Transitive parallelism of residues in buildings

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Abstract

We study the buildings in which parallelism of residues is an equivalence relation. If the building admits a group action, we describe how parallel residues are related to residues with equal stabilizers. This permits to retrieve the fact that in a Coxeter group or in a graph product, intersections of parabolic subgroups are parabolic.

1   Introduction

In a building Δ, residues are convex subsets equipped with a natural building structure directly inherited from Δ. In [Tit74], J. Tits has introduced the notion of *projection* on residues that has been used extensively to study the abstract structure of buildings (see for instance [Ron89] or [AB08]). In particular, residues are sufficiently nicely embedded in Δ so that we can project the entire building on them *i.e* for any chamber $x \in \Delta$ and any residue $R \subset \Delta$ there exists a unique chamber $\text{proj}_R(x) \in R$ realizing the distance between x and R.

Two residues R and Q are *parallel* if

$$\text{proj}_R(Q) = R \text{ and } \text{proj}_Q(R) = Q.$$

This notion has been introduced by J. Tits in [Tit92] and is the object of an extensive study in [MPW15, Chapter 21]. The goal of this article is to study parallel residues and to relate this notion to residues with equal stabilizers under a group action.

1.1   Main results

A simple geometric argument (see Proposition 3.5 for details) shows that in a thin building parallelism is a transitive relation and thus is an equivalence relation on the set of all the residues. In the thick case, this holds if and only if Δ is right-angled (see [Cap14, Proposition 2.10]). We will study the intermediate case and characterize the buildings in which parallelism is a transitive relation by the structure of their residues of rank 2.

Theorem 1.1 (Theorem 3.7). *In a building Δ, parallelism is an equivalence relation on the set of all the residues if and only if any spherical residue of rank 2 is either thin or right-angled.*

The second result is a group theoretical application of Theorem 1.1. Therefore we will consider group actions on buildings. In this paper, all actions are assumed to be type-preserving. We recall that under this assumption, residues with equal stabilizers are parallel (see [MPW15, Proposition 22.3]). On the other hand, if the converse is true, then parallelism is an equivalence relation on the residues. In the thin and right-angled cases, the group is a Coxeter group or a graph product and we obtain the following corollary.

Corollary 1.2 (Corollary 4.5). *Let G be a Coxeter group or a graph product. Then, in G intersections of parabolic subgroups are parabolic.*

In the case of Coxeter groups, this Corollary is a classical fact due to J. Tits (see for instance [Dav08, Lemma 5.3.6] for another proof). In the case of graph products this corollary has been established recently by Y. Antolín and A. Minasyan by the means of Bass-Serre theory (see [AM15, Proposition 3.4]). The present article highlights in particular that these properties of Coxeter groups and graph products are true for the same reasons.

1.2   Organization of the article

In Section 2, we recall generalities about buildings, insisting on the notions of projections and right-angled buildings. Then, in Section 3 we discuss the notion of parallel residues and describe the buildings in which parallelism is an equivalence relation. Eventually, in Section 4, we study the situation where parallel residues admit the same stabilizers under a chamber-transitive group action.

1.3   Terminology and notation

All along this article, we will use the following conventions. The identity element in a group will always be designated by e. For a set E, the *cardinality* of E is designated by #E. If $\mathcal{G}$ is a graph then $\mathcal{G}^{(0)}$ is the *set of all the vertices* of $\mathcal{G}$ and $\mathcal{G}^{(1)}$ is the *set of all the edges* of $\mathcal{G}$. For $v, w \in \mathcal{G}^{(0)}$, we write $v \sim w$ if there exists an edge in $\mathcal{G}$ whose extremities are v and w.

Acknowledgment

2 Buildings

Buildings are both combinatorial and geometric objects introduced by J. Tits to study Lie groups of exceptional types. In this section, we give a quick introduction to buildings. We emphasize the notion of a projection and the particular case of right-angled buildings. Buildings are the objects of extensive introductions in [Ron89] and [AB08] to which we refer for details.

2.1 Chamber systems

Throughout this paper S is a fixed set.

Definition 2.1. *A chamber system X over S is a set endowed with a family of partitions indexed by S. The elements of X are called* chambers.

In this subsection, X is a chamber system over S. For $s \in S$, two chambers $c, c' \in X$ are said to be *s-adjacent* if they belong to the same subset of X in the partition associated with s. In this case, we write $c \sim_s c'$ and s is called the *type* of the adjacency relation. Usually, omitting the type we refer to *adjacent* chambers and we write $c \sim c'$. Note that any chamber is adjacent to itself.

A map $f : X \longrightarrow X'$ between two chamber systems X, X' over S is a called a *morphism* if it preserves the adjacency relations. If a morphism $f : X \longrightarrow X$ is a bijection, it is called an *automorphism* and if moreover f preserves the types of the adjacency relations, we say that f is a *type preserving automorphism*. We designate by $\mathrm{Aut_T}(X)$ the group of *type preserving automorphisms* of X. Given a subset Y of X, then Y inherits naturally the structure of a chamber system.

We call *gallery*, a finite sequence $\{c_k\}_{k=1,\dots,\ell}$ of chambers such that $c_k \sim c_{k+1}$ for $k = 1, \dots, \ell - 1$. The galleries induce a metric on X.

