On the Frobenius vector of some simplicial affine semigroups

Ali Mahdavi Farhad Rahmati*

Abstract

We give a formula for a Frobenius vector of a Gorenstein simplicial affine semigroup S, and when the semigroup is Cohen-Macaulay we give an algorithm computing the set of minimal Frobenius vectors of S for a special class of semigroups.

1 Introduction and basic notions

The set of nonnegative integers will be denoted by $\mathbb{N}$. An affine semigroup is a finitely generated submonoid of $\mathbb{N}^r$ for some positive integer r. Let $S = \langle a_1, \ldots, a_{r+m} \rangle$ be an affine semigroup generated by $A = \{a_1, \ldots, a_{r+m}\} \subset \mathbb{N}^r$, that is to say, $S = \mathbb{N}a_1 + \mathbb{N}a_2 + \cdots + \mathbb{N}a_{r+m}$. In such a case, A will be said to be a system of generators of S. Moreover, if no proper subset of A generates S, the set A is a minimal system of generators of S. Every affine semigroup has a unique minimal system of generators (see [9, Chapter 3]). Let K be a field. The ring $K[S]$ is defined as the subalgebra of $K[y_1, \ldots, y_r]$ generated by $y^{a_1}, y^{a_2}, \ldots$ and $y^{a_{r+m}}$ with $y^\alpha := y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_r^{\alpha_r}$ where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$. The semigroup S is said to be Cohen-Macaulay (Gorenstein) if $K[S]$ is. Let I_S, called the semigroup ideal of S, be the kernel of K-algebra homomorphism from $K[x_1, \ldots, x_{r+m}]$ to $K[S]$ defined by $x_i \mapsto y^{a_i}$. For $u = (u_1, \ldots, u_{r+m}) \in \mathbb{N}^{r+m}$, we define the S-degree of the monomial x^u by $\sum_{i=1}^{r+m} u_i a_i$ and denote by $\deg_S(x^u)$. The semigroup S is said to be complete intersection if I_S is a complete intersection ideal. It is well-known that

*Corresponding author.

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I_S is a binomial prime ideal ([5, Proposition 1.4]). When $r = 1$ and $a_1, \ldots, a_{m+1}$ are relatively prime positive integers, the semigroup is called numerical semigroup. In this case $\mathbb{N} \setminus S$ is a finite set. For a numerical semigroup S the largest integer $f^*(S)$ in $\mathbb{N} \setminus S$ is called the Frobenius number of S, and the problem of finding this number is called the Frobenius problem. The Frobenius number occurs in many branches of mathematics and is one of the most studied invariants in the theory of numerical semigroups. This problem has attracted substantial attention in the last 100+ years (see [4], [7], [8]). There is no general formula for the Frobenius number for m greater than one. Sylvester in [14] proved that for $m = 1$, $f^*(S) = a_1a_2 - a_1 - a_2$.

The Frobenius problem is generalized to the higher dimensional cases (see [1], [2], [15], [16]). The vector Frobenius problem of Cohen-Macaulay and Gorenstein simplicial affine semigroup is studied in the next section. It is shown that every simplicial affine semigroup has at least one minimal Frobenius vector and an algorithm is presented for computing minimal Frobenius vectors of some Cohen-Macaulay simplicial affine semigroup.

2 Frobenius vector

Let S be the affine semigroup generated by $A = \{a_1, \ldots, a_{r+m}\}$ in $\mathbb{N}^r$ and $\mathbf{G}(S)$ be the group generated by S in $\mathbb{Z}^r$, that is, $\mathbf{G}(S) = \{a - b \,|a, b \in S\}$. We use $\mathbf{G}(a_1, \ldots, a_n)$ to denote the group generated by $\{a_1, \ldots, a_n\}$.

