $G$ a finite group acting on a finite set $M$ of cardinality $n$.

Consider a (continuous) map $f : \tilde{X} \to X$. The *fixed-point set* and *homotopy fixed-point set* of $f/p$

$$\mathrm{Fix}(f/p) \subseteq \text{h-Fix}(f/p) \xrightarrow{\pi} \tilde{X}$$

are defined as in Definitions 1.1 and 1.2. There are associated multivalued maps $F : X \multimap X$ and $\tilde{F} : \tilde{X} \multimap \tilde{X}$, given by

$$F(x) = f(p^{-1}(x)) \text{ and } \tilde{F}(\tilde{x}) = p^{-1}(f(\tilde{x})), \quad (x \in X, \tilde{x} \in \tilde{X}).$$

Suppose that $U \subseteq \tilde{X}$ is an open subspace such that $U \cap \mathrm{Fix}(f/p)$ is compact. We shall first construct a *topological Lefschetz index*

$$t\text{-}L(f/p \,|\, U) \in \tilde{\omega}_0(U^{-\tau(p)})$$

in the stable homotopy group of the Thom space of the restriction $-\tau(p)|U$ of the virtual bundle $-\tau(p)$ to the subspace $U$.

We choose a fibrewise smooth embedding $j : \tilde{X} \hookrightarrow X \times F$, over $X$, for some Euclidean space $F$. The (fibrewise) normal bundle $\nu$ of $j$ satisfies $\tau(p) \oplus \nu = \tilde{X} \times F$ (up to homotopy). Using the Riemannian metric (on $F$) we can construct a fibrewise tubular neighbourhood of $j$ over $X$: $D(\nu) \hookrightarrow X \times F$. (Here $D(\nu)$ is the closed unit disc bundle in an appropriately scaled metric.) We also need to choose an embedding $i : X \hookrightarrow X' \subseteq E$ of the ENR $X$ as a retract of an open subspace $X'$ of a Euclidean space $E$, with a retraction $r : X' \to X$. Then construct the pullback $p' : \tilde{X}' = r^*\tilde{X} \to X'$ of $p : \tilde{X} \to X$ and the corresponding tubular neighbourhood $j' : D(\nu') \hookrightarrow X' \times F$, where $\nu' = r^*\nu$, of $\tilde{X}'$. The map $f : \tilde{X} \to X$ extends to a map $f' = i \circ f \circ r : \tilde{X}' \to X'$.

Let $U'$ denote the open subset $r^{-1}(U)$ of $\tilde{X}'$. Then $U' \cap \mathrm{Fix}(f'/p') = U \cap \mathrm{Fix}(f/p)$ is compact. To avoid complicating the notation, we shall regard $D(\nu')$, using $i$ and $j'$, as a subspace of $E \oplus F$ and $X'$ as a subspace of $E$.

The Lefschetz index will be represented by a pointed map

$$E^+ \wedge F^+ \to E^+ \wedge U^\nu ,$$

where the superscript '+' means the one-point compactification so that $E^+$ and $E^+ \wedge F^+ = (E \oplus F)^+$ are spheres with basepoint at infinity. The Thom space $U^\nu$ is the topological quotient of the disc bundle $D(\nu|U)$ by the unit sphere bundle $S(\nu|U)$.

By the compactness of $U' \cap \mathrm{Fix}(f'/p')$, we may choose an open neighbourhood $V$ in $\tilde{X}'$ with compact closure $\overline{V}$ such that

$$U' \cap \mathrm{Fix}(f'/p') \subseteq V \subseteq \overline{V} \subseteq U'$$

and then a real number $\epsilon > 0$ such that $\|p'(\tilde{x}) - f'(\tilde{x})\| \geqslant \epsilon$ for all $\tilde{x} \in \overline{V} - V$.

Now we can write down an explicit pointed map

$$\varphi : E^+ \wedge F^+ = (E \oplus F)^+ \to E^+ \wedge U^\nu = E^+ \wedge (D(\nu|U)/S(\nu|U))$$

as follows. A point $v$ in the closed subspace $D(\nu'|\overline{V})$ of $(E \oplus F)^+$ lies in the fibre of $\nu'$ over some point $\tilde{x} \in \overline{V}$: $v \in D(\nu'_{\tilde{x}})$. We define

$$\varphi(v) = [c_\epsilon(p'(\tilde{x}) - f'(\tilde{x})), r(v)] ,$$

where $c_\epsilon : E \to E^+$ is given by

$$c_\epsilon(u) = \begin{cases} (\epsilon^2 - \|u\|^2)^{-1/2} u & \text{if } \|u\| < \epsilon, \\ * \ (= \infty) & \text{otherwise.} \end{cases}$$

Notice that, if $\tilde{x} \in \overline{V} - V$, then $\|p'(\tilde{x}) - f'(\tilde{x})\| \geqslant \epsilon$, so that $\varphi(v) = *$. This means that $\varphi$ takes the value $*$ on the boundary $S(\nu'|\overline{V}) \cup D(\nu'|\overline{V} - V)$ of $D(\nu'|\overline{V})$ and we can extend $\varphi$ over $(E \oplus F)^+$ to take the value $*$ on the closed complement of the open unit disc bundle $B(\nu'|V)$.

Forming the class of $\varphi$ as a stable map from $F^+$ to $U^\nu$, we obtain the *topological Lefschetz index*

$$t\text{-}L(f/p \,|\, U) \in \tilde{\omega}_0(U^{-\tau(p)}) = \omega^0\{F^+; U^\nu\} .$$

(Here we use the notation $\omega^0\{A; B\}$ for the group of stable maps from a pointed space $A$ to a pointed space $B$.)

To construct the homotopy Lefschetz index, we choose $U$ to be an open neighbourhood of $\mathrm{Fix}(f/p)$ such that the straight line segment joining $p(\tilde{x})$ to $f(\tilde{x})$ lies in $X'$ for all $\tilde{x} \in U$ (that is, $(1-t)ip(\tilde{x}) + tif(\tilde{x}) \in X'$ for $0 \leqslant t \leqslant 1$). Then we have a map

$$U \to \text{h-Fix}(f/p) : \tilde{x} \mapsto (\tilde{x}, \alpha), \text{ where } \alpha(t) = r((1-t)ip(\tilde{x}) + tif(\tilde{x})),$$

extending the inclusion of $\text{Fix}(f/p)$ in $\text{h-Fix}(f/p)$.

We define the *homotopy Lefschetz index*, or *Nielsen-Reidemeister index*,

$$h\text{-}L(f/p) \in \tilde{\omega}_0(\text{h-Fix}(f/p)^{-\pi^*\tau(p)})$$

of $f/p$ to be the image of $t\text{-}L(f/p \mid U)$ under the induced map

$$\tilde{\omega}_0(U^{-\pi^*\tau(p)}) \to \tilde{\omega}_0(\text{h-Fix}(f/p)^{-\pi^*\tau(p)})\,.$$

It determines the global *topological Lefschetz index* of $f/p$

$$t\text{-}L(f/p) \in \tilde{\omega}_0(\tilde{X}^{-\tau(p)}),$$

which is defined as $\pi_*(h\text{-}L(f/p))$.