Definition 2.2. *The distance between two chambers x and y is the length of the shortest gallery connecting x to y and is designated by $d_c(x, y)$. A shortest gallery between two chambers is called* minimal.

For $I \subset S$, a subset C of X is said to be *I-connected* if for any pair of chambers $c, c' \in C$ there exists a gallery $c = c_1 \sim \cdots \sim c_\ell = c'$ such that for any $k = 1, \dots, \ell - 1$, the chambers c_k and c_{k+1} are i_k-adjacent for some $i_k \in I$.

Definition 2.3. *The I-connected components are called the I-residues or the residues of type I. The rank of an I-residue is the cardinality of I. The residues of rank 1 are called* panels.

We observe that a *I*-residue of a chamber system has a natural structure of a chamber system over *I*.

A subset *C* of *X* is called *convex* if every minimal gallery whose extremities belong to *C* is entirely contained in *C*. Convexity is stable by intersection and for $A \subset X$, the *convex hull* of *A* is the smallest convex subset containing *A*. In particular, convex subsets of *X* are subsystems and residues are convex.

The following example is crucial because it will be used to equip Coxeter groups and graph products with structures of chamber systems (see Definition 2.7 and Theorem 2.15).

Example 2.4. *Let G be a group, B a subgroup and $\{H_i\}_{i \in I}$ a family of subgroups of G containing B. The set of all the left cosets of H_i/B defines a partition of G/B. We denote by $C(G, B, \{H_i\}_{i \in I})$ this chamber system over I. This chamber system comes with a natural action of G. The group G is a group of type-preserving automorphisms of $C(G, B, \{H_i\}_{i \in I})$ and the action is chamber-transitive.*

In this paper we shall primarily be concerned with the case where $B = \{e\}$.

2.2 Coxeter systems

A *Coxeter matrix over S* is a symmetric matrix $M = \{m_{r,s}\}_{r,s \in S}$ whose entries are elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ for any $s \in S$ and $\{m_{r,s}\} \geq 2$ for any $r, s \in S$ distinct. Let *M* be a Coxeter matrix. The *Coxeter group* of type *M* is the group given by the following presentation

$$W = \langle s \in S | (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in S \rangle .$$

We call *special subgroup* a subgroup of *W* of the form

$$W_I = \langle s \in I | (rs)^{m_{r,s}} = 1 \text{ for any } r, s \in I \rangle \text{ with } I \subset S.$$

Definition 2.5. *A* parabolic subgroup *of W is a subgroup of the form $wW_I w^{-1}$ where $w \in W$ and $I \subset S$. An involution of the form wsw^{-1} for $w \in W$ and $s \in S$ is called a* reflection.

Example 2.6. *Let $\mathbb{X}^d = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$. A* Coxeter polytope *is a convex polytope of $\mathbb{X}^d$ such that any dihedral angle is of the form $\frac{\pi}{k}$ with k not necessarily constant. Let D be a Coxeter polytope and let $\sigma_1, \ldots, \sigma_n$ be the codimension 1 faces of D. We set $M = \{m_{i,j}\}_{i,j=1,\ldots,n}$ the matrix defined by $m_{i,i} = 1$, if σ_i and σ_j do not meet in a codimension 2 face $m_{i,j} = \infty$, and if σ_i and σ_j meet in a codimension 2 face $\frac{\pi}{m_{i,j}}$ is the dihedral angle between σ_i and σ_j.*

A theorem of Poincaré (see for instance [GP01, Theorem 1.2.]) says that the reflection group of $\mathbb{X}^d$ generated by the codimension 1 faces of D is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^d)$ *and is isomorphic to the Coxeter group of type M.*

Definition 2.7. *With the notation introduced in Example 2.4, the* Coxeter system *associated with W is the chamber system over S given by $C(W, \{e\}, \{W_{\{s\}}\}_{s \in S})$. We use the notation $\Sigma(W, S)$ to designate this chamber system.*

The chambers of $\Sigma(W, S)$ are the elements of W and two distinct chambers $w, w' \in W$ are s-adjacent if and only if $w = w's$. For $I \subset S$, notice that the I-residues of $\Sigma(W, S)$ are the left-cosets of W_I in W. Again W is a group of automorphisms of $\Sigma(W, S)$ and the action is chamber-transitive.

Now we recall classical terminology about Coxeter systems.

Definition 2.8. *i) Let $r = wsw^{-1}$ be a reflection for some $w \in W$ and $s \in S$. The wall M_r in $\Sigma(W, S)$ is the set of all the panels stabilized by r.*

ii) Let M be a wall and R be a residue. We say that M crosses R if one of the panels of M is contained in R.

In the particular case where W is a finite group, we refer to a *spherical* Coxeter group and system. If $M = \{m_{r,s}\}_{r,s \in S}$ with $\{m_{r,s}\} \in \{2, \infty\}$ for any $r \neq s$, then we refer to a *right-angled* Coxeter group or system.

2.3 Buildings

Hereafter (W, S) is a fixed Coxeter system.

Definition 2.9 ([Tit74, Definition 3.1.]). *A chamber system Δ over S is a building of type (W, S) if it admits a maximal family $\mathcal{A}p(\Delta)$ of subsystems isomorphic to $\Sigma(W, S)$, called apartments, such that*

- *any two chambers lie in a common apartment,*

- *for any pair of apartments A and B, there exists an isomorphism from A to B fixing $A \cap B$.*

If the group W is a spherical (resp. right-angled) Coxeter group then Δ is called a *spherical* (resp. *right-angled*) building.

