Density by moduli and Wijsman statistical convergence

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Abstract

In this paper, we have generalized the Wijsman statistical convergence of closed sets in metric space by introducing the f-Wijsman statistical convergence of these sets, where f is an unbounded modulus. It is shown that the Wijsman convergent sequences are precisely those sequences which are f-Wijsman statistically convergent for every unbounded modulus f. We have also introduced a new concept of Wijsman strong Cesàro summability with respect to a modulus f, and investigate the relationship between the f-Wijsman statistically convergent sequences and the Wijsman strongly Cesàro summable sequences with respect to f.

1 Introduction and background

The idea of statistical convergence was first introduced by Fast [16] and Steinhaus [31] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors, namely Šalát [27], Fridy [17], Connor [13], Aizpuru et al. [1], Küçükaslan et al. [20], and many others.

Statistical convergence depends on the natural density of subsets of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. The natural density $d(K)$ of set $K \subseteq \mathbb{N}$ (see [25, Chapter 11]) is defined by

$$d(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n \colon k \in K\}\right|, \tag{1.1}$$

Received by the editors in November 2016 - In revised form in February 2017.
Communicated by E. Colebunders.
2010 *Mathematics Subject Classification* : 40A35; 46A45; 40G15.
Key words and phrases : modulus function; natural density; statistical convergence; strong Cesàro summability; Wijsman convergence.

where $|\{ k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. Obviously we have $d(K) = 0$ provided that K is finite.

In what follows we write $(x_k) \subset A$ if all elements of the sequence (x_k) belong to A.

Definition 1.1. A sequence $(x_k) \subset \mathbb{R}$ is said to be statistically convergent to $l \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} \colon |x_k - l| \geq \varepsilon\}$ has the zero natural density.

A new concept of density by moduli was introduced by Aizpuru *et al.*[1] that enabled them to obtain a nonmatrix method of convergence, namely, the f-statistical convergence which is a generalization of statistical convergence.

We recall that a modulus is a function $f \colon [0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$,

(iii) f is increasing,

(iv) f is continuous.

The functions f satisfying condition (ii) are called subadditive. If f, g are moduli and a, b are positive real numbers, then

$$f \circ g, \quad af + bg, \quad \text{and} \quad f \vee g \quad \text{are moduli.}$$

If $f \colon [0, \infty) \to [0, \infty)$ is a modulus and ρ is a metric on a set X, then $(x, y) \mapsto f(\rho(x, y))$ produces a metric uniformly equivalent to ρ. In particular $f(x) = x/(1 + x)$ produces one of the standard bounded metrics equivalent to an initial metric $\rho \colon X \times X \to [0, \infty)$ which can be unbounded. It is also interesting to note that $f \colon [0, \infty) \to [0, \infty)$ is a modulus if and only if there is an uniformly continuous, non-constant function $g \colon [0, \infty) \to [0, \infty)$ such that

$$f(t) = \sup_{\substack{|x-y| \leq t \\ x,y \in [0,\infty)}} |g(x) - g(y)| \tag{1.2}$$

holds for every $t \in [0, \infty)$. The details can be found in Dovgoshey and Martio [15].

Remark 1.2. For bounded moduli, characterization (1.2) has been, in fact, known Lebesgue [21] in 1910.

The idea of replacing of natural density with density by moduli, has motivated Bhardwaj and Dhawan [9, 10] to look for some new generalizations of statistical convergence. Using the density by moduli Bhardwaj *et al.* [11] have also introduced the concept of f-statistical boundedness which is a generalization of the concept of statistical boundedness [18] and intermediate between the usual boundedness and the statistical boundedness.

The concept of convergence of sequences of points has been extended by several authors [3, 4, 5, 7, 19, 29, 30, 33, 34] to convergence of sequences of sets. One of such extensions considered in this paper is the concept of Wijsman convergence. Nuray and Rhoades [26] extended the notion of Wijsman convergence of sequences of sets to that of Wijsman statistical convergence and introduced the notion of Wijsman strong Cesàro summability of sequences of sets and discussed its relations with Wijsman statistical convergence.

In this paper we extend the Wijsman statistical convergence to a f-Wijsman statistical convergence, where f is an unbounded modulus.

Let us recall the basic definitions of f-density and f-statistical convergence.

Definition 1.3 ([1])**.** Let $f\colon [0,\infty) \to [0,\infty)$ be an unbounded modulus. The f-density $d^f(K)$ of a set $K \subseteq \mathbb{N}$ is defined as

$$d^f(K) := \lim_{n\to\infty} \frac{f(|\{k \le n\colon k \in K\}|)}{f(n)} \tag{1.3}$$

if this limit exists. A sequence $(x_k) \subset \mathbb{R}$ is said to be f-statistically convergent to $l \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set $\{k \in \mathbb{N}\colon |x_k - l| \ge \varepsilon\}$ has the zero f-density.

Remark 1.4. For each unbounded modulus f, the finite sets have the zero f-density and

$$(d^f(K) = 0) \Rightarrow (d^f(\mathbb{N} - K) = 1)$$

holds for every $K \subseteq \mathbb{N}$ but, in general, the implication

$$(d^f(\mathbb{N} - K) = 1) \Rightarrow (d^f(K) = 0)$$

does not hold. For example if we take

$$f(x) := \log(1 + x) \text{ and } K := \{2n\colon n \in \mathbb{N}\},$$

then

$$d^f(K) = d^f(\mathbb{N} - K) = 1.$$

Example 1.5. A set having the zero natural density may have a non-zero f-density. In particular, we have

$$d(K) = 0 \quad \text{and} \quad d^f(K) = 1/2$$

for $f(x) := \log(1 + x)$ and $K := \{n^2\colon n \in \mathbb{N}\}$.

Now we pause to collect some definitions related to Wijsman convergence of sequences of sets in a metric space.

Let (X, ρ) be a metric space with a metric ρ. For any $x \in X$ and any non-empty set $A \subseteq X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y\in A} \rho(x, y).$$

In what follows we denote by $CL(X)$ the set of all non-empty closed subsets of (X, ρ).