It is, of course, necessary to verify that the classes so constructed are independent of the choices made. This is best done by placing the definition in the wider context of fibrewise maps and simultaneously establishing the homotopy invariance of the index. When $p$ is the identity map $1 : \tilde{X} \to X$, so that $f$ is a map $X \to X$, the construction reduces to Dold's definition of the fixed-point index of a single-valued map as described, for example, in [9, 6]: we take $F$ to be the zero vector space and $j$ to be the identity map $1 : X \to X = X \times 0$. The verification proceeds as in this special case, and the details will be omitted here.

It is clear from the construction that the index $h\text{-}L(f/p)$ vanishes if the fixed-point set $\text{Fix}(f/p)$ is empty. The standard properties of the Lefschetz index (localization at the fixed-point set, additivity, homotopy invariance, multiplicativity) also follow essentially as in the classical case. Commutativity, which is more subtle, will be the subject of the next section. In the remainder of this section we look at two special features of the theory for multivalued maps.

**Trivial bundles.** We consider, first, the case in which the bundle $p : \tilde{X} \to X$ is trivial. Suppose that $p$ is the projection $X \times M \to X$, where $M$ is a closed smooth manifold of dimension $m$. The fibrewise tangent bundle $\tau(p)$ is the pullback of the tangent bundle $\tau M$ of $M$.

From $f$, which is now a map $f : X \times M \to X$, we can construct a fibrewise map

$$f^\sharp : X \times M \to X \times M, \qquad (x, y) \mapsto (f(x, y), y)$$

over the compact manifold $M$, that is, a family of maps $f^\sharp_y : X \to X$ parametrized by $y \in M$: $f^\sharp_y(x) = f(x, y)$.

The fibrewise fixed-point set

$$\text{Fix}_M(f^\sharp) = \{(x, y) \in X \times M \mid f^\sharp_y(x) = x\}$$

and homotopy fixed-point set $\text{h-Fix}_M(f^\sharp)$, defined as

$$\{((x, y), \alpha) \mid (x, y) \in X \times M,\ \alpha : [0, 1] \to X,\ \alpha(0) = x,\ \alpha(1) = f^\sharp_y(x)\}\,,$$

are transparently the same as the fixed-point sets $\mathrm{Fix}(f/p)$ and $\text{h-Fix}(f/p)$ of $f/p$. We shall show, by comparing the definitions, that the index $h\text{-}L(f/p)$ of the fraction coincides with the fibrewise fixed-point index $h\text{-}L_M(f^\sharp)$ of the fibrewise map.

The fibrewise homotopy Lefschetz index $h\text{-}L_M(f^\sharp)$ is an element of the group

$$\omega_M^0\{M \times S^0; \text{h-Fix}_M(f^\sharp)_{+M}\}$$

of fibrewise stable maps over $M$ from $M \times S^0$ to the fibrewise pointed space obtained by adjoining a disjoint basepoint to each fibre of $\text{h-Fix}_M(f^\sharp) \to M$. The two indices are related by the Poincaré-Atiyah duality isomorphism

$$\lambda_M : \omega_M^0\{M \times S^0; \text{h-Fix}_M(f^\sharp)_{+M}\} \xrightarrow{\cong} \tilde{\omega}_0(\text{h-Fix}(f^\sharp)^{-\pi^*\tau M}).$$

(See, for example, [7, Proposition 4.1] and the references given there.)

**Proposition 2.1.** *Suppose that $p : \tilde{X} = X \times M \to X$ is trivial, as described in the text. Then the fixed-point index $h\text{-}L(f/p)$ of $f/p$ is equal to the image under the duality isomorphism $\lambda_M$*

$$\omega_M^0\{M \times S^0; \text{h-Fix}_M(f^\sharp)_{+M}\} \xrightarrow{\cong} \tilde{\omega}_0(\text{h-Fix}(f/p)^{-\pi^*\tau M})$$

*of the fibrewise fixed-point index $h\text{-}L_M(f^\sharp)$ of the fibrewise map $f^\sharp$ determined by $f$.*

*Outline proof.* This will be verified by following through the explicit geometric definitions. We choose an embedding of $X$ as a retract $r : X' \to X$ of an open subspace $X' \subseteq E$ of a Euclidean space $E$, a smooth embedding of $M$ in a Euclidean space $F$, with normal bundle $\nu$, and a tubular neighbourhood $D(\nu) \hookrightarrow F$. This allows us to treat $X$ and $M$ as subspaces of $E$ and $F$, respectively.

Suppose that $U \subseteq X \times M$ is an open neighbourhood of $\mathrm{Fix}(f/p) = \mathrm{Fix}(f^\sharp)$. Let $V$ be in an open neighbourhood of $\mathrm{Fix}(f/p)$ in $X' \times M$ such that $\overline{V}$ is compact and contained in $(r \times 1)^{-1}U$. There is some $\epsilon > 0$ such that $\|x - f(r(x), y)\| \geqslant \epsilon$ for all $(x, y) \in \overline{V} - V$.

The topological fibrewise Lefschetz index $t\text{-}L_M(f^\sharp \mid U)$ is a stable map $M \times S^0 \to U_{+M}$ over $M$. (See, for example, [9, Part II, Section 6].) It is represented by the map

$$E^+ \times M \to (E^+ \times M) \wedge_M U_{+M}$$

sending $(x, y) \in \overline{V}$ to $[c_\epsilon(x - f(r(x), y)), (r(x), y)]$ and a point $(x, y)$ in the complement of $V$ in $E^+ \times M$ to the basepoint over $y \in M$. (Here, again, $U_{+M}$ is the fibrewise pointed space $U \sqcup M$ obtained by adjoining a basepoint in each fibre.

The duality isomorphism $\lambda_M$ is constructed in three steps by taking the smash product over $M$ with the identity map $\nu_M^+ \to \nu_M^+$ on the fibrewise one-point compactification of $\nu$ over $M$ to get a fibrewise stable map

$$(E^+ \times M) \wedge_M \nu_M^+ \to (E^+ \times M) \wedge_M U_{+M} \wedge_M \nu_M^+,$$

then collapsing fibrewise basepoints to a single point to get a map of pointed spaces

$$E^+ \wedge M^\nu \to E^+ \wedge U^\nu,$$

and finally composing with the product of the Pontryagin-Thom map $F^+ \to M^\nu$ with the identity on $E^+$ to produce an explicit map

$$E^+ \wedge F^+ \to E^+ \wedge U^\nu.$$

This is exactly the map defining $t\text{-}L(f/p \mid U)$. ∎

Of course, the bundle $\tilde{X} \to X$ is locally trivial and there is a similar local result. Suppose that $U \subseteq \tilde{X}$ is an open set such that $U \cap \mathrm{Fix}(f/p)$ is a component $N$ of the fibre $M$ of $p$ at a point $x_0 \in X$. Choose a local trivialization $p^{-1}(W) = W \times M \to W$ of $\tilde{X} \to X$ over some open neighbourhood $W$ of $x_0$. To keep the notation simple, we shall identify $p^{-1}(W)$ with $W \times M$. By replacing $U$ by a smaller neighbourhood of $N$ we may assume that it has the form $U = V \times N$, where $V \subseteq W$ is an open neighbourhood of $x_0$, and that $f(U) \subseteq W$. Then $f$ determines a fibrewise map $f^\sharp : V \times N \to W \times N$ over $N$ with fixed-point set $\mathrm{Fix}(f^\sharp) = \{x_0\} \times N$.