Hereafter, Δ is a fixed building of type (W, S). A straightforward application of this definition is the existence of retraction maps of the building over apartments.

Definition 2.10. *Let $x \in \Delta$ and $A \in \mathcal{A}p(\Delta)$. Assume that x is contained in A. We call retraction onto A centered x the map $\pi_{A,x} : \Delta \longrightarrow A$ defined by the following property.*

> *For $c \in \Delta$, there exists a chamber $\pi_{A,x}(c) \in A$ such that for any apartment A' containing x and c, for any isomorphism $f : A' \longrightarrow A$ that fixes $A \cap A'$, then $f(c) = \pi_{A,x}(c)$.*

Example 2.11. *i) Any infinite tree without leaf is a building of type (W, S) where W is the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $S = \{(1, 0), (0, 1)\}$.*

ii) For $n \geq 1$ and $\mathbf{k}$ a field, the flags of subspaces of a n dimensional vector space over $\mathbf{k}$ is a spherical building (see [Ron89, Chapter 1]). On Figure 1 is represented the geometric realization of the building of $\mathbf{k}^3$ where $\mathbf{k}$ is the finite field of order 2.

The building Δ is called a *thin* (resp. *thick*) building if any panel contains exactly two (resp. at least three) chambers. Note that thin buildings are Coxeter systems. We recall that an *I*-residue of Δ is itself a building of type (W_I, I). Hence it makes sense to talk about *thin*, *thick*, *spherical* or *right-angled* residues.

For x and y two chambers of Δ, the convex hull of the pair $\{x, y\}$ in Δ is the convex hull of $\{x, y\}$ in any apartment containing x and y (see [Tit74, Proposition 3.18.]). This fact permits to build projections on residues.

Proposition 2.12 ([Tit74, Proposition 3.19.3.]). *Let R be a residue and x be a chamber in Δ. There exists a unique chamber $\mathrm{proj}_R(x) \in R$ such that $d_c(x, \mathrm{proj}_R(x)) = \mathrm{dist}(x, R)$. Moreover, for any chamber y in R there exists a minimal gallery from x to y passing through $\mathrm{proj}_R(x)$.*

Observe that not all convex subsets of a building admit projection maps. Indeed, let P be a panel of Δ. Then any subset C of P is convex. However, the projection of Δ onto C exists if and only if $\#C = 1$ or $C = P$.

2.4 Graph products and right-angled buildings

Let $\mathcal{G}$ denote a *simplicial graph i.e* no edge is a loop and no edge is double. If in $\mathcal{G}$ two distinct vertices v and v' are connected by an edge, we write $v \sim v'$. A group G_v is associated with each $v \in \mathcal{G}^{(0)}$ and we denote by $F_{\mathcal{G}}$ the free product of the family $\{G_v\}_{v \in \mathcal{G}^{(0)}}$.

Definition 2.13. *The* graph product *given by the pair $(\mathcal{G}, \{G_v\}_{v \in \mathcal{G}^{(0)}})$ is the group defined by the following quotient*

$$\Gamma = F_{\mathcal{G}} / R,$$

where R is the normal subgroup $\langle\langle gg'g^{-1}g'^{-1} : g \in G_v, g' \in G_{v'} \text{ and } v \sim v' \rangle\rangle$.

Example 2.14. *If all the groups $\{G_v\}_{v \in \mathcal{G}^{(0)}}$ are of order 2 then Γ is a right-angled Coxeter group. If all the groups $\{G_v\}_{v \in \mathcal{G}^{(0)}}$ are infinite cycles, then Γ is a right-angled Artin group (see [Cha07]). In fact, all right-angled Coxeter and Artin groups may be obtained as a graph product. If $\mathcal{G}$ has no edge Γ is a free product and if $\mathcal{G}$ is a complete graph Γ is a free abelian product.*

Now we designate by S the set $\mathcal{G}^{(0)}$. This is motivated by the fact that a Coxeter group is canonically associated to a graph product. From now on, we fix a graph product Γ given by a pair $(\mathcal{G}, \{G_s\}_{s \in S})$. Then, the graph product defined by the pair $(\mathcal{G}, \{\mathbb{Z}/2\mathbb{Z}\}_{s \in S})$ is isomorphic to the right-angled Coxeter group defined by the matrix $M = \{m_{s,t}\}_{s,t \in S}$ given by: $m_{s,t} = 2$ if $s \sim t$ and $m_{s,t} = \infty$ if $s \nsim t$ in $\mathcal{G}$. We denote by W this Coxeter group and by (W, S) the associated Coxeter system.

With this notation, the following theorem associates a right-angled building to a graph product.

Theorem 2.15 ([Dav98, Theorem 5.1.]). *Let Δ be the chamber system $C(\Gamma, \{e\}, \{G_s\}_{s \in S})$ (see Example 2.4). Then Δ is a building of type (W, S).*

Right-angled buildings in which all the panels of same type have same cardinality have been classified by F. Haglund and F. Paulin.