Definition 1.6. Let (X, ρ) be a metric space, $(A_k) \subset CL(X)$ and $A \in CL(X)$. Then (A_k) is said to be:

- *Wijsman convergent to A*, if the numerical sequence $(d(x, A_k))$ is convergent to $d(x, A)$ for each $x \in X$;
- *Wijsman statistically convergent* to $A \in CL(X)$, if the numerical sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$ for each $x \in X$;
- *Wijsman bounded* if
$$\sup_k d(x, A_k) < \infty \tag{1.4}$$
holds for each $x \in X$;
- *Wijsman Cesàro summable to A* if, for each $x \in X$, the sequence $(d(x, A_k))$ is Cesàro summable to $d(x, A)$, i.e.,
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A);$$
- *Wijsman strongly Cesàro summable to A* if, for each $x \in X$, the sequence $(d(x, A_k))$ is strongly Cesàro summable to $d(x, A)$, i.e.,
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

The Wijsman convergence of sequences $(A_k) \subset CL(X)$ is the convergence with respect to the so-called Wijsman topology which is the weakest topology on $CL(X)$ such that the function $d(x, \cdot)$ is continuous for each $x \in X$. The Wijsman topology was first studied by Lechicki and Levi [22]. It is interesting that a sequence $(A_k) \subset CL(X)$ is convergent with respect to the Vietoris topology generated by a metric $\rho\colon X \times X \to [0, \infty)$ if and only if (A_k) is Wijsman convergent for all metrics which are equivalent to ρ [8]. Note also that the uniform convergence of distance functions $d(x, \cdot)$ determined by a metric ρ is equivalent to convergence in the Hausdorff distance generated by ρ. (See, for example [6, § 3.2].)

Remark 1.7. The sets A_k belonging to a Wijsman bounded sequence (A_k) can be unbounded subsets of (X, ρ), i.e.,
$$\operatorname{diam} A_k = \sup\{\rho(x, y)\colon x, y \in A_k\} = \infty.$$
Moreover, the triangle inequality implies that (A_k) is Wijsman bounded if there exists at least one point $p \in X$ such that (1.4) holds with $x = p$.

Example 1.8. Let (X, ρ) be the complex plane $\mathbb{C}$ with the standard metric. Let us consider the sequence (A_k) defined as follows:
$$A_k := \begin{cases} \left\{z \in \mathbb{C}\colon |z - 1| = \frac{1}{k}\right\}, & \text{if } k \text{ is a square,} \\ \{0\}, & \text{otherwise.} \end{cases}$$
This sequence is Wijsman statistically convergent to $\{0\}$ but not Wijsman convergent.

Definition 1.9. Let (X,ρ) be a metric space, let $(A_k) \subset CL(X)$ and let $f\colon [0,\infty) \to [0,\infty)$ be an unbounded modulus. The sequence (A_k) is said to be f-Wijsman statistically convergent to $A \in CL(X)$ if the sequence $(d(x,A_k))$ is f-statistically convergent to $d(x,A)$ for each $x \in X$.

We write

$$[WS^f] - \lim A_k = A$$

if (A_k) is f-Wijsman statistically convergent to A. In the case where $f(x) = ax$, $a > 0$, the f-Wijsman statistical convergence reduces to the Wijsman statistical convergence.

We prove that the Wijsman convergent sequences are precisely those sequences which are f-Wijsman statistically convergent for every unbounded modulus f. We also introduce a new concept of Wijsman strong Cesàro summability with respect to a modulus and show that if a sequence is Wijsman strongly Cesàro summable, then it is Wijsman strongly Cesàro summable with respect to all moduli f. The moduli f for which the converse is true are investigated. Finally, we study a relation between Wijsman strong Cesàro summability with respect to a modulus f and f-Wijsman statistical convergence.

2 f-Wijsman statistical convergence

The results of this section are closely related with paper [1].

Theorem 2.1. *Let $f\colon [0,\infty) \to [0,\infty)$ be an unbounded modulus, (X,ρ) be a metric space, $A \in CL(X)$ and let $(A_k) \subset CL(X)$ such that*

$$[WS^f] - \lim A_k = A. \tag{2.1}$$

Then (A_k) is Wijsman statistically convergent to A.

Proof. For all $x \in X$, $\varepsilon > 0$ and $n \in \mathbb{N}$ we write

$$K_{x,\varepsilon}(n) := \{k \le n\colon |d(x,A_k) - d(x,A)| \ge \varepsilon\}.$$

If (A_k) is not Wijsman statistically convergent to A, then there are $x \in X$ and $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \frac{|K_{x,\varepsilon}(n)|}{n} > 0.$$

Hence there exist $p \in \mathbb{N}$ and a sequence $(n_m) \subset \mathbb{N}$, such that

$$\lim_{m\to\infty} n_m = \infty \tag{2.2}$$

and

$$\frac{1}{n_m}|K_{x,\varepsilon}(n_m)| \ge \frac{1}{p}$$

for every $m \in \mathbb{N}$. The last inequality is equivalent to

$$n_m \le p\,|K_{x,\varepsilon}(n_m)|. \tag{2.3}$$

Using the subadditivity of f and (2.3) we obtain

$$f(n_m) \leq p\, f(|K_{x,\varepsilon}(n_m)|).$$

Consequently the inequality

$$\frac{f(|K_{x,\varepsilon}(n_m)|)}{f(n_m)} \geq \frac{1}{p} \tag{2.4}$$

holds for every $m \in \mathbb{N}$. Equality (2.2) and inequality (2.4) imply

$$\limsup_{n\to\infty} \frac{f(|K_{x,\varepsilon}(n)|)}{f(n)} \geq \frac{1}{p},$$

contrary to (2.1). ∎

Remark 2.2. Using Example 1.5 it is easy to construct a Wijsman statistically convergent sequence which is not f-Wijsman statistically convergent with $f(x) = \log(1+x)$.

Theorem 2.3. *Let (X, ρ) be a metric space and f, g be unbounded moduli. Then for all A, $B \in CL(X)$ and every $(A_k) \subset CL(X)$ the equalities*

$$[WS^f] - \lim A_k = A \quad \textit{and} \quad [WS^g] - \lim A_k = B \tag{2.5}$$

imply $A = B$.

Proof. Let (X, ρ) be a metric space, let $(A_k) \subset CL(X)$ and let (2.5) hold. By Theorem 2.1 the sequence (A_k) is Wijsman statistically convergent to A and to B. Using the uniqueness of statistical limits of numerical sequences we obtain that $d(x, A) = d(x, B)$ holds for every $x \in X$. It implies the equality $A = B$ because A, $B \in CL(X)$. ∎

Corollary 2.4. *Let (X, ρ) be a metric space and let $(A_k) \subset CL(X)$. Then for every unbounded modulus $f\colon [0, \infty) \to [0, \infty)$, the limit*

$$[WS^f] - \lim A_k$$

is unique if it exists.