The argument outlined above expresses $t\text{-}L(f/p \mid U)$ as the image of the fibrewise topological Lefschetz index $t\text{-}L_N(f^\sharp \mid U)$ under the duality isomorphism

$$\omega^0_N\{N \times S^0; V_+ \times N\} = \omega^0\{N_+; V_+\} \xrightarrow{\cong} \tilde{\omega}_0(V_+ \wedge N^{-\tau N}).$$

Because $X$ is an ENR, there is a smaller open neighbourhood $V_0 \subseteq V$ of $x_0$ inside $V$ such that the inclusion $V_0 \hookrightarrow V$ is homotopic, through a homotopy inside $V$ that fixes $x_0$, to the constant map at $x_0$. It follows that $t\text{-}L_N(f^\sharp \mid U)$ is the image of the fibrewise Lefschetz index

$$L_N(f^\sharp \mid U) \in \omega^0(N) = \omega^0\{N_+; S^0\} = \tilde{\omega}_0(N^{-\tau N})$$

under the map induced by the inclusion of $S^0$ as $\{x_0\}_+$ in $V_+$. We shall refer to this class $L_N(f^\sharp \mid U)$ as a *local index*. It contributes to the index $h\text{-}L(f/p)$ through the homomorphism

$$k_* : \tilde{\omega}_0(N^{-\tau N}) \to \tilde{\omega}_0(\text{h-Fix}(f/p)^{-\pi^*\tau(p)})$$

induced by the inclusion $k : N \hookrightarrow \mathrm{Fix}(f/p) \hookrightarrow \text{h-Fix}(f/p)$.

**Proposition 2.2.** *Consider a general (so locally trivial) fibrewise manifold $p : \tilde{X} \to X$. Suppose that* $\mathrm{Fix}(f/p)$ *is the disjoint union of a finite number of connected components $N_i$, $i \in I$, of the fibres over points $x_i \in X$. Let $k_i : N_i \hookrightarrow$* $\mathrm{Fix}(f/p) \hookrightarrow \text{h-Fix}(f/p)$ *denote the inclusion and let $L_i \in \omega^0(N_i) = \tilde{\omega}_0(N_i^{-\tau N_i})$ be the local Lefschetz index described above. Then*

$$h\text{-}L(f/p) = \sum_{i \in I} (k_i)_*(L_i).$$

*Proof.* It suffices to show that, for any neighbourhood $U$ of $\mathrm{Fix}(f/p)$, the index $t\text{-}L(f/p \mid U)$ is the sum of terms $(k_i^U)_*(L_i)$, ($i \in I$), where $k_i^U : N_i \hookrightarrow U$ is the inclusion. This follows, by the additivity of the index, from the discussion above applied to disjoint neighbourhoods $U_i$ of $N_i$ in $U$. ∎

**Corollary 2.3.** *Suppose that $p : \tilde{X} \to X$ is a finite n-fold cover and that $\#F(x) = n$ for all $x \in X$. Then h-$L(f/p)$ coincides with the fixed-point index of the n-valued map $F$ defined by Schirmer in [16].*

*Proof.* Schirmer's definition proceeds by reduction, through a homotopy, to the case in which $\mathrm{Fix}(f/p)$ is a finite set. In that case, we can apply Proposition 2.2 to express the index as a sum of local Lefschetz indices as in [16]. The assertion then follows from the homotopy invariance of the index. ∎

**Smooth fibre bundles.** We consider next the case in which $X$ is a closed manifold and $p : \tilde{X} \to X$ is a smooth fibre bundle. The fixed-point set $\mathrm{Fix}(f/p)$ is just the *coincidence set* $\{\tilde{x} \in \tilde{X} \mid p(\tilde{x}) = f(\tilde{x})\}$ of $p$ and $f$, and we shall show that h-$\mathrm{Fix}(f/p)$ is exactly the homotopy coincidence index, in the terminology of [7], of $p$ and $f$. We form the trivial bundle $E = \tilde{X} \times X \to \tilde{X}$ over $\tilde{X}$ with a preferred *null section* $z$, $z(\tilde{x}) = (\tilde{x}, p(\tilde{x}))$, and section $s$, $s(\tilde{x}) = (\tilde{x}, f(\tilde{x}))$, associated with $f$. The pullback $\nu = z^*\tau_{\tilde{X}}E$ of the fibrewise tangent bundle of $E$ over $\tilde{X}$ is identified with the tangent bundle $\tau X$ of $X$.

The coincidence set is called in [7] the *null set* $\mathrm{Null}(s) = \{\tilde{x} \in \tilde{X} \mid s(\tilde{x}) = z(\tilde{x})\}$ of the section $s$. The *homotopy null set* h-$\mathrm{Null}(s)$, defined in [7, Definition 2.3] as the space of pairs $(\tilde{x}, \alpha)$ where $\tilde{x} \in \tilde{X}$ is a point of the base and $\alpha$ is a path in the fibre over $\tilde{x}$ from $z(\tilde{x})$ to $s(\tilde{x})$, is exactly h-$\mathrm{Fix}(f/p)$.

The *homotopy Euler index* of $s$ is constructed in [7, Definition 2.4] as an element

$$h\text{-}\gamma(s) \in \omega^0_{\tilde{X}}\{\tilde{X} \times S^0; \text{h-Null}(s)^{\pi^*\nu}_{\tilde{X}}\},$$

which is the (asymmetric) *homotopy coincidence index* of $p$ and $f$. Again we have a duality isomorphism

$$\lambda_{\tilde{X}} : \omega^0_{\tilde{X}}\{\tilde{X} \times S^0; \text{h-Null}(s)^{\pi^*\nu}_{\tilde{X}}\} \xrightarrow{\cong} \tilde{\omega}_0(\mathrm{Null}(s)^{\pi^*\nu - \pi^*\tau\tilde{X}}).$$

Now the tangent bundle $\tau\tilde{X}$ of the total space $\tilde{X}$ of the bundle is identified, up to homotopy, with $p^*\tau X \oplus \tau(p)$. This allows us to substitute $-\pi^*\tau(p)$ for $\pi^*\nu - \pi^*\tau\tilde{X}$.

**Proposition 2.4.** *Suppose $p : \tilde{X} \to X$ is a smooth fibre bundle, as described in the text. Then h-$L(f/p)$ is the image under the isomorphism*

$$\lambda_{\tilde{X}} : \omega^0_{\tilde{X}}\{\tilde{X} \times S^0; \text{h-Null}(s)^{\pi^*\nu}_{\tilde{X}}\} \xrightarrow{\cong} \tilde{\omega}_0(\text{h-Fix}(f/p)^{-\pi^*\tau(p)})$$

*of the homotopy coincidence index h-$\gamma(s)$ of $p$ and $f$.*

*Proof.* This is again established by a careful comparison of the two definitions. See [7, Proposition 5.4]. ∎

When $f$ is smooth and transverse to $p$, that is, when $p \times f : \tilde{X} \to X \times X$ is transverse to the diagonal, we can use Koschorke's definition of the coincidence index from [14, Section 4] to give a geometric description of *h-$L(f/p)$*. The set $\mathrm{Fix}(f/p)$ is then an $m$-dimensional closed submanifold, $N$ say, of $\tilde{X}$, with tangent bundle $\tau N$ identified with the restriction of $\tau(p)$.