Theorem 2.16 ([HP03, Proposition 5.1.]). *Let Δ be a right-angled building of type (W, S). If in Δ all panels of the same type are of the same cardinality, then Δ is uniquely determined, up to isomorphism, by these cardinalities.*

Along with Theorem 2.15, this implies that all the right-angled buildings in which all panels of the same type are of the same cardinality are obtained from graph-products. Namely, if Δ is such a building of type (W, S), let $\mathcal{G}$ be the graph of relations associated to (W, S). Then, for each $s \in S$, fix a group G_s of cardinality equal to the cardinality of the s-panels of Δ. Then Δ is isomorphic to the building associated to the graph product given by the pair $(\mathcal{G}, \{G_s\}_{s \in S})$.

By analogy with Definition 2.5, we define parabolic subgroups in Γ.

Definition 2.17. *For $I \subset S$ we write $\Gamma_I = \langle G_s : s \in I \rangle$ and a subgroup of the form $g\Gamma_I g^{-1}$, with $g \in \Gamma$, is called a* parabolic subgroup *of Γ.*

3 Parallel residues

Parallel residues have been defined by J. Tits in [Tit92]. This notion derives from the notion of opposite residues. We refer to [MPW15, Chapter 21] for details about parallel residues in general and to [Wei03, Chapters 5 and 9] for details about opposite residues.

The goal of this section is to study the buildings in which parallelism of residues is a transitive relation.

3.1 Definition and first properties

In the rest of the paper, Δ is a building of type (W, S).

Definition 3.1. *Let R and Q be two residues in Δ. We say that R is* parallel *to Q if*

$$\mathrm{proj}_R(Q) = R \text{ and } \mathrm{proj}_Q(R) = Q.$$

The following proposition summarizes some basic properties of parallel residues.

Proposition 3.2 ([MPW15, Propositions 21.8 and 21.17]). *Let R and Q be respectively a I-residue and a J-residue in Δ. Let $Q' = \mathrm{proj}_Q(R)$ and $R' = \mathrm{proj}_R(Q)$. Then the following properties hold.*

   i) R is parallel to Q if and only if for any apartment A containing a chambers of both R and Q the residues $R \cap A$ and $Q \cap A$ are parallel in A.

   ii) R' and Q' are parallel residues.

   iii) The maps $\mathrm{proj}_{R|Q'} : Q' \longrightarrow R'$ and $\mathrm{proj}_{Q|R'} : R' \longrightarrow Q'$ are reciprocal bijections.

iv) *For any $x, y \in R'$, $d_c(x, \text{proj}_Q(x)) = d_c(y, \text{proj}_Q(y))$.*

v) *There exists a unique $w(R, Q) \in W$ such that for any apartment A containing a chamber of both R' and Q', for any chamber x in $R' \cap A$ one has in A : $w(R, Q)x = \text{proj}_{Q'}(x)$.*

vi) *Let $w = w(R, Q)$, then R' (resp. Q') is of type $I' = \{s \in I : w^{-1}sw = t$ for some $t \in J\}$ (resp. $J' = \{s \in J : w^{-1}sw = t$ for some $t \in I\}$).*

3.2 Opposite residues

In this subsection, Δ is a spherical building. If not further specified, the proofs of the following claims are contained in [Wei03, Chapters 5 and 9].

For a chamber x in Δ, a chamber y is called *opposite* to x if $d_c(x, y) = \text{diam}\,\Delta$, where d_c is the distance over the chambers given by Definition 2.2. Clearly this definition is empty in the non-spherical case. However, it is very rich in the spherical case. Indeed, for any chamber x and apartment A containing x, there exists a unique chamber y opposite to x contained in A. We denote by $\text{op}_A : A \longrightarrow A$ the map sending a chamber to its opposite chamber in A.

Definition 3.3. *Let R and Q be two residues in Δ. We say that R and Q are* opposite *residues if there exists an apartment A intersecting both R and Q such that*

$$\text{op}_A(R) = Q \cap A \text{ and } \text{op}_A(Q) = R \cap A.$$

In fact two residues are opposite if and only if the condition of the preceding definition is satisfied for any apartment A intersecting both R and Q.

Opposite residues are parallel (see for instance[MPW15, Proposition 21.24]). However the converse is false as any two chambers are always parallel. In fact, the notions of opposite and parallel residues in a spherical building are connected by the following proposition.

Proposition 3.4 ([MPW15, Proposition 21.26]). *Two parallel residues R and Q are opposite if and only if for some chamber $x \in R$ there exists a chamber $y \in Q$ opposite to x in Δ.*

Moreover, it appears that for any residue $R \subsetneq \Delta$ and apartment A, there exists a residue Q such that A intersects Q and R is opposite to Q. Thus, any residue admits an opposite (and thus a parallel) residue. This is not true in the non-spherical case. Indeed, in the thin building associated with the infinite dihedral group, no panel admits a parallel residue.

3.3 Parallel residues defining an equivalence relation

By simple geometric arguments, we observe that in a thin building parallel residues are characterized by walls.