We will say that a modulus $f\colon [0, \infty) \to [0, \infty)$ is slowly varying if the limit relation

$$\lim_{x\to\infty} \frac{f(ax)}{f(x)} = 1 \tag{2.6}$$

holds for every $a > 0$. (See Seneta [28, Chapter 1] for the properties of slowly varying functions.) It is clear that all bounded modulus are slowly varying. The function $f(x) = \log(1+x)$ is an example of unbounded, slowly varying modulus.

The following lemma is a refinement of Lemma 3.4 from [1].

Lemma 2.5. *Let K be an infinite subset of $\mathbb{N}$. Then there is an unbounded, concave and slowly varying modulus $f\colon [0,\infty) \to [0,\infty)$ such that*

$$d^f(K) = 1. \tag{2.7}$$

Proof. For every $n \in \mathbb{N}$ write

$$K(n) := \{m \in K \colon m \leq n\}.$$

Since K is infinite, there is a sequence $(n_k) \subset \mathbb{N}$ such that:

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty \tag{2.8}$$

and

$$n_{k+1} - n_k < n_{k+2} - n_{k+1}, \quad 2n_k < n_{k+1} \tag{2.9}$$

and

$$n_k < |K(n_{k+1})| \tag{2.10}$$

hold for every $k \in \mathbb{N}$.

Write $n_0 = 0$ and define a function $f\colon [0,\infty) \to [0,\infty)$ by the rule: if $x \in [n_{k-1}, n_k]$, $k \in \mathbb{N}$, then

$$f(x) = \frac{x - n_{k-1}}{n_k - n_{k-1}} + k - 1. \tag{2.11}$$

In particular, we have

$$f(n_k) = k \tag{2.12}$$

for every $k \in \mathbb{N} \cup \{0\}$. We claim that f has all desirable properties.

(i) *f is unbounded modulus.* It is clear that $f(0) = 0$ holds and f is strictly increasing and unbounded. For subadditivity of f it suffices to show that the function $\frac{f(t)}{t}$ is decreasing on $(0,\infty)$. Indeed, if $\frac{f(t)}{t}$ is decreasing, then

$$f(x+y) = x\frac{f(x+y)}{x+y} + y\frac{f(x+y)}{x+y} \leq x\frac{f(x)}{x} + y\frac{f(y)}{y} = f(x) + f(y).$$

(See, for example, Timan [32, 3.2.3].) The function $\frac{f(x)}{x}$ is decreasing on $(0,\infty)$ if and only if this function is decreasing on (n_{k-1}, n_k) for every $k \in \mathbb{N}$. Using (2.9) we see that the last condition trivially holds on (n_0, n_1), because in this case, the right hand side in (2.11) is

$$\frac{x - n_0}{n_1 - n_0} - (1-1) = \frac{x}{n_1}.$$

Moreover, for $k \geq 2$ the restriction $f|_{(n_{k-1}, n_k)}$ is decreasing if and only if

$$\frac{(k-1)(n_k - n_{k-1}) - n_{k-1}}{n_k - n_{k-1}} \geq 0. \tag{2.13}$$

Since we have

$$(k-1)(n_k - n_{k-1}) - n_{k-1} \geq n_k - 2n_{k-1}$$

for $k \geq 2$ the second inequality in (2.9) implies (2.6). Equality (2.12) implies that f is unbounded. Thus f is an unbounded modulus.

(ii) *f is concave*. Since f is a piecewise affine function, the one-sided derivatives of f exist at all points $x \in [0, \infty)$. Using (2.11) and the first inequality in (2.9) we see that these derivatives are decreasing. Hence f is concave. (For the proof of concavity of functions with decreasing one-sided derivatives see, for example, Artin [2, p. 4].)

(iii) *f is slowly varying*. It is easy to see that (2.6) holds for all $a > 0$ if it holds for all $a > 1$. Since f is increasing, the inequality $a > 1$ implies that

$$\liminf_{x\to\infty} \frac{f(ax)}{f(x)} \geq 1.$$

Thus f is slowly varying if and only if

$$\limsup_{x\to\infty} \frac{f(ax)}{f(x)} \leq 1. \tag{2.14}$$

Let $a > 1$ and $x > 0$. Suppose that

$$x \in [n_{k-1}, n_k] \text{ and } ax \in [n_{k+p}, n_{k+p+1}]$$

for some $p, k \in \mathbb{N}$. It implies that

$$a = \frac{ax}{x} \geq \frac{n_{k+p}}{n_k}. \tag{2.15}$$

Using (2.13) and (2.15) we obtain

$$(x \in [n_{k-1}, n_k]) \Rightarrow (ax \in [n_{k-1}, n_k] \text{ or } ax \in [n_k, n_{k+1}]) \tag{2.16}$$

for all sufficiently large x. Now it follows from (2.11) and (2.16) that

$$f(ax) \leq f(x) + 2. \tag{2.17}$$

Since we have $\lim_{x\to\infty} f(x) = \infty$, inequality (2.17) implies (2.14).

(iv) *Equality* (2.7) *holds*. We must prove the equality

$$\lim_{m\to\infty} \frac{f(|K(m)|)}{f(m)} = 1. \tag{2.18}$$

Let $m \in \mathbb{N}$ such that $m \geq n_2$. Then there is $k \geq 3$ for which

$$n_{k-1} \leq m \leq n_k. \tag{2.19}$$

The last double inequality and (2.12) imply

$$k - 1 = f(n_{k-1}) \leq f(m) \leq f(n_k) = k. \tag{2.20}$$

From (2.19) it follows that

$$|K(n_{k-1})| \leq |K(m)| \leq |K(n_k)|. \tag{2.21}$$

Using (2.10), (2.21) and the inequality $|K(n_k)| \leq n_k$ we obtain

$$n_{k-2} \leq |K(m)| \leq n_k,$$

which implies

$$k-2 = f(n_{k-2}) \leq |K(m)| \leq f(n_k) = k. \tag{2.22}$$

Limit relation (2.18) follows from (2.20) and (2.22). ■

Example 2.6. The ternary Cantor function $G\colon [0,1] \to [0,1]$ leads to an interesting example of unbounded modulus which is not concave. Indeed, G is subadditive (see, for example, Doboš [14] and Timan [32, 3.2.4]) and can be characterized as the unique real-valued, continuous, increasing function $f\colon [0,1] \to \mathbb{R}$ satisfying the functional equations

$$f\left(\frac{x}{3}\right) = \frac{1}{2}f(x) \text{ and } f(1-x) = 1 - f(x)$$

(see Chalice [12] for the proof). Now we define a sequence of functions G_k, such that $G_1 = G$ and, for every $k \geq 2$, $\mathrm{dom}(G_k) = [0, 3^{k-1}]$ and

$$G_k(x) = 2G_{k-1}\left(\frac{x}{3}\right), \quad x \in [0, 3^{k-1}].$$

Then the extended Cantor function

$$G_e\colon [0,\infty) \to [0,\infty), \quad G_e(x) = G_k(x), \text{ if } x \in [0, 3^{k-1}]$$

is a correctly defined, unbounded modulus which is not concave.