**Corollary 2.5.** *Suppose that $p : \tilde{X} \to X$ is a smooth fibre bundle and that $f : \tilde{X} \to X$ is smooth and transverse to $p$. Then the inclusion of $N = \mathrm{Fix}(f/p)$ in* h-Fix$(f/p)$

$$\tilde{\omega}_0(N^{-\tau N}) \to \tilde{\omega}_0(\text{h-Fix}(f/p)^{-\pi^*\tau(p)})$$

*maps the fundamental class $[N]$ of $N$ to h-$L(f/p)$.*

*Proof.* The interpretation of $\lambda_{\tilde{X}}(\gamma(s))$ as the class represented by $N$ is explained in [7, Proposition 4.6]. ∎

## 3 Commutativity

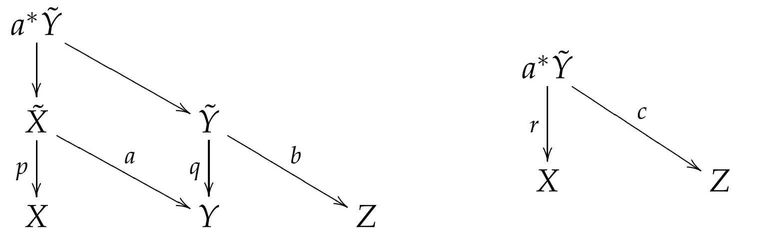
We begin by introducing an informal category of fractions in which the objects are compact ENRs and a morphism $a/p$ from $X$ to $Y$ is a pair $(a, p)$ consisting of a fibrewise manifold $p : \tilde{X} \to X$, with fibres closed manifolds of some dimension, $m$ say, and a map $a : \tilde{X} \to Y$. (To be formal, a morphism should be specified by such a pair up to equivalence, so that the morphisms form a set.) Associated with the fraction $a/p$ is the multivalued map $A : X \multimap Y$ given by

$$A(x) = \{a(\tilde{x}) \mid \tilde{x} \in \tilde{X},\ p(\tilde{x}) = x\} \quad \text{(that is, } a(p^{-1}(x))).$$

Composition is defined as follows. Suppose that $Z$ is another compact ENR and that $b/q$ is a morphism from $Y$ to $Z$ prescribed by a fibrewise manifold $q : \tilde{Y} \to Y$ with fibres of dimension $n$ and a map $b : \tilde{Y} \to Z$. We form the pullback

$$a^*\tilde{Y} = \{(\tilde{x}, \tilde{y}) \in \tilde{X} \times \tilde{Y} \mid a(\tilde{x}) = q(\tilde{y})\},$$

together with maps $r : a^*\tilde{Y} \to X$, specifying a fibrewise manifold of dimension $m + n$, and $c : a^*\tilde{Y} \to Z$ given by $r(\tilde{x}, \tilde{y}) = p(\tilde{x})$ and $c(\tilde{x}, \tilde{y}) = b(\tilde{y})$. The *composition $b/q \circ a/p$* is defined to be $c/r$, as illustrated in the following diagram.




If $B : Y \multimap Z$ and $C : X \multimap Z$ are the multivalued maps associated with $b/q$ and $c/r$, then $C = B \circ A$, that is,

$$C(x) = \bigcup_{y \in A(x)} B(y)\,.$$

The identity morphism on $X$ is the fraction $1/1$ given by the fibrewise manifold $1 : \tilde{X} = X \to X$ of dimension 0 and the identity map $1 : \tilde{X} = X \to X$.

The index was defined in Section 2 for endomorphisms in this category. Turning to a discussion of commutativity, we now suppose that $Z = X$ so that we can form the compositions $b/q \circ a/p$, which is an endomorphism of $X$, and $a/p \circ b/q$, an endomorphism of $Y$. There is also an associated endomorphism of $X \times Y$ given by the fibrewise manifold $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$ of dimension $m + n$ and the map $\tau \circ (a \times b) : \tilde{X} \times \tilde{Y} \to X \times Y$, $(\tilde{x}, \tilde{y}) \mapsto (b(\tilde{y}), a(\tilde{x}))$, where $\tau : Y \times X \to X \times Y$ interchanges the two factors.

**Proposition 3.1.** *There are homotopy equivalences*

$$\text{h-Fix}(b/q \circ a/p) \xrightarrow{\sim} \text{h-Fix}(\tau \circ (a \times b)/(p \times q)) \xleftarrow{\sim} \text{h-Fix}(a/p \circ b/q)$$

*under which the fixed-point indices h-$L(b/q \circ a/p)$, h-$L(\tau \circ (a \times b)/(p \times q))$ and h-$L(a/p \circ b/q)$ coincide.*

*Proof.* The homotopy fixed-point set

$$\text{h-Fix}(b/q \circ a/p) = \{(\tilde{x}, \tilde{y}, \alpha) \mid \tilde{x} \in \tilde{X},\ \tilde{y} \in \tilde{Y},\ \alpha : [0,1] \to X,$$
$$\alpha(0) = p(\tilde{x}),\ \alpha(1) = b(\tilde{y}),\ a(\tilde{x}) = q(\tilde{y})\}$$

is included in

$$\text{h-Fix}(\tau \circ (a \times b)/(p \times q)) = \{(\tilde{x}, \tilde{y}, \alpha, \beta) \mid \tilde{x} \in \tilde{X},\ \tilde{y} \in \tilde{Y},\ \alpha : [0,1] \to X,$$
$$\beta : [0,1] \to Y,\ \alpha(0) = p(\tilde{x}),\ \alpha(1) = b(\tilde{y}),\ \beta(0) = q(\tilde{y}),\ \beta(1) = a(\tilde{x})\}$$

as the subspace of quadruples $(\tilde{x}, \tilde{y}, \alpha, \beta)$ with $\beta$ a constant path. Both contain the fixed subspace (abbreviated for the purposes of this proof to 'Fix')

$$\text{Fix} = \text{Fix}(b/q \circ a/p) = \text{Fix}(\tau \circ (a \times b)/(p \times q)) =$$
$$\{(\tilde{x}, \tilde{y}) \mid a(\tilde{x}) = q(\tilde{y}),\ b(\tilde{y}) = p(\tilde{x})\} \subseteq \tilde{X} \times \tilde{Y}$$

with both $\alpha$ and $\beta$ constant.

Since $q : \tilde{Y} \to Y$ is a fibration, we can choose a lifting function

$$\ell : \{(\tilde{y}, \beta) \mid \tilde{y} \in \tilde{Y},\ \beta : [0,1] \to Y,\ \beta(0) = q(\tilde{y})\} \to \text{map}([0,1], \tilde{Y})$$

such that $\tilde{\beta} = \ell(\tilde{y}, \beta)$ is a path lifting $\beta$ with $\tilde{\beta}(0) = \tilde{y}$.