Proposition 3.5 ([MPW15, Proposition 21.19]). *In a thin building, two residues are parallel if and only if the set of walls that cross them are equal.*

A consequence of the preceding proposition is that in a thin building, parallelism is an equivalence relation on the residues. In general, the relation induced by the parallelism may not be transitive as illustrated by the example of Figure 1. In fact, in the thick case, this happens if and only if the building is right-angled (see [Cap14, Proposition 2.10]).
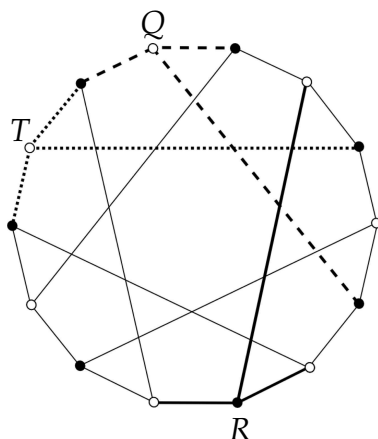

Figure 1: In this spherical building, R is parallel to Q and T but Q is not parallel to T.

In the following we observe that this strong property leaves only few examples between thin and right-angled buildings. To this end we will use several times the following fact: two panels σ and σ' are parallel if and only if there exists an apartment A in which $\sigma \cap A$ is parallel to $\sigma' \cap A$. This follows directly from the definition of the projections.

We start by establishing a short lemma about thin and right-angled residues in Δ.

Lemma 3.6. *Let R be a thin or a right-angled residue in Δ, let σ and σ' be two parallel panels in R and let δ be a panel in Δ. If δ is parallel to σ then it is parallel to σ'.*

Proof. If R is thin, then any apartment containing σ also contains σ'. Hence the lemma is satisfied by Propositions 3.2.i) and 3.5.

Now we assume that R is right-angled. We define $\delta' := \text{proj}_R(\delta)$ and we observe that, as σ is contained in R, one has $\text{proj}_\sigma(\delta) = \text{proj}_\sigma(\delta')$. If δ' is not parallel to σ then $\text{proj}_\sigma(\delta')$ is a single chamber. But this is absurd because $\text{proj}_\sigma(\delta') = \text{proj}_\sigma(\delta)$ and σ is parallel to δ. Likewise, if δ' is not parallel to δ then δ' is a single chamber. This implies again that $\text{proj}_\sigma(\delta)$ is a single chamber which is absurd. Hence δ' is parallel to both σ and δ.

As in R parallelism is an equivalence relation, δ' is parallel to σ'. By convexity of the apartments and by Proposition 2.12, any apartment intersecting both δ and σ' intersects δ'. Then, by transitivity of parallelism in the apartments and by Propositions 3.2.i), we obtain that δ is parallel to σ. ∎

To characterize buildings in which parallelism is an equivalence relation on the set of all the residues, we will use the following notation. For σ and δ two panels in Δ, for an apartment A intersecting both σ and δ, we write r and t for

the reflections in A stabilizing respectively $\sigma \cap A$ and $\delta \cap A$. We call *order* of the pair (σ, δ), and we write $\mathrm{Ord}(\sigma, \delta)$, the order of rt in W and we observe that $\mathrm{Ord}(\sigma, \delta)$ is well defined *i.e* it does not depend on the choice of A. Moreover, as a consequence of Proposition 3.2.*i*), σ is parallel to δ, if and only if $\mathrm{Ord}(\sigma, \delta) = 1$.

Theorem 3.7. *In a building Δ, the following are equivalent:*

  i) Parallelism is an equivalence relation on the set of all the residues.

  ii) Parallelism is an equivalence relation on the set of all the panels.

  iii) For any pair of panels σ and δ, if σ is thick then $\mathrm{Ord}(\sigma, \delta) \in \{1, 2, \infty\}$.

  iv) Any spherical residue of rank 2 is either thin or right-angled.

Proof. The implications $i) \implies ii)$ and $iii) \implies iv)$ are immediate and we start by proving $ii) \implies i)$. Let R, Q and T be residues such that R and Q are parallel to T. Let A be an apartment intersecting both R and Q. Let M be a wall crossing $A \cap R$. By Propositions 3.2.*i*) and 3.5, it is enough to prove that M crosses $A \cap Q$. To this end let $\sigma \subset R$ be a panel such that $\sigma \cap A$ is crossed by M. Then, as R is parallel to T, there exists a panel σ_T in T that is parallel to σ.

Now pick an apartment A' intersecting both σ_T and $A \cap Q$. As $A' \cap T$ is parallel to $A' \cap Q$, there exists a panel $\delta_{A'}$ in $A' \cap Q$ that is parallel to $\sigma_T \cap A'$. We observe that the panel $\delta \subset Q$ containing $\delta_{A'}$ is parallel to σ_T. Then, by assumption, σ is parallel to δ and by Proposition 3.5, M crosses $\delta \cap A$.

We prove $ii) \implies iii)$ by contradiction (this step is essentially the same as the proof of [Cap14, Proposition 2.10]). Let σ be a thick panel and δ be a panel such that $\mathrm{Ord}(\sigma, \delta) = n > 2$. In an apartment A intersecting both σ and δ, the wall crossing $\sigma \cap A$ intersects the wall crossing $\delta \cap A$. As a consequence, Δ contains a residue R of rank 2 that is not right-angled nor thin.