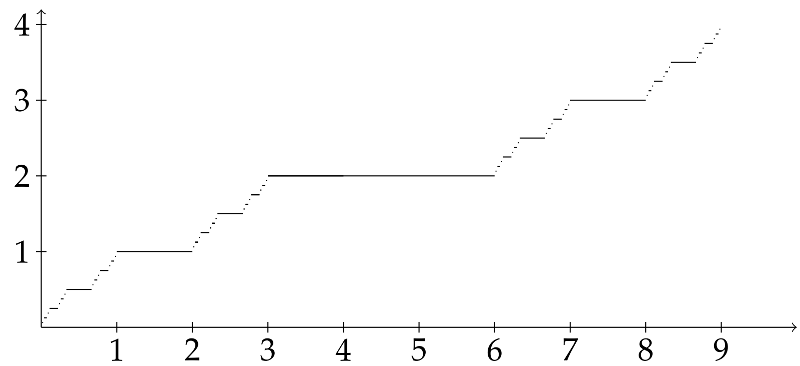

Figure 1: The graph of G_e

Let us denote by *MUCS* the set of all unbounded, concave and slowly varying moduli.

Theorem 2.7. *Let (X,ρ) be a metric space, $(A_k) \subset CL(X)$ and $A \in CL(X)$. Then the following statements are equivalent:*

(i) (A_k) is Wijsman convergent to A;

(*ii*) *The equality*

$$[WS^f] - \lim A_k = A \tag{2.23}$$

holds for every unbounded modulus f*;*

(*iii*) *Equality* (2.23) *holds for every* $f \in MUCS$*.*

Proof. $(i) \Rightarrow (ii)$ Let (i) hold. Since (A_k) is Wijsman convergent to A, the set

$$K_{x,\varepsilon} := \{k \in \mathbb{N}\colon |d(x, A_k) - d(x, A)| \geq \varepsilon\}$$

is finite for all $x \in X$ and $\varepsilon > 0$. Let $f\colon [0, \infty) \to [0, \infty)$ be an unbounded modulus. The equality

$$\lim_{n\to\infty} \frac{f(|K_{x,\varepsilon}|)}{f(n)} = 0,$$

holds because f is unbounded and increasing. Thus, (2.23) holds.

$(ii) \Rightarrow (iii)$ It is trivial.

$(iii) \Rightarrow (i)$ Let (iii) hold. Suppose, (A_k) is not Wijsman convergent to A. Then the set $K_{x,\varepsilon}$ is infinite for some $x \in X$ and $\varepsilon > 0$. Now by Lemma 2.5 there exists $f \in MUCS$ such that $d^f(K_{x,\varepsilon}) = 1$, which contradicts (2.23). ∎

Remark 2.8. The sequence (A_k) in Example 1.8 is f-Wijsman statistically convergent with $f(x) = x$ but not Wijsman convergent.

Theorem 2.7 leads us to formulate the following problem.

Problem 2.9. Let M be a set of all unbounded moduli. Describe the sets $S \subseteq M$ for which the conditions:

- (A_k) is Wijsman convergent to A

and

- $[WS^f] - \lim A_k = A$ holds for every $f \in S$

are equivalent for all metric spaces (X, ρ), $(A_k) \subset CL(X)$ and $A \in CL(X)$.

The following theorem is similar to Theorem 3.1 from [1].

Theorem 2.10. *Let* (X, ρ) *be a metric space,* $f\colon [0, \infty) \to [0, \infty)$ *be an unbounded modulus,* $(A_i) \subset CL(X)$ *and* $A \in CL(X)$*. Then*

$$[WS^f] - \lim A_i = A$$

holds if and only if, for each $x \in X$*, there exists* $K_x \subseteq \mathbb{N}$ *such that*

$$d^f(K_x) = 0 \quad \text{and} \quad \lim_{k\in\mathbb{N}-K_x} d(x, A_i) = d(x, A).$$

Proof. For every $K \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we write $K(n)$ for the set

$$K \cap \{1, \ldots, n\}.$$

Suppose

$$[WS^f] - \lim A_i = A \tag{2.24}$$

holds. For every $x \in X$ we must find a set $K_x \subseteq \mathbb{N}$ such that

$$\lim_{i \in \mathbb{N} - K_x} d(x, A_i) = d(x, A) \tag{2.25}$$

and

$$\lim_{n \to \infty} \frac{f(|K_x(n)|)}{f(n)} = 0 \tag{2.26}$$

holds.

Let $x \in X$. For every $j \in \mathbb{N}$ define the set $B_j \subseteq \mathbb{N}$ by the rule:

$$(i \in B_j) \Leftrightarrow \left(|d(x, A_i) - d(x, A)| \geq \frac{1}{j} \right). \tag{2.27}$$

It is clear that $B_{j_1} \subseteq B_{j_2}$ holds whenever $j_2 \geq j_1$. If all B_j are finite, then (2.25) and (2.26) are valid with $K_x = \varnothing$. Suppose B_j are infinite for some $j \in \mathbb{N}$. If there is B_{j_1} satisfying the condition

- $B_j - B_{j_1}$ is finite for every $j \in \mathbb{N}$,

then (2.25) and (2.26) follows from (2.24) with $K_x = B_{j_1}$. (Note that (2.26) follows from (2.24).)

Let us consider the case when, for every B_j, there is l such that $B_{l+j} - B_j$ is infinite. Define a sequence $(j_k) \subseteq \mathbb{N}$ recursively by the rule:

- if $k = 1$, then j_1 is the smallest j for which B_j is infinite,
- if $k \geq 2$, then j_k is the smallest j with infinite $B_j - B_{j_{k-1}}$.