Following [8, Proposition 3.10], we construct a deformation retraction of h-Fix$(\tau \circ (a \times b)/(p \times q))$ to the subspace h-Fix$(b/q \circ a/p)$:

$$(\tilde{x}, \tilde{y}, \alpha, \beta) \mapsto (\tilde{x}, \tilde{\beta}(t), \alpha_t, \beta_t), \qquad (0 \leqslant t \leqslant 1),$$

where $\tilde{\beta} = \ell(\tilde{y}, \beta)$ and

$$\alpha_t(u) = \begin{cases} \alpha((1+t)u) & \text{if } 0 \leqslant u \leqslant (1+t)^{-1}, \\ b(\tilde{\beta}((1+t)u - 1)) & \text{if } (1+t)^{-1} \leqslant u \leqslant 1, \end{cases}$$

and $\beta_t(u) = \beta(t + (1-t)u)$ for $0 \leqslant u \leqslant 1$.

This establishes that inclusion

$$\text{h-Fix}(b/q \circ a/p) \hookrightarrow \text{h-Fix}(\tau \circ (a \times b)/(p \times q))$$

is a homotopy equivalence. Notice that the pullback of the fibrewise tangent bundle $\tau(p \times q)$ restricts to the pullback of the fibrewise tangent bundle of $a^*\tilde{Y} \to X$: the fibre at $(\tilde{x}, \tilde{y}, \alpha, \beta)$ is equal to $\tau_{\tilde{x}}(p) \oplus \tau_{\tilde{y}}(q)$.

To prove commutativity, we shall essentially follow Dold's argument in the classical case [10] and reduce to working on subspaces of Euclidean spaces.

It suffices to show that, for any 'small' open neighbourhood $U$ of the fixed-point set Fix in $\tilde{X} \times \tilde{Y}$, the inclusion

$$\tilde{\omega}_0((U \cap a^*\tilde{Y})^{-(\tau(p)\oplus\tau(q))}) \to \tilde{\omega}_0(U^{-(\tau(p)\oplus\tau(q))})$$

maps $t\text{-}L(b/q \circ a/p \mid U \cap a^*\tilde{Y})$ to $t\text{-}L(\tau \circ (a \times b)/(p \times q) \mid U)$.

As in the construction of the index, we first choose embeddings $X \hookrightarrow X' \hookrightarrow E_X$ and $Y \hookrightarrow Y' \hookrightarrow E_Y$ of $X$ and $Y$ as retracts $r : X' \to X$ and $s : Y' \to Y$ of open subspaces of Euclidean spaces $E_X$ and $E_Y$, and form the pullbacks $\tilde{X}' = r^*\tilde{X}$, $\tilde{Y}' = s^*\tilde{Y}$. Then $a$ and $b$ are extended, using the retractions, to maps $a' : \tilde{X}' \to Y$ and $b' : \tilde{Y}' \to X$. Thus

$$(a')^*\tilde{Y} = \{(\tilde{x}, \tilde{y}) \in \tilde{X}' \times \tilde{Y} \mid a'(\tilde{x}) = q(\tilde{y})\} \subseteq \tilde{X}' \times \tilde{Y}.$$

Now choose an open neighbourhood $V$ of

$$\text{Fix} = \{(\tilde{x}, \tilde{y}) \in \tilde{X}' \times \tilde{Y} \mid a'(\tilde{x}) = q(\tilde{y}),\, b(\tilde{y}) = p'(\tilde{x})\}$$

in $(a')^*\tilde{Y}$ such that $\overline{V}$ is compact and contained in $U' = (r \times s)^{-1}(U)$.

Writing $D_\delta(E_Y)$ for a closed disc of radius $\delta > 0$ centred at 0 in $E_Y$, we can use the compactness of $Y$ to choose $\delta$ such that $y + w \in Y'$ for all $(y, w) \in Y \times D_\delta(E_Y) = Z$, say. Then we have a map

$$(y, w) \mapsto y + w : Z = Y \times D_\delta(E_Y) \to Y'$$

and can pull back $q' : \tilde{Y}' \to Y'$ to get a fibrewise manifold $\tilde{Z} \to Z$ over $Z$. Since $D_\delta(E_Y)$ is contractible, there is a fibrewise equivalence

$$\theta : \tilde{Y} \times D_\delta(E_Y) \to \tilde{Z}$$

of fibrewise manifolds over $Y \times D_\delta(E_Y) = Z$ with $\theta(\tilde{y}, 0) = (\tilde{y}, 0)$ for $\tilde{y} \in \tilde{Y}$.

The equivalence $\theta$ provides a family of diffeomorphisms

$$\theta_{y,w} : \tilde{Y}_y \to \tilde{Y}'_{y+w}, \qquad (y, w) \in Z,$$

such that $q'(\theta_{y,w}(\tilde{y})) = y + w$ and $\theta_{y,0}(\tilde{y}) = \tilde{y}$. It is convenient to be slightly imprecise and abbreviate $\theta_{y,w}(\tilde{y})$ to $\theta(\tilde{y}, w)$. This allows us to write down a crucial map

$$\psi : \overline{V} \times D_\delta(E_Y) \to \tilde{X}' \times \tilde{Y}', \qquad ((\tilde{x}, \tilde{y}), w) \mapsto (\tilde{x}, \theta(\tilde{y}, w)).$$

Notice that, for $(\tilde{x}, \tilde{y}) \in \text{Fix}$, $\psi$ maps $((\tilde{x}, \tilde{y}), 0)$ to $(\tilde{x}, \tilde{y})$. The image of the open neighbourhood $V \times B_\delta(E_Y)$ of $\text{Fix} \times \{0\}$ is an open neighbourhood $W$ of Fix in $\tilde{X}' \times \tilde{Y}'$ with closure $\overline{W}$ equal $\psi(\overline{V} \times D_\delta(E_Y))$.

We shall use the neighbourhood $V$ for the description of the topological Lefschetz index of $b/q \circ a/p$ and the neighbourhood $W$ for $\tau \circ (a \times b)/(p \times q)$.

The index $t\text{-}L(b/q \circ a/p) \mid U \cap a^*\tilde{Y}$ is determined by the map

$$g : \overline{V} \to E_X, \quad (\tilde{x}, \tilde{y}) \mapsto p'(\tilde{x}) - b(\tilde{y}),$$

and the index $t\text{-}L(\tau \circ (a \times b) \mid U)$ by the map

$$h_0 : \overline{W} \to E_X \times E_Y$$

taking $\psi((\tilde{x}, \tilde{y}), w)$, for $((\tilde{x}, \tilde{y}), w) \in \overline{V} \times D_\delta(E_Y)$, to

$$(p'(\tilde{x}) - b'(\theta(\tilde{y}, w)), q'(\theta(\tilde{y}, w)) - a'(\tilde{x})) = (p'(\tilde{x}) - b'(\theta(\tilde{y}, w)), w).$$

In order to relate $h_0$ to $g$, we deform this map by a homotopy $h_t : \overline{W} \to E_X \times E_Y$:

$$\psi((\tilde{x}, \tilde{y}), w) \mapsto (p'(\tilde{x}) - (1-t)b'(\theta(\tilde{y}, w)) - tb(\tilde{y}), w), \qquad 0 \leqslant t \leqslant 1,$$

which is nowhere zero on the boundary $\overline{W} - W$. (For, if $w = 0$, we have $\theta(\tilde{y}, w) = \tilde{y}$, so that $p'(\tilde{x}) - (1-t)b'(\theta(\tilde{y}, w)) - tb(\tilde{y}) = p'(\tilde{x}) - b(\tilde{y})$, which is non-zero on $\overline{V} - V$.) By compactness, there is an $\epsilon > 0$ such that $\|h_t((\tilde{x}, \tilde{y}), w)\| \geqslant \epsilon$ for all points $((\tilde{x}, \tilde{y}), w)$ in $\overline{W} - W$.