Then, we set $\sigma' = \mathrm{proj}_R(\sigma)$ and we choose two distinct panels σ_1, σ_2 of the same type, contained in R, lying at a minimal distance and containing a chamber of σ'. Choose an apartment A and a chamber $x \in A$ such that $\pi_{A,x}(\sigma_1) = \pi_{A,x}(\sigma_2)$. In $R \cap A$, let δ'_A be the panel opposite to $\pi_{A,x}(\sigma_1)$ and let δ' be the panel in Δ such that $T \cap A = T_A$. As $\pi_{A,x}$ decrease the distance over the chambers, then δ' is opposite to both σ_1 and σ_2 in R and thus is parallel to them in Δ.

Here we prove $iv) \implies ii)$. Let σ, σ' and δ be three panels such that σ and σ' are parallel to δ. We prove the implication by induction on $d = \max\{\mathrm{dist}(\sigma, \delta), \mathrm{dist}(\sigma', \delta)\}$. If $d = 0$ there is nothing to prove.

If $d > 0$, consider R a residue of rank 2 containing σ and such that $\mathrm{dist}(\delta, R) < \mathrm{dist}(\delta, \sigma)$ and choose an apartment A intersecting both σ and δ. There exists a panel T_A in $R \cap A$ such that T_A is parallel to $\sigma \cap A$ and $\mathrm{dist}(\delta, R) = \mathrm{dist}(\delta \cap A, T_A)$. This panel is the panel opposite to $\sigma \cap A$ in $R \cap A$. We designate by T the panel in R containing T_A and we check that T is parallel to both σ and δ. We do the same with σ' and we obtain T' parallel to both σ' and δ and such that $\mathrm{dist}(T', \delta) < d$.

Now, by the induction assumption, we obtain that T is parallel to T'. To finish, we observe that σ and T (resp. σ' and T') are contained in thin or right-angled residues and the proof is achieved by Lemma 3.6. ∎

3.4 A generalization of a result of Haglund and Paulin

As it is suggested by Theorem 3.7, the buildings in which parallelism is an equivalence relation are made of thin and right-angled residues. In fact, these buildings are obtained from right-angled buildings by substituting a given Coxeter system for chambers. In this subsection, we explain this in detail and observe that this implies a generalization of Theorem 2.16.

In the rest of this section, Δ is a building of type (W, S) satisfying the equivalent conditions of Theorem 3.7. Let $M = \{m_{s,r}\}_{s,r \in S}$ be the Coxeter matrix associated to (W, S). We set:

- $S_\perp := \{s \in S : m_{s,r} \in \{2, \infty\} \text{ for any } r \neq s\}$,

- $S_T := S \backslash S_\perp$,

- $S'_\perp = \{wsw^{-1} \in W : w \in W_{S_T} \text{ and } s \in S_\perp\}$.

The set $S_\perp$ is the set of possibly thick types of Δ. If we designate by R_T a S_T-residue in $\Sigma(W, S)$, then $S'_\perp$ is the set of all the walls that bound R_T. Observe that $S_\perp$ is not always equal to $S'_\perp$. For instance, if S is finite and if there exists $s \in S_\perp$ and $r, t \in S_T$ such that $m_{s,t} = m_{r,t} = \infty$, then $S'_\perp$ is infinite.

By assumptions, two reflections $r, r' \in S'_\perp$ either commute or generate an infinite dihedral subgroup in W. In particular, the subgroup $\langle S'_\perp \rangle$ is a right-angled Coxeter group and we designate by $(W_\perp, S'_\perp)$ the associated Coxeter system. Now we observe that the set of all the S_T-residues of Δ inherits from Δ a structure of right-angled building of type $(W_\perp, S'_\perp)$. We denote by $\Delta_\perp$ this building.

Applying to $\Delta_\perp$ the classification of right-angled buildings due to F. Haglund and F. Paulin, we obtain the following.

Proposition 3.8. *Let Δ be a building of type (W, S) satisfying the equivalent conditions of Theorem 3.7 and let $\Delta_\perp$ be the right-angled building of the S_T-residues of Δ. If in $\Delta_\perp$ all panels of the same type are of the same cardinality, then Δ is uniquely determined, up to isomorphism, by these cardinalities.*

Proof. Let Δ and Δ' be two buildings of same type (W, S) satisfying the equivalent conditions of Theorem 3.7. Let $\Delta_\perp$ and $\Delta'_\perp$ be the two right-angled buildings associated to them and assume that in $\Delta_\perp$ and $\Delta'_\perp$ all panels of the same type are of the same cardinality. If these cardinalities are equal then, by Theorem 2.16, $\Delta_\perp$ and $\Delta'_\perp$ are isomorphic.

We consider an isomorphism $F : \Delta_\perp \longrightarrow \Delta'_\perp$. We fix two base chambers $x_0 \in \Delta$ and $x'_0 \in \Delta'$ and two apartments A_0 and A'_0 respectively in Δ and Δ' containing the base chambers. We also designate by $f : A_0 \longrightarrow A'_0$ the isomorphism mapping $x_0 \longmapsto x'_0$.

Then we define the map $\Phi : \Delta \longrightarrow \Delta'$ extending f as follow. For $x \in \Delta$, let $R \in \Delta_\perp$ be the S_T-residue containing x. We define $\Phi(x)$ to be the only chamber in $F(R)$ such that $f \circ \pi_{A_0, x_0}(x) = \pi_{A'_0, x'_0} \circ \Phi(x)$. In other word

$$\Phi(x) := \pi^{-1}_{A'_0, x'_0}(f \circ \pi_{A_0, x_0}(x)) \cap F(R).$$

It is immediate to verify that Φ is a well defined bijection and preserves the adjacency relations. ∎

As a particular case we obtain the following corollary that generalizes Theorem 2.16.