Write $B_1^* := B_{j_1}$ and, for $k \geq 2$, $B_k^* := B_{j_k} - B_{j_{k-1}}$. It follows from (2.27) that

$$(i \in B_1^*) \Leftrightarrow \left(|d(x, A_i) - d(x, A)| \geq \frac{1}{j_1} \right) \tag{2.28}$$

and, for $k \geq 2$,

$$(i \in B_k^*) \Leftrightarrow \left(\frac{1}{j_k} \leq |d(x, A_i) - d(x, A)| < \frac{1}{j_{k-1}} \right). \tag{2.29}$$

It is easily seen that $B_{k_1}^*$ and $B_{k_2}^*$ are disjoint for all distinct $k_1, k_2 \in \mathbb{N}$.

Let $(n_k) \subseteq \mathbb{N}$ be a infinite strictly increasing sequence. Write

$$B^* := \bigcup_{k=1}^{\infty} (B_k^* - \{1, \ldots, n_k\}). \tag{2.30}$$

We claim that (2.25) holds with $K_x = B^*$. To prove (2.25) it is suffices to show that the set

$$K_{x,\varepsilon}^* := \{ i \in (\mathbb{N} - B^*) \colon |d(x, A_i) - d(x, A)| \geq \varepsilon \} \tag{2.31}$$

is finite for every $\varepsilon > 0$. If $\varepsilon > 0$, then we have either

$$\varepsilon \geq \frac{1}{j_1} \tag{2.32}$$

or there is $k \geq 2$ such that

$$\frac{1}{j_{k-1}} > \varepsilon \geq \frac{1}{j_k}. \tag{2.33}$$

Let $\varepsilon \geq \frac{1}{j_1}$ and let $i \in K^*_{x,\varepsilon}$. Then $i \in (\mathbb{N} - B^*)$ and

$$|d(x, A_i) - d(x, A)| \geq \frac{1}{j_1} \tag{2.34}$$

hold. Since

$$\mathbb{N} - B^* = \bigcap_{k=1}^{\infty} (\{1, \ldots, n_k\} \cup (\mathbb{N} - B^*_k)),$$

the condition $i \in \mathbb{N} - B^*$ implies

$$i \in \{1, \ldots, n_1\} \text{ or } i \in (\mathbb{N} - B^*_1).$$

If $i \in (\mathbb{N} - B^*_1)$, then using (2.28) we obtain

$$|d(x, A_i) - d(x, A)| < \frac{1}{j_1},$$

which contradicts (2.34). Hence $i \in \{1, \ldots, n_1\}$ holds. Thus if $\varepsilon \geq \frac{1}{j_1}$, then $K^*_{x,\varepsilon}$ is finite with $|K^*_{x,\varepsilon}| \leq n_1$. Similarly if

$$\frac{1}{j_{k-1}} > \varepsilon \geq \frac{1}{j_k} \text{ with } k \geq 2,$$

then, using (2.29) instead of (2.28), we can prove the inequality $|K^*_{x,\varepsilon}| \leq n_k$. Limit relation (2.25) follows.

Now we prove that there exists an increasing, infinite sequence $(n_k) \subseteq \mathbb{N}$ such that (2.26) holds for $K_x = B^*$ with B^* defined by (2.30).

Equality (2.24) implies that $d^f(B_j) = 0$ holds for every $j \in \mathbb{N}$. Hence for given $\varepsilon_1 > 0$ there is $n_1 \in \mathbb{N}$ such that

$$\frac{f(|B_{j_1}(n)|)}{f(n)} \leq \varepsilon_1$$

is valid for every $n \geq n_1$. Let $0 < \varepsilon_2 \leq \frac{1}{2}\varepsilon_1$. Using the equality $d^f(B_{j_2}) = 0$ we can find $n_2 > n_1$ such that

$$\frac{f(|B_{j_2}(n)|)}{f(n)} \leq \varepsilon_2$$

for all $n \geq n_2$. By induction on k we can find $n_k > n_{k-1}$ which satisfies

$$\frac{f(|B_{j_k}(n)|)}{f(n)} \leq \frac{1}{2}\varepsilon_{k-1} \leq \left(\frac{1}{2}\right)^{k-1} \varepsilon_1$$

for all $n \geq n_k$. It follows (2.30), that, for every $k \in \mathbb{N}$, the inclusion

$$B^*(n) \subseteq B_{j_k}(n)$$

holds if $n \in [n_{k+1}, n_k)$. Hence we have

$$\frac{f(|B^*(n)|)}{f(n)} \leq \left(\frac{1}{2}\right)^{k-1} \varepsilon_1$$

if $n \in [n_{k+1}, n_k)$, $k \in \mathbb{N}$. The equality

$$\lim_{n\to\infty} \frac{f(|B^*(n)|)}{f(n)} = 0$$

follows.

Assume now that, for every $x \in X$, there is $K_x \subset \mathbb{N}$ such that

$$d^f(K_x) = 0 \text{ and } \lim_{i\in\mathbb{N}-K_x} d(x, A_i) = d(x, A).$$

Let $x \in X$ and $\varepsilon > 0$. Then there is $i_0 \in \mathbb{N} - K_x$ such that

$$|d(x, A_i) - d(x, A)| \leq \varepsilon$$

for all $i \in (\mathbb{N} - K_x) - \{1, \dots, i_0\}$. Hence

$$\{i \in \mathbb{N}\colon\ |d(x, A_i) - d(x, A)| > \varepsilon\} \subseteq K_x \cup \{1, \dots, i_0\}.$$

Equality $d^f(K_x) = 0$ implies $d^f(K_x \cup \{1, \dots, i_0\}) = 0$. The limit relation

$$[WS^f] - \lim A_i = A$$

follows. ∎

3 Wijsman statistical convergence and Wijsman Cesàro summability

The following example shows that Wijsman statistical convergence does not imply Wijsman Cesàro summability.

Example 3.1. Let $(X, \rho) = \mathbb{R}$ with the standard metric and let (A_k) be defined as

$$A_k = \begin{cases} \{k\}, & \text{if } k \text{ is a square,} \\ \{0\}, & \text{otherwise.} \end{cases}$$

This sequence is Wijsman statistically convergent to the set $\{0\}$ since

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n\colon |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}| = 0$$

holds for all $x \in \mathbb{R}$ and $\varepsilon > 0$. Now, we show that this sequence is not Wijsman Cesàro summable. For the sequence $(\sigma_k(0))$ of Cesàro means of order one of the sequence $(d(0, A_k))$ we have

$$\sigma_k(0) = \begin{cases} \frac{(1^2+2^2+\cdots+n^2)}{n^2}, & \text{if } k = n^2, \text{ for some } n \in \mathbb{N} \\ \frac{(1^2+2^2+\cdots+n^2)}{k}, & \text{if } n^2 < k < (n+1)^2, \text{ for some } n \in \mathbb{N}. \end{cases}$$

The sequence $(\sigma_k(0))$ is not convergent because

$$\lim_{n\to\infty} \frac{\sum_1^n k^2}{n^2} = \lim_{n\to\infty} \frac{1}{6} \frac{n(n+1)(2n+1)}{n^2} = \infty.$$

We now give an example of sequence $(A_k) \subset CL(X)$ such that the sequence $(\sigma_k(x))$ of Cesàro means of the sequence $(d(x, A_k))$ has a finite limit for every $x \in X$ but (A_k) is not Wijsman Cesàro summable to A for any $A \in CL(X)$.