We can thus use the map $h_1$, taking $\psi((\tilde{x}, \tilde{y}), w)$ to $(g(\tilde{x}, \tilde{y}), w)$, rather than $h_0$, to realize the index.

To complete the construction we need to choose fibrewise embeddings $\tilde{X} \hookrightarrow X \times F_p : \tilde{x} \mapsto (\tilde{x}, j_p(\tilde{x}))$, $\tilde{Y} \hookrightarrow Y \times F_q : \tilde{y} \mapsto (\tilde{y}, j_q(\tilde{y}))$, and tubular neighbourhoods $D(\nu_p) \subseteq X \times F_p$, $D(\nu_q) \subseteq Y \times F_q$, which we pull back to tubular neighbourhoods of $\tilde{X}'$ and $\tilde{Y}'$.

The map $h_1$ and the embedding of the disc bundle $D(\nu_p \oplus \nu_q|\overline{W})$ into $(E_X \oplus E_Y) \times (F_p \oplus F_q)$ determine the index of $\tau \circ (a \times b)/(p \times q)$.

Using $\theta$ we can pull back the embedding of $\tilde{Y}'$ into $Y' \times F_q$ to an embedding of $(a')^*\tilde{Y} \times D_\delta(E_Y)$ into $\tilde{X}' \times D_\delta(E_Y) \times F_q$ over $\tilde{X}' \times D_\delta(E_Y)$. Combining this with the embedding of $\tilde{X}'$ into $X' \times F_p$, we obtain a fibrewise embedding of $(a')^*\tilde{Y} \times D_\delta(E_Y)$ into $X' \times D_\delta(E_Y) \times (F_p \oplus F_q)$ over $X' \times D_\delta(E_Y)$. Then we can include $X'$ into $E_X$ and $D(E_Y)$ into $E_Y$ to embed $(a')^*\tilde{Y} \times D_\delta(E_Y)$ into $(E_X \oplus E_Y) \times (F_p \oplus F_q)$. A tubular neighbourhood of this embedding and the map $g$ on $\overline{V}$ then determine the index of $(b/q) \circ (a/p)$, or, to be precise, $g \times 1$ on $\overline{V} \times D_\delta(E_Y)$ determines the index of $((b/q) \circ (a/p)) \times (z/1)$, where $z/1$ is the fraction given by the identity and zero maps $1, z : D_\delta(E_Y) \to D_\delta(E_Y)$.

The maps $h_1$ on $\overline{W}$ and $g \times 1$ on $\overline{V} \times D_\delta(E_Y)$ correspond via $\psi$. However, the embeddings into $(E_X \oplus E_Y) \times (F_p \oplus F_q)$ do not correspond: they map $((\tilde{x}, \tilde{y}), w) \in \overline{V} \times D_\delta(E_Y)$ to $(p'(\tilde{x}),\ a'(\tilde{x}) + w,\ j_p(\tilde{x}),\ j_p(\tilde{y}))$ and $(p'(\tilde{x}),\ w,\ j_p(\tilde{x}),\ j_p(\tilde{y}))$, respectively. But the two are connected, as in the classical proof [10], by a second homotopy:

$$((\tilde{x}, \tilde{y}), w) \mapsto (p'(\tilde{x}),\ (1-t)a'(\tilde{x}) + w,\ j_p(\tilde{x}),\ j_q(\tilde{y})), \qquad 0 \leqslant t \leqslant 1.$$

We conclude that the topological index of $\tau \circ (a \times b)/(p \times q)$ on $\overline{W}$ coincides with the topological index of $(b/q \circ a/p) \times (z/1)$ on $\overline{V} \times D_\delta(E_Y)$. But taking the product with the index of $z/1$ amounts to suspending by the identity map on $E_Y^+$. So this completes the proof of the commutativity property of the index.

Another, rather more elementary, proof for the special case of finite covers can be found in [8]. ■

Let us now revert to the setting of Section 2 in which $p : \tilde{X} \to X$ is a fibrewise manifold and $f : \tilde{X} \to X$ is a map. We shall apply Proposition 3.1 to the case in which $Y$ is equal to $\tilde{X}$, $q : \tilde{Y} = \tilde{X} \to Y = \tilde{X}$ and $a : \tilde{X} \to Y = \tilde{X}$ are both the identity on $\tilde{X}$, and $b : \tilde{Y} = \tilde{X} \to X$ is the given map $f$. Then $b/q \circ a/p = f/1 \circ 1/p$ is just $f/p$. The composition $a/p \circ b/q = 1/p \circ f/1$ is given by the fibrewise manifold

$$f^*\tilde{X} = \{(\tilde{x}, \tilde{y}) \in \tilde{X} \times \tilde{X} \mid f(\tilde{x}) = p(\tilde{y})\} \to \tilde{X}, \quad (\tilde{x}, \tilde{y}) \mapsto \tilde{x}$$

and map $(\tilde{x}, \tilde{y}) \mapsto \tilde{y} : f^*\tilde{X} \to \tilde{X}$, and it determines the multivalued map $\tilde{F} : \tilde{X} \multimap \tilde{X}$ given by $\tilde{F}(\tilde{x}) = \{\tilde{y} \in \tilde{X} \mid f(\tilde{x}) = p(\tilde{y})\}$.

Now suppose that $p : \tilde{X} \to X$ is a finite $n$-fold cover. The equality $h\text{-}L(1/p \circ f/1) = h\text{-}L(f/1 \circ 1/p) = h\text{-}L(f/p)$, together with Corollary 2.3 applied to the fraction $1/p \circ f/1$, demonstrates that Schirmer's fixed-point index of the $n$-valued map $\tilde{F}$ constructed by Brown coincides with the fixed-point index of $f/p$. Equality of the Nielsen numbers is an immediate consequence.

**Proposition 3.2.** *Suppose that $p : \tilde{X} \to X$ is an $n$-fold covering space and that $f : \tilde{X} \to X$ is a map. Then the Nielsen number as defined by Schirmer, $N(\tilde{F})$, of the $n$-valued map $\tilde{F} : \tilde{X} \multimap \tilde{X}$, $\tilde{F}(\tilde{x}) = p^{-1}(f(\tilde{x}))$, is equal to the Nielsen number, $N(f/p)$, of $f/p$.* ∎

## 4 Projective spaces

We shall apply Proposition 2.2 to compute the fixed-point indices in some examples involving projective spaces.

First of all, we take $X$ to be the quaternionic projective space $\mathbb{H}P^n = \mathbb{H}P(\mathbb{H}^{n+1})$ of 1-dimensional (left) $\mathbb{H}$-subspaces of $\mathbb{H}^{n+1}$ ($n \geqslant 1$), $\tilde{X}$ to be the sphere $S^{4n+3} = S(\mathbb{H}^{n+1})$ and $p : \tilde{X} \to X$ to be the principal $\mathrm{Sp}(1)$-bundle $S(\mathbb{H}^{n+1}) \to \mathbb{H}P(\mathbb{H}^{n+1})$. The fibrewise tangent bundle $\tau(p)$ is trivial with fibre the Lie algebra $\mathbb{R}\mathrm{i} \oplus \mathbb{R}\mathrm{j} \oplus \mathbb{R}\mathrm{k}$ of $\mathrm{Sp}(1) \subseteq \mathbb{H}$. In coordinates, $p(z_0, \ldots, z_n) = [z_0, \ldots, z_n]$, where $z_i \in \mathbb{H}$, $\sum |z_i|^2 = 1$.