Definition 1. The affine semigroup S is called *simplicial* if there exist $a_{i_1}, \ldots, a_{i_r} \in A$ such that
(1) $a_{i_1}, \ldots, a_{i_r}$ are linearly independent over $\mathbb{Q}$ and
(2) for every $a \in S$, there exists $0 \neq n \in \mathbb{N}$ such that $na \in \mathbb{N}a_{i_1} + \cdots + \mathbb{N}a_{i_r}$.

If r is lesser than three, every affine semigroup is simplicial. From now on, we will suppose that S is a simplicial affine semigroup. Assume without loss of generality that $\{i_1, \ldots, i_r\} = \{1, \ldots, r\}$. The Apéry set of $a \neq 0$ in S is defined as $\mathrm{Ap}(S, a) = \{x \in S \,|x - a \notin S\}$. Let k_i be the smallest natural number such that $k_ia_{r+i} \in \sum_{i=1}^r \mathbb{N}a_i$, for $i = 1, \ldots m$. By definition,

$$\bigcap_{i=1}^{r} \mathrm{Ap}(S, a_i) \subseteq \left\{ \sum_{i=1}^{m} t_i a_{r+i} \,|0 \leq t_i < k_i \right\},$$

so $\cap_{i=1}^r \mathrm{Ap}(S, a_i)$ is finite. The set $\cap_{i=1}^r \mathrm{Ap}(S, a_i)$ is also called the Apéry set of S relative to $E := \{a_1, \ldots, a_r\}$. Observe that $\mathrm{Ap}(S, E) := \{s \in S \mid s - e \notin S, \forall e \in E\} = \cap_{i=1}^r \mathrm{Ap}(S, a_i)$. The set $\{x^\alpha \,|\deg_S(x^\alpha) \in \mathrm{Ap}(S, E)\}$ is a basis for $\frac{K[x_1,\ldots,x_{r+m}]}{\langle I_S, x_1,\ldots,x_r\rangle}$ as a $K-$vector space.

The following proposition gives a useful criterion for determining whether or not a simplicial affine semigroup is Cohen-Macaulay (see [11, Corollary 1.6]).

Proposition 1. *If S is a simplicial affine semigroup, the following statements are equivalent:*
- *$K[S]$ is Cohen-Macaulay;*
- *For all $\omega_1, \omega_2 \in \mathrm{Ap}(S, E)$, if $\omega_1 \neq \omega_2$, then $\omega_1 - \omega_2 \notin \mathbf{G}(a_1, \ldots, a_r)$.*

By definition, every element $a \in S$ can be written as $a = \sum_{i=1}^{r} \alpha_i a_i + \omega$, with $\omega \in \mathrm{Ap(S, E)}$ and $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, r$. Let $a = \sum_{i=1}^{r+m} z_i a_i \in \mathbf{G}(S)$ and $z_k < 0$ for some $k \in \{r+1, \ldots, r+m\}$. Since S is simplicial, there exists $n_k \in \mathbb{N}$ such that $(n_k - z_k)a_k \in \mathbb{N}a_1 + \cdots + \mathbb{N}a_r$. So $a = \sum_{i=1, i\neq k}^{r+m} z_i a_i + (z_k - n_k)a_k + n_k a_k$ and $(z_k - n_k)a_k \in \mathbf{G}(a_1, \ldots, a_r)$. Repeating this process, we see that a can be written as $\sum_{i=1}^{r} z'_i a_i + \sum_{i=r+1}^{r+m} n_i a_i, n_i \in \mathbb{N}$. Without loss of generality one may assume $\sum_{i=r+1}^{r+m} n_i a_i \in \mathrm{Ap(S, E)}$. Hence every element $a \in \mathbf{G}(S)$ can be written as $a = \sum_{i=1}^{r} z_i a_i + \omega$ where $\omega \in \mathrm{Ap(S, E)}$. The next proposition, which is Corollary 1.7 from [11], assert that when S is Cohen-Macaulay, this expression is unique.