Corollary 3.9. *Let Δ be a building of type (W, S) satisfying the equivalent conditions of Theorem 3.7. If in Δ all panels of the same type are of the same cardinality, then Δ is uniquely determined, up to isomorphism, by these cardinalities.*

4   Parallel residues and stabilizers

In this section, G is a subgroup of $\mathrm{Aut}_T(\Delta)$ acting chamber-transitively. Here we discuss the relationship between the fact that two residues are parallel and the fact that these two residues have same stabilizers under the action of G.

4.1   Parallel residues with equal stabilizers

First, we recall that as the action of G is chamber-transitive, then two residues with equal stabilizers are parallel (see [MPW15, Proposition 22.3]). The following proposition describes the situation where the converse is true.

Proposition 4.1. *Suppose that $\mathrm{Stab}_G(P) = \mathrm{Stab}_G(P')$ for any pair P, P' of parallel residues. Then the following properties are satisfied:*

  i) *Parallelism is an equivalence relation on the residues. In particular, Δ satisfies the equivalent conditions of Theorem 3.7.*

 ii) *The action is free.*

iii) *For any pair of residues R, Q one has*

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) = \mathrm{Stab}_G(\mathrm{proj}_R(Q)) = \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

Proof. i) Under the hypothesis of the proposition, two residues have same stabilizer if and only if they are parallel.

ii) Let x be a chamber in Δ. As any pair of chambers are parallel residues, for all $y \in \Delta$, one has $\mathrm{Stab}_G(x) = \mathrm{Stab}_G(y)$. Thus $\mathrm{Stab}_G(x) = \{e\}$.

iii) By symmetry, it is sufficient to prove, that

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) = \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

Let $g \in \mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q)$. As g is an automorphism of Δ that stabilizes both R and Q, the map $\mathrm{proj}_{Q|R}(\cdot)$ is equivariant by g. Then $g(\mathrm{proj}_Q(R)) = \mathrm{proj}_Q(R)$ and

$$\mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) < \mathrm{Stab}_G(\mathrm{proj}_Q(R)).$$

Let $g \in \mathrm{Stab}_G(\mathrm{proj}_Q(R))$. As g preserves the types, if Q is a I-residue then $g(Q)$ is also a I-residue. In particular, Q and $g(Q)$ are two I-residues containing $\mathrm{proj}_Q(R)$, thus $g(Q) = Q$. As $\mathrm{proj}_Q(R)$ is parallel to $\mathrm{proj}_R(Q)$, under our assumption $g \in \mathrm{Stab}_G(\mathrm{proj}_R(Q))$. We can use the previous argument to prove that $g(R) = R$ and

$$\mathrm{Stab}_G(\mathrm{proj}_Q(R)) < \mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q). \qquad \blacksquare$$

In the rest of the section, we assume that the action of G is chamber-transitive and that the assumption of the preceding proposition hold.

In the thin case, it is clear that G is isomorphic to W. In the right-angled case, the next proposition says that it is isomorphic to a graph product of stabilizers of panels. To this end, we will use the following notation. For a right-angled Coxeter group W, we designate by $\mathcal{G}_W$ the simplicial graph such that the graph product given by $(\mathcal{G}_W, \{\mathbb{Z}/2\mathbb{Z}\}_{s \in S})$ is isomorphic to W.

Proposition 4.2. *Let Δ be a right-angled building of type (W, S) and G a group of type preserving automorphisms acting freely and chamber-transitively on Δ. Let x_0 be a chamber in Δ and let G_s be the stabilizer in G of the s-panel containing x_0. Then G is isomorphic to the graph product given by the pair $(\mathcal{G}_W, \{G_s\}_{s \in S})$.*

Proof. Let Γ be the graph product given by the pair $(\mathcal{G}_W, \{G_s\}_{s \in S})$. To Γ we associate the right-angled building Δ_Γ given by Theorem 2.15. We observe that Δ_Γ is of type (W, S) and that for $\sigma_s(\Delta)$ and $\sigma_s(\Delta_\Gamma)$ two panels of type $s \in S$ respectively in Δ and in Δ_Γ one has:

$$\#\sigma_s(\Delta) = \#\sigma_s(\Delta_\Gamma).$$

Hence, by Theorem 2.16, Δ and Δ_Γ are isomorphic and we both denote them Δ. As a consequence, Γ is the subgroup of $\mathrm{Aut}_T(\Delta)$ generated by the set $\{G_s\}_{s \in S}$. In particular, this proves that $\Gamma < G$.