We shall look at the case in which $f$ admits a lift to a map $\tilde{f} : \tilde{X} \to \tilde{X}$.

**Proposition 4.1.** *Let $p : \tilde{X} \to X$ be the principal* $\mathrm{Sp}(1)$*-bundle $S^{4n+3} \to \mathbb{H}P^n$, where $n \geqslant 1$, and suppose that $f : \tilde{X} \to X$ is equal to $p \circ \tilde{f}$, where $\tilde{f} : S^{4n+3} \to S^{4n+3}$ is a map of degree $d$. Then the topological Lefschetz index $t\text{-}L(f/p)$ in*

$$\omega_3(S^{4n+3}) = \omega_3(*) = (\mathbb{Z}/24\mathbb{Z})\nu,$$

*where $\nu$ is the standard generator represented by the framed manifold* $\mathrm{Sp}(1)$ *with the left invariant framing, is equal to $(1 + nd)\nu$.*

*Proof.* Let $r, s \geqslant 1$ be positive integers such that $r - s = d$. We shall make the computation using the specific map $f$ given by

$$f(z_0, \ldots, z_n) = [z_0^r \overline{z_0}^s, a_1 z_1, \ldots, a_n z_n],$$

where $a_i \in \mathbb{R}$ and $1 < a_1 < a_2 < \ldots < a_n$. It is elementary to check that the fixed-point set is the union of $n+1$ fibres $N_i$, $i = 0, \ldots, n$, over the points $x_i = [e_i]$, where $e_0, \ldots, e_n$ is the standard basis of $\mathbb{H}^{n+1}$.

The fixed submanifold $N_0$ is non-degenerate. To calculate the local index $L_0 \in \omega^0(N_0)$ we can, therefore, linearize and reduce to consideration of the map, $f_0^\sharp$ say,

$$S(\mathbb{H}) \times \mathbb{H}^n \to \mathbb{H}^n : (z_0, (v_1, \ldots, v_n)) \mapsto (a_1 z_0^{1-d} v_1, \ldots, a_n z_0^{1-d} v_n).$$

The index is determined by $1 - f_0^\sharp$:

$$S(\mathbb{H}) \times (\mathbb{H}^n - \{0\}) \to \mathbb{H}^n - \{0\}, \quad (z_0, (v_i)) \mapsto ((1 - a_i z_0^{1-d}) v_i),$$

which is homotopic to the map $l_0$ taking $(z_0, (v_i))$ to $(z_0^{1-d} v_i)$. Thus, $L_0$ is the image under the $J$-homomorphism

$$J : KO^{-1}(N_0) \to \omega^0(N_0)^\times$$

(to the group of units in the stable cohomotopy ring $\omega^0(N_0)$) of the class determined by $l_0$. The relevant groups are $KO^{-1}(S^3) = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ and $\omega^0(S^3) = \mathbb{Z} \oplus (\mathbb{Z}/24\mathbb{Z})\nu$. The class $[l_0]$ is equal to $(0, n(d-1)) \in KO^{-1}(S^3)$ (with an appropriate choice of signs for the generators) and maps under $J$ to $1 + n(d-1)\nu$. (Care is needed to distinguish the generators $\nu$ and $-\nu$.)

A similar, but easier, computation for $L_i$, $i > 0$, shows that $L_i = 1 \in \omega^0(N_i)$.

The composition of the duality isomorphism determined by the left invariant framing of $S^3 = \mathrm{Sp}(1)$ and the homomorphism induced by projection to a point $*$:

$$\mathbb{Z} \oplus (\mathbb{Z}/24\mathbb{Z})\nu = \omega^0(S^3) \xrightarrow{\cong} \omega_3(S^3) \to \omega_3(*) = (\mathbb{Z}/24\mathbb{Z})\nu$$

maps 1 to $\nu$ and $\nu$ to $\nu$. Hence the sum of the $n+1$ local indices $L_i$ maps to $(1 + n(d-1))\nu + n\nu = (1 + nd)\nu \in \omega_3(*)$. By Proposition 2.2, this is equal to the global index in $\omega_3(S^{4n+3}) = \omega_3(*)$.

(For the degree zero case $d = 0$, we can also take $f$ to be the constant map at $[e_0]$. The fixed-point set is a single fibre $N_0$ at $x_0$ and $L_0 = 1$. The single term $L_0$ gives the global index as $\nu \in \omega_3(*)$. This confirms the sign of the generator $\nu$ in the general calculation above.) ∎

In this example, the homotopy fixed-point index contains no more information than the topological index. For a map $f$ with $S(\mathbb{H}e_0) \subseteq \mathrm{Fix}(f/p)$, the projection $\pi : \text{h-Fix}(f/p) \to S(\mathbb{H}^{n+1})$ is locally fibre homotopy trivial with fibre $\Phi = \Omega(\mathbb{H}P(\mathbb{H}^{n+1}), x_0)$ and is actually a product $S(\mathbb{H}e_0) \times \Phi$ over $S(\mathbb{H}e_0)$. The map $\Phi \to \mathrm{Sp}(1)$ coming from the fibration induces an isomorphism $\pi_i(\Phi) \to \pi_i(\mathrm{Sp}(1))$ for $i < 4n+2$. It follows that $\omega_3(\text{h-Fix}(f/p)) = \omega_3(\mathrm{Sp}(1)) = (\mathbb{Z}/24\mathbb{Z}) \oplus \mathbb{Z}$ and, because the inclusion $S(\mathbb{H}e_0) \hookrightarrow S(\mathbb{H}^{n+1})$ is null homotopic, that $h\text{-}L(f/p)$ lies in the $\mathbb{Z}/24\mathbb{Z}$ summand.

*Remark* 4.2. If $d \neq 1$, the map $\tilde{f} : S(\mathbb{H}^{n+1}) \to S(\mathbb{H}^{n+1})$ has a fixed point and thus h-Fix$(f/p)$ is non-empty. If $d = 1$, but $n+1$ is not divisible by 24, the computation of the index shows that h-Fix$(f/p)$ is non-empty, that is, that there exists a point $v \in S(\mathbb{H}^{n+1})$ such that $\tilde{f}(v) \in \mathbb{H}v$. (Note, however, that there may be a map $f : \tilde{X} \to X$ with $\mathrm{Fix}(f/p) = \emptyset$ that does not admit a lift $\tilde{f}$. For example, if $n = 1$, so that $\mathbb{H}P^4 = S^4$, the antipodal involution on $S^4$ provides a map $f$ with no fixed points.) It is, therefore, natural to ask whether, when $n+1$ is a multiple of 24, there is some map $\tilde{f}$ of degree 1 such that $\tilde{f}(v) \notin \mathbb{H}v$ for all $v \in S(\mathbb{H}^{n+1})$. Certainly there is no example in which $\tilde{f}$ is of the form $\tilde{f}(v) = A(v)/\|A(v)\|$ for some non-singular $\mathbb{R}$-linear endomorphism of $\mathbb{H}^{n+1}$. (For, if $Av \notin \mathbb{H}v$ for all non-zero $v$, we have a family of maps $A_t : S(\mathbb{H}) \to \mathrm{GL}(\mathbb{R}^{4(n+1)})$, $A_t(z)v = \cos(\pi t/2)zv + \sin(\pi t/2)Av$, $0 \leqslant t \leqslant 1$, with $A_1$ constant, but $A_0$ representing a non-trivial element in $\pi_3(\mathrm{GL}(\mathbb{R}^{4(n+1)})) \cong \mathbb{Z}$. See [17].)