Example 3.2. Let $(X, \rho) = \mathbb{R}$ with the standard metric and let (A_k) be defined as

$$A_k = \begin{cases} \{-1\}, & \text{if k is even,} \\ \{1\}, & \text{if k is odd.} \end{cases}$$

Let $x \in \mathbb{R}$. For the sequence $(\sigma_k(x))$ of Cesàro means of order one of the sequence $(d(x, A_k))$ we have

$$\sigma_k(x) = \begin{cases} |x|, & \text{if } k \text{ is even and } x \notin [-1, 1], \\ 1, & \text{if } k \text{ is even and } x \in [-1, 1], \\ \left| x - \frac{1}{k} \right|, & \text{if } k \text{ is odd and } x \notin [-1, 1], \\ 1 - \frac{x}{k}, & \text{if } k \text{ is odd and } x \in [-1, 1]. \end{cases}$$

Consequently

$$\lim_{k\to\infty} \sigma_k(x) = \begin{cases} |x|, & \text{if } x \notin [-1, 1], \\ 1, & \text{if } x \in [-1, 1]. \end{cases} \tag{3.1}$$

Now we prove that (A_k) is not Wijsman Cesàro summable. Indeed, suppose contrary that there is $A \in CL(X)$ with

$$\lim_{k\to\infty} \sigma_k(x) = d(x, A) \tag{3.2}$$

for every $x \in \mathbb{R}$. Since A is non-empty, there is $x_0 \in A$. Using (3.1) and (3.2) we obtain

$$0 = d(x_0, A) = \lim_{k\to\infty} \sigma_k(x_0) \text{ and } \lim_{k\to\infty} \sigma_k(x_0) \geq 1,$$

which is a contradiction.

Remark 3.3. It seems to be interesting to find a criteria guaranteeing the Wijsman Cesàro summability of $(A_k) \subset CL(X)$ to some $A \in CL(X)$ if the sequence $\big(\sigma_k(x)\big)$ of Cesàro means of $\big(d(x, A_k)\big)$ is Cesàro summable for every $x \in X$.

In the next theorem we show that the Wijsman statistical convergence implies the Wijsman Cesàro summability in case of Wijsman bounded sequences.

Theorem 3.4. *Let (X, ρ) be a metric space, let $A \in CL(X)$ and let $(A_k) \subset CL(X)$. If (A_k) is Wijsman bounded and Wijsman statistically convergent to A, then (A_k) is Wijsman Cesàro summable to A.*

Proof. Let $\varepsilon > 0$, $x \in X$, and let (A_k) be Wijsman bounded. For every $n \in \mathbb{N}$ define the sets $K_{x,\varepsilon}(n)$, $K'_{x,\varepsilon}(n)$ and M_x as

$$K_{x,\varepsilon}(n) := \{k \leq n \colon |d(x, A_k) - d(x, A)| \geq \varepsilon\},$$
$$K'_{x,\varepsilon}(n) := \{1, \dots, n\} - K_{x,\varepsilon}(n) \text{ and } M_x := \sup_k |d(x, A_k)|.$$

Suppose (A_k) is Wijsman statistically convergent to A. Then the limit relation

$$\lim_{n\to\infty} \frac{|K_{x,\varepsilon}(n)|}{n} = 0,$$

holds. Now we have

$$\left| d(x,A) - \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) \right| \le \frac{1}{n}\sum_{k=1}^{n} |(d(x,A_k) - d(x,A))|$$
$$= \frac{1}{n}\left(\sum_{k\in K'_{x,\varepsilon}(n)} |d(x,A_k) - d(x,A)| + \sum_{k\in K_{x,\varepsilon}(n)} |d(x,A_k) - d(x,A)| \right)$$
$$\le \frac{(n - |K_{x,\varepsilon}(n)|)\varepsilon}{n} + \frac{1}{n}|K_{x,\varepsilon}(n)|\, M_x \le \varepsilon + M_x \frac{|K_{x,\varepsilon}(n)|}{n}.$$

It implies the inequality

$$\limsup_{n\to\infty} \left| d(x,A) - \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) \right| \le \varepsilon.$$

Letting ε to 0 we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = d(x,A).$$

Since x is an arbitrary point of X, the sequence (A_k) is Wijsman Cesàro summable to A. ■

Corollary 3.5. *Let (X,ρ) be a bounded metric space, $A \in CL(X)$, $(A_k) \subset CL(X)$ and let $f\colon [0,\infty) \to [0,\infty)$ be an unbounded modulus. If*

$$[WS^f] - \lim A_k = A,$$

then (A_k) is Wijsman Cesàro summable to A.

It follows from Theorem 2.1 and Theorem 3.4 because in each bounded metric space every sequence of non-empty, closed sets is Wijsman bounded.

4 Wijsman strong Cesàro summability with respect to a modulus

The well-known space w of strongly Cesàro summable sequences is defined as:

$$w := \left\{ (x_k)\colon \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_k - l| = 0, \text{ for some } l \in \mathbb{R} \right\}.$$

Maddox [23] extended the strong Cesàro summabllity to that of strong Cesàro summabllity with respect to a modulus f and studied the space

$$w(f) := \left\{ (x_k) \colon \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|x_k - l|) = 0, \text{ for some } l \in \mathbb{R} \right\}.$$

In the year 2012, Nuray and Rhoades [26] introduced the notion of Wijsman strong Cesàro summability of sequences of sets and discussed its relation with Wijsman statistical convergence.

In this section, we introduce a new concept of Wijsman strong Cesàro summability with respect to a modulus f. It is shown that, under certain conditions on f, the Wijsman strong Cesàro summability with respect to f implies the f-Wijsman statistical convergence and that the concepts of f-Wijsman statistical convergence and of Wijsman strong Cesàro summability with respect to f are equivalent for Wijsman bounded sequences.