Now, for $g \in G$, we prove by induction on $n = d_c(x_0, gx_0)$ that g is a product of elements of $\{G_s\}_{s \in S}$. If $n = 0$ there is nothing to prove. If $n > 0$ consider a minimal gallery:

$$x_0 \sim \cdots \sim x_{n-1} \sim x_n = gx_0.$$

Let $h \in G$ be such that $hx_{n-1} = x_n$. As h preserves the type, $h \in \mathrm{Stab}_G(\sigma)$ where σ is the s-panel containing $\{x_{n-1}, x_n\}$. Let $\gamma \in G$ be such that $\gamma x_0 = x_{n-1}$. In particular, $\sigma = \gamma \sigma_s$, $\mathrm{Stab}_G(\sigma) = \gamma G_s \gamma^{-1}$ where σ_s is the s-panel containing x_0 and $h = \gamma g_s \gamma^{-1}$ for one $g_s \in G_s$. Then, by freeness of the action, $g = h\gamma$ and with $\mathrm{dist}(x_0, \gamma x_0) = n - 1$ the proof is achieved. ∎

4.2 Application to intersection of parabolic subgroups

In this section, we apply Proposition 4.1 to thin and right-angled buildings under the action of Coxeter groups and graph products.

First we verify that the assumption of the theorem are satisfied in the case of a Coxeter groups.

Proposition 4.3. *If Δ is a thin building, then parallel residues have equal stabilizers.*

Proof. Let R be a residue. Here we prove that the stabilizer of R under the action of W is the subgroup $G < W$ generated by the reflections about the walls that cross R. This will imply the proposition by Proposition 3.5.

As W is type preserving, it is clear that $G < \mathrm{Stab}_W(R)$. Now we fix $x_0 \in R$ and for $g \in \mathrm{Stab}_W(R)$ we consider a minimal gallery

$$x_0 \sim x_1 \sim \cdots \sim x_n = gx_0.$$

By convexity of the residues, this gallery is contained in R. Let $r_i \in W$ be the reflection that maps x_i to x_{i+1}. Then, by simple chamber-transitivity of the action, $g = r_n \dots r_0$ and the proof is complete. ∎

In the right-angled case we establish an analogue proposition.

Proposition 4.4. *Let Γ be the graph-product given by a pair $(\mathcal{G}, \{G_s\}_{s \in S})$ and let Δ be the associated right-angled building. Then any two parallel residues of Δ have equal stabilizers.*

Proof. Let R and Q be two parallel residues. Up to a conjugation, we can assume that x_0 is in R. According to Proposition 3.2.*vii)*, R and Q are of same type I. We write
$$I^\perp = \{s \in S \backslash I : v_s \sim v_i \text{ for all } i \in I\}.$$
By [Cap14, Proposition 2.8.*ii)*], R and Q are both contained in T a J-residue where $J = I \cup I^\perp$. We observe that $\Gamma_J = \Gamma_I \times \Gamma_{I^\perp}$ and that $\mathrm{Stab}_\Gamma(T) = \Gamma_J$. As a consequence, Γ_J acts transitively on the set of I residues contained in T. Thus, there exists $g \in \Gamma_J$ such that $gR = Q$. Hence $\mathrm{Stab}_\Gamma(Q) = g\mathrm{Stab}_\Gamma(R)g^{-1}$ and with $\mathrm{Stab}_\Gamma(R) = \Gamma_I$ the proposition is proved. ∎

Now we know that both actions of Coxeter groups and of graph-products on their associated buildings satisfy the assumption of Proposition 4.1. In the next proposition we obtain from this fact that intersections of parabolic subgroups are parabolic.

From now on, Δ is either a thin or a right-angled building of type (W, S). We fix a base chamber $x_0 \in \Delta$ and for $s \in S$ we denote by σ_s the s-panel containing x_0. The group G is a group acting freely and chamber-transitively on Δ. In fact, G is either W in the thin case or a graph product Γ in the right-angled case (see Proposition 4.2). For $I \subset S$ we set
$$G_I := \langle \mathrm{Stab}_G(\sigma_s) : s \in I \rangle.$$
In fact, G_I is either W_I in the thin case or Γ_I in the right-angled case (see Definitions 2.5 and 2.17). We recall that a parabolic subgroup $gG_Ig^{-1} < G$ stabilizes the I-residue $R = gG_Ix_0$. We also recall that, according to Proposition 3.2.*v)*, for R and Q two residues, $w(R, Q) \in W$ is such that for any apartment A containing a chamber of both $\mathrm{proj}_R(Q)$ and $\mathrm{proj}_Q(R)$ and for any chamber x in $\mathrm{proj}_R(Q) \cap A$ one has in A: $w(R, Q)x = \mathrm{proj}_Q(x)$.

Corollary 4.5. *For $g \in G$, and $I, J \subset S$, let $R = G_Ix_0$ and $Q = gG_Jx_0$. Then*
$$G_I \cap gG_Jg^{-1} = \gamma G_K \gamma^{-1},$$
where $\gamma \in G_I$ and $K = \{s \in I : w^{-1}sw = t \text{ for some } t \in J\}$ with $w = w(R, Q)$.

Proof. Let $P = G_I \cap gG_Jg^{-1}$, we choose $\gamma \in G_I$ such that $\mathrm{dist}(\gamma x_0, Q) = \mathrm{dist}(R, Q)$. As in Δ parallel residues have equal stabilizers, with Proposition 4.1
$$P = \mathrm{Stab}_G(R) \cap \mathrm{Stab}_G(Q) = \mathrm{Stab}_G(\mathrm{proj}_R(Q)) = \gamma G_K \gamma^{-1}.$$
On the other hand, the type K of the residue $\mathrm{proj}_R(Q)$ is given by Proposition 3.2.*vi)* which finishes the proof. ∎

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