There is a similar result for complex projective spaces.

**Proposition 4.3.** *Let $p : \tilde{X} \to X$ be the principal $\mathrm{U}(1)$-bundle $S^{2n+1} \to \mathbb{C}P^n$, where $n \geqslant 1$, and suppose that $f : \tilde{X} \to X$ is equal to $p \circ \tilde{f}$, where $\tilde{f} : S^{2n+1} \to S^{2n+1}$ is a map of degree $d$. Then the topological Lefschetz index t-$L(f/p)$ in*

$$\omega_1(S^{2n+1}) = \omega_1(*) = (\mathbb{Z}/2\mathbb{Z})\eta,$$

*where $\eta$ is the Hopf element represented by $\mathrm{U}(1)$ with the left invariant framing, is equal to $(1+nd)\eta$.* ∎

In view of Proposition 2.4, these results on quaternionic and complex projective spaces provide examples of coincidence indices in codimension 3 and 1. The corresponding problem for real projective spaces is much simpler.

**Proposition 4.4.** *Let $p : \tilde{X} \to X$ be the principal $\mathrm{O}(1)$-bundle $S^n \to \mathbb{R}P^n$, where $n \geqslant 1$, and suppose that $f : \tilde{X} \to X$ is equal to $p \circ \tilde{f}$, where $\tilde{f} : S^n \to S^n$ is a map of degree $d$. Then* h-Fix$(f/p)$ *is the disjoint union of two components:*

$$\text{h-Fix}(f/p) = \text{h-Fix}(\tilde{f}) \sqcup \text{h-Fix}(-\tilde{f})$$

*and the homotopy Lefschetz index h-$L(f/p)$ in*

$$\omega_0(\text{h-Fix}(f/p)) = \omega_0(\text{h-Fix}(\tilde{f})) \oplus \omega_0(\text{h-Fix}(-\tilde{f})) = \mathbb{Z} \oplus \mathbb{Z}$$

*is equal to $(1+(-1)^n d, 1-d)$.*

*The Nielsen number is thus: $N(f/p) = 0$ if $d = 1$ and $n$ is odd; $N(f/p) = 1$ if either $d = 1$ and $n$ is even, or $d = -1$ and $n$ is odd; and otherwise $N(f/p) = 2$.*

*Proof.* As in the other cases, we could write down an explicit map $f^\sharp$ with $\mathrm{Fix}(f/p) = \mathrm{Fix}(\tilde{f}) \sqcup \mathrm{Fix}(-\tilde{f})$ finite and sum the local indices. But the sums on the two components must give the classical Lefschetz numbers of $\tilde{f}$ and $-\tilde{f}$, which can be computed as traces in rational cohomology: $L(\tilde{f}) = 1 + (-1)^n d$, $L(-\tilde{f}) = 1 + (-1)^n((-1)^{n+1}d)$. ∎

*Remark* 4.5. In this section we were able to compute a non-trivial stable homotopy index in dimension $m = 3$, but in general the stable homotopy indices are likely to be difficult to calculate if the dimension $m$ is greater than zero. More tractable indices can be obtained by taking the Hurewicz image of the stable homotopy index in ordinary homology. In particular, using the Thom isomorphism for $\tau(p)$, we get indices in mod 2 homology $h\text{-}L^H(f/p) \in H_m(\text{h-Fix}(f/p); \mathbb{F}_2)$ and $t\text{-}L^H(f/p) \in H_m(\tilde{X}; \mathbb{F}_2)$. There is even a trace formula for $t\text{-}L^H(f/p)$ in $H_m(\tilde{X}; \mathbb{F}_2)$:

$$\text{for } a \in H^m(\tilde{X}; \mathbb{F}_2), \quad \langle a, t\text{-}L^H(f/p)\rangle = \sum_i (-1)^i \operatorname{tr} A_i \in \mathbb{F}_2,$$

where $A_i$ is the composition

$$H^i(\tilde{X}; \mathbb{F}_2) \xrightarrow{a\cdot} H^{i+m}(\tilde{X}; \mathbb{F}_2) \xrightarrow{p_!} H^i(X; \mathbb{F}_2) \xrightarrow{f^*} H^i(\tilde{X}; \mathbb{F}_2)$$

involving the Umkehr map $p_!$ for the fibrewise manifold. (Compare [15, Section 2].)

## References

[1] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: IV.* Annals of Math. **93** (1971), 119–138.

[2] R. F. Brown, *The Lefschetz number of an n-valued multimap.* JP J. Fixed Point Theory Appl. **2** (2007), 53–60.

[3] R. F. Brown, *Nielsen numbers of n-valued fiber maps.* J. Fixed Point Theory Appl. **4** (2008), 183–201.

[4] R. F. Brown, *Construction of multiply fixed n-valued maps.* Topology Appl. **196** (2015), 249–259.

[5] R. F. Brown and K. Kolahi, *Nielsen coincidence, fixed point and root theories of n-valued maps.* J. Fixed Point Theory Appl. **14** (2013), 309–324.

[6] M. C. Crabb, *Equivariant fixed-point indices of iterated maps.* J. Fixed Point Theory Appl. **2** (2007), 171–193.

[7] M. C. Crabb, *The homotopy coincidence index.* J. Fixed Point Theory Appl. **7** (2010), 1–32.

[8] M. C. Crabb, *Lefschetz indices for n-valued maps.* J. Fixed Point Theory Appl. **17** (2015), 153–186.

[9] M. C. Crabb and I. M. James, *Fibrewise homotopy theory.* Springer, Berlin, 1998.

[10] A. Dold, *Lectures on Algebraic Topology.* Springer, Berlin-Heidelberg-New York, 1972.

[11] A. Dold, *The fixed point index of fibre-preserving maps.* Invent. math. **25** (1974), 281–297.

[12] A. Dold, *The fixed point transfer of fibre-preserving maps.* Math. Zeit. **148** (1976), 215–244.

[13] A. Granas and J. Dugundji, *Fixed Point Theory.* Springer, New York, 2003.

[14] U. Koschorke, *Nielsen coincidence theory in arbitrary codimensions.* J. Reine Angew. Math. **598** (2006), 211–236.

[15] K. Ponto, *Coincidence invariants and higher Reidemeister traces.* J. Fixed Point Theory Appl. **18** (2016), 147–165.

[16] H. Schirmer, *An index and Nielsen number for n-valued multifunctions.* Fund. Math. **124** (1984), 207–219.

[17] R. M. W. Wood, *Quaternionic eigenvalues.* Bull. London Math. Soc. **17** (1985), 137–138.

Institute of Mathematics,
University of Aberdeen,
Aberdeen AB24 3UE, UK
email: m.crabb@abdn.ac.uk