Definition 4.1. Let (X, ρ) be a metric space and let $f\colon [0,\infty) \to [0,\infty)$ be a modulus. A sequence $(A_k) \subset CL(X)$ is said to be Wijsman strongly Cesàro summable to $A \in CL(X)$ with respect to f, if the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(|d(x, A_k) - d(x, A)|\right) = 0$$

holds for each $x \in X$.

We write

$$[Ww^f] - \lim A_k = A$$

if (A_k) is Wijsman strongly Cesàro summable to A with respect to f.

Remark 4.2. For $f(x) = x$, the concept of Wijsman strong Cesàro summability with respect to f reduces to that of Wijsman strong Cesàro summability.

Theorem 4.3. *Let (X, ρ) be a metric space, $(A_k) \subset CL(X)$, $A \in CL(X)$ and let $f\colon [0,\infty) \to [0,\infty)$ be a modulus. If (A_k) is Wijsman strongly Cesàro summable to A, then*

$$[Ww^f] - \lim A_k = A. \tag{4.1}$$

Proof. Suppose that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0 \tag{4.2}$$

holds for each $x \in X$. Let $\varepsilon > 0$ and choose $\delta \in (0,1)$ such that $f(t) < \varepsilon$ for all $t \in [0, \delta]$. Consider

$$\sum_{k=1}^{n} f(|d(x, A_k) - d(x, A)|) = \sum_1 + \sum_2,$$

where the first summation is over the set $\{k \leq n\colon |d(x, A_k) - d(x, A)| \leq \delta\}$ and the second is over $\{k \leq n\colon |d(x, A_k) - d(x, A)| > \delta\}$. It is clear that $\sum_1 \leq n\varepsilon$. To estimate $\sum_2$ we use the inequalities

$$\left|d(x, A_k) - d(x, A)\right| < \frac{\left|d(x, A_k) - d(x, A)\right|}{\delta} \leq \left\lceil |d(x, A_k) - d(x, A)|\delta^{-1} \right\rceil,$$

where $\lceil \cdot \rceil$ is the ceiling function. The modulus functions are increasing and subadditive. Hence

$$\begin{aligned} f(|d(x, A_k) - d(x, A)|) \leq f(1) \left\lceil |d(x, A_k) - d(x, A)|\, \delta^{-1} \right\rceil \\ \leq 2f(1)\, |d(x, A_k) - d(x, A)|\, \delta^{-1} \end{aligned}$$

holds whenever $|d(x, A_k) - d(x, A)| > \delta$. Thus we have

$$\sum_2 \leq 2f(1)\delta^{-1} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)|\,,$$

which together with $\sum_1 \leq n\varepsilon$ yields

$$\frac{1}{n} \sum_{k=1}^{n} f\big(|d(x, A_k) - d(x, A)|\big) \leq \varepsilon + 2\, f(1)\, \delta^{-1}\, \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)|.$$

Now using (4.2) we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\big(|d(x, A_k) - d(x, A)|\big) \leq \varepsilon.$$

Equality (4.1) follows by letting ε to 0. ∎

The next example shows that (4.1) does not imply that (A_k) is Wijsman strongly Cesàro summable to A.

Example 4.4. Let $(X, \rho) = [0, \infty)$ with the standard metric and let $f(x) = \log(1 + x)$. Let us consider a sequence (A_k) defined by

$$A_k = \begin{cases} \{k\}, & \text{if } k \in \{2^r : r \in \mathbb{N}\}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then, for every $x \in [0, \infty)$, we have

$$d(x, A_k) = \begin{cases} |x - k|, & \text{if } k \in \{2^r : r \in \mathbb{N}\}, \\ x, & \text{otherwise.} \end{cases} \tag{4.3}$$

For any numerical sequence $(x_i) \subset [0, \infty)$, the limit relation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = 0$$

holds if and only if

$$\lim_{r\to\infty} \frac{1}{2^r} \sum_{i=2^r}^{2^{r+1}-1} f(x_i) = 0.$$

(See Maddox [24, p. 523].) Hence

$$[Ww^f] - \lim A_k = \{0\}$$

holds if and only if we have

$$\lim_{r\to\infty} \frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}-1} \log\Big(1 + \big|d(x, A_k) - d(x, \{0\})\big|\Big) = 0 \tag{4.4}$$

for every $x \in [0, \infty)$. Using (4.3) we see that

$$\sum_{k=2^r}^{2^{r+1}-1} \log\Big(1 + \big|d(x, A_k) - d(x, \{0\})\big|\Big) = \log\Big(1 + \big||x - 2^r| - x\big|\Big).$$

For sufficiently large r we have

$$1 + \big||x - 2^r| - x\big| = 2^r - 2x + 1.$$

Consequently the left-hand of (4.4) is equal to

$$\lim_{r\to\infty} \frac{1}{2^r} \log(2^r - 2x + 1).$$

The last limit is 0. Thus (A_k) is Wijsman strongly Cesàro summable to $\{0\}$ with respect to f. Now, using (4.3) we obtain

$$\frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}-1} \big|d(x, A_k) - d(x, \{0\})\big| = \frac{2^r - 2x}{2^r}$$

for sufficiently large r. Thus

$$\lim_{r\to\infty} \frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}-1} \big|d(x, A_k) - d(x, \{0\})\big| = 1,$$

which implies that (A_k) is not Wijsman strongly Cesàro summable to $\{0\}$.

The following lemma was proved by Maddox in [24].

Lemma 4.5. *Let $f\colon [0, \infty) \to [0, \infty)$ be a modulus. Then there is a finite $\lim_{t\to\infty} \frac{f(t)}{t}$ and the equality*

$$\lim_{t\to\infty} \frac{f(t)}{t} = \inf\{t^{-1} f(t)\colon t \in (0, \infty)\} \tag{4.5}$$

holds.

Proof. Write

$$\beta := \inf\{t^{-1}f(t) \colon t \in (0,\infty)\}. \tag{4.6}$$

It suffices to show that

$$\limsup_{t\to\infty} \frac{f(t)}{t} \le \beta. \tag{4.7}$$

Let $\varepsilon > 0$ and let $t_0 \in (0,\infty)$ such that

$$\beta \ge \frac{f(t_0)}{t_0} - \varepsilon.$$

The last inequality is equivalent to

$$f(t_0) \le t_0(\beta+\varepsilon). \tag{4.8}$$

For every $t \in (0,\infty)$ we have

$$t = t_0 \left\lfloor \frac{t}{t_0} \right\rfloor + \left(t - t_0 \left\lfloor \frac{t}{t_0} \right\rfloor\right) \le \left(t_0 \left\lfloor \frac{t}{t_0} \right\rfloor + 1\right), \tag{4.9}$$

where $\lfloor\cdot\rfloor$ is the floor function. Using the increase and subadditivity of f and (4.8)–(4.9) we obtain

$$\frac{f(t)}{t} \le \frac{f(t_0)\left\lfloor \frac{t}{t_0} \right\rfloor + f(1)}{t} \le \frac{t_0(\beta+\varepsilon)\left\lfloor \frac{t}{t_0} \right\rfloor + f(1)}{t}$$

for all sufficiently large t. Hence

$$\limsup_{t\to\infty} \frac{f(t)}{t} \le (\beta+\varepsilon) \limsup_{t\to\infty} \frac{t_0\left\lfloor \frac{t}{t_0} \right\rfloor}{t} = \beta+\varepsilon.$$

Inequality (4.7) follows by letting ε to 0. ∎

Theorem 4.6. *Let (X,ρ) be a metric space, $A \in CL(X)$ and $(A_k) \subset CL(X)$. If $f\colon [0,\infty) \to [0,\infty)$ is a modulus such that*

$$\beta := \lim_{t\to\infty} \frac{f(t)}{t} > 0 \textit{ and } [Ww^f] - \lim A_k = A, \tag{4.10}$$

then (A_k) is Wijsman strongly Cesàro summable to A.

Proof. Let a modulus f satisfy condition (4.10). By Lemma 4.5 we have

$$\beta = \inf\{t^{-1}f(t) \colon t > 0\}.$$

Consequently

$$f(t) \ge \beta t \tag{4.11}$$

holds for every $t \ge 0$. It follows from (4.11) that

$$\frac{1}{n}\sum_{k=1}^{n} |d(x,A_k) - d(x,A)| \le \beta^{-1}\frac{1}{n}\sum_{k=1}^{n} f(|\,d(x,A_k) - d(x,A)|),$$

holds for every $x \in X$. Using the second term of (4.10) we see that (A_k) is Wijsman strongly Cesàro summable to A. ∎

Theorem 4.7. *Let (X,ρ) be a metric space, $A \in CL(X)$ and $(A_k) \subset CL(X)$. Suppose that $f\colon [0,\infty) \to [0,\infty)$ is an unbounded modulus which satisfies the inequalities*

$$\lim_{t\to\infty} \frac{f(t)}{t} > 0 \text{ and } f(xy) \geq c\, f(x)\, f(y) \tag{4.12}$$

with some $c \in (0,\infty)$ for all $x, y \in [0,\infty)$. Then the following statements hold:

(*i*) *If (A_k) is Wijsman strongly Cesàro summable to A with respect to f, then (A_k) is f-Wijsman statistically convergent to A;*

(*ii*) *If (A_k) is Wijsman bounded and f-Wijsman statistically convergent to A, then (A_k) is Wijsman strongly Cesàro summable to A with respect to f.*

Proof. Let

$$K_{x,\varepsilon}(n) := \{k \leq n\colon |d(x,A_k) - d(x,A)| \geq \varepsilon\}$$

for all $x \in X$, $\varepsilon \in (0,\infty)$ and $n \in \mathbb{N}$.

(*i*) Let $[Ww^f] - \lim A_k = A$. By subadditivity of moduli we have

$$\sum_{k=1}^{n} f(|d(x,A_k) - d(x,A)|) \geq f\left(\sum_{k=1}^{n} |d(x,A_k) - d(x,A)|\right)$$

for every $x \in X$. Using the second inequality from (4.12) we obtain

$$f\left(\sum_{k\in K_{x,\varepsilon}(n)} |d(x,A_k) - d(x,A)|\right) \geq f\big(|K_{x,\varepsilon}(n)|\,\varepsilon\big) \geq cf\big(|K_{x,\varepsilon}(n)|\big) f(\varepsilon).$$

Hence

$$\frac{1}{n}\sum_{k=1}^{n} f(|\,d(x,A_k) - d(x,A)|) \geq c\left(\frac{f\left(|K_{x,\varepsilon}(n)|\right)}{f(n)}\right)\left(\frac{f(n)}{n}\right) f(\varepsilon). \tag{4.13}$$

This inequality, the first inequality from (4.12), $[Ww^f] - \lim A_k = A$ and $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$ imply $[WS^f] - \lim A_k = A$.

(*ii*) Let (A_k) be Wijsman bounded and let $[WS^f] - \lim A_k = A$. Since (A_k) is Wijsman bounded, we have

$$M_x := \sup_k |d(x,A_k)| + d(x,A)| < \infty. \tag{4.14}$$

For all $n \in \mathbb{N}$, $x \in X$ and $\varepsilon > 0$, we write $K'_{x,\varepsilon}(n) := \{1,\ldots,n\} - K_{x,\varepsilon}(n)$. Now,

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{n} f(|\,d(x,A_k) - d(x,A)|) \\ &\quad = \frac{1}{n}\sum_{k\in K_{x,\varepsilon}(n)} f(|d(x,A_k) - d(x,A)|) + \frac{1}{n}\sum_{k\in K'_{x,\varepsilon}(n)} f(|d(x,A_k) - d(x,A)|) \\ &\qquad \leq \frac{|K_{x,\varepsilon}(n)|}{n} f(M_x) + \frac{1}{n} n\, f(\varepsilon). \end{aligned}$$

Letting $n \to \infty$ we get

$$\frac{1}{n}\sum_{k=1}^{n} f(|\, d(x, A_k) - d(x, A)|) \leq f(\varepsilon),$$

in view of Theorem 2.1 and (4.14). Now the equality

$$[Ww^f] - \lim A_k = A$$

follows from $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$. ∎

Remark 4.8. If we take $f(x) = x$ in Theorem 4.7, we obtain Theorem 6 of Nuray and Rhoades [26].

It seems to be interesting to find a solution of the following problem.

Problem 4.9. Find characteristic properties of moduli f for which the equalities $[WS^f] - \lim A_k = A$ and $[Ww^f] - \lim A_k = A$ are equivalent for all bounded metric spaces (X, ρ), $(A_k) \subset CL(X)$ and $A \in CL(X)$.

Acknowledgments. We would like to thank the anonymous referee whose report helped in the improvement of the manuscript. Finally, the third author was supported by grant of the State Fund for Fundamental Research (project F71/20570) and partially supported by grant 0115U000136 of the Ministry Education and Science of Ukraine.

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