Proposition 2. *If S is a Cohen-Macaulay simplicial affine semigroup, then*
(1) Every element in $\mathbf{G}(S)$ is equal to an unique expression of the form $z_1 a_1 + \ldots + z_r a_r + \omega$ with $z_i \in \mathbb{Z}$ and $\omega \in \mathrm{Ap(S, E)}$.
(2) The element $\sum_{i=1}^{r} z_i a_i + \omega$ with $z_i \in \mathbb{Z}$ and $\omega \in \mathrm{Ap(S, E)}$ is in S if and only if $z_i \geq 0$ for all i.

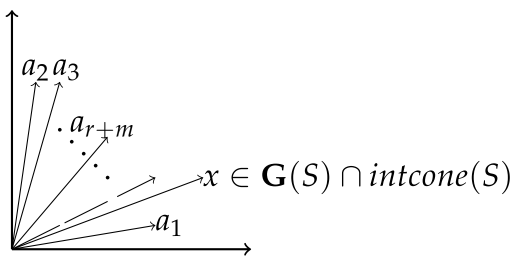
The cone spanned by S and interior of cone S, are denoted by:

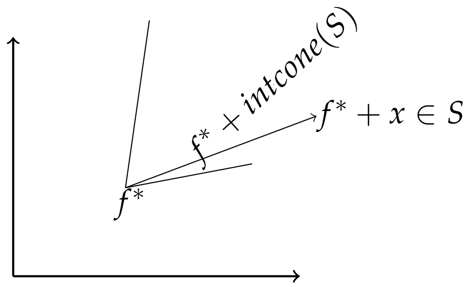
$$cone(S) = \left\{ \sum_{i=1}^{r} r_i a_i \,|r_i \in \mathbb{Q}_{\geq 0} \right\}, intcone(S) = \left\{ \sum_{i=1}^{r} r_i a_i \,|r_i \in \mathbb{Q}_{>0} \right\}$$

respectively. From the definition it is easy to see that

$$\mathbf{G}(S) \cap intcone(S) = \left\{ \sum_{i=1}^{r} z_i a_i + \omega_j \,\middle|\, \gamma_1^j > -z_1, \ldots, \gamma_r^j > -z_r, j = 1, \ldots, t \right\}.$$

Definition 2. Let S be an affine semigroup. The vector $f^* \in \mathbf{G}(S) \setminus S$ is called a *Frobenius vector* for S if for all $x \in \mathbf{G}(S) \cap intcone(S)$, $f^* + x \in S$.





The set of Frobenius vectors of S will be denoted by F(S). We define a cone ordering on F(S) by writing $f_1^* \leq f_2^*$ if $f_2^* + cone(S) \subseteq f_1^* + cone(S)$. We will denote by $\mathrm{MF}(S)$ the set of minimal Frobenius vectors of S with respect to $\leq$.

Let $\mathrm{Ap(S, E)} = \{\omega_1 = 0, \omega_2, \ldots, \omega_t\}$. Since S is simplicial, there exist non-negative rational numbers γ_i^j, $i = 1, \ldots, r$, $j = 1, \ldots, t$, such that $\omega_j = \sum_{i=1}^{r} \gamma_i^j a_i$. Let M and M_i^j are $r \times r$ matrices, with column vectors $a_1, a_2, \ldots, a_r$ and $a_1, a_2, \ldots, \hat{a}_i, \ldots, a_r, \omega_j$, respectively, where $\hat{a}_i$ means that a_i is omitted. It is not hard to see that

$$\gamma_i^j = \left| \frac{\det M_i^j}{\det M} \right|. \tag{2.1}$$

Now, by Euclidean division, there exists a unique integer $\mu_i^j \geq -1$ and a unique rational number $0 < \beta_i^j \leq 1$ such taht $\gamma_i^j = \mu_i^j + \beta_i^j$, for each $i = 1, \ldots, r$. Define

$$\xi_j = \sum_{i=1}^{r} \beta_i^j a_i = \sum_{i=1}^{r} (-\mu_i^j) a_i + \sum_{i=1}^{r} \gamma_i^j a_i = \sum_{i=1}^{r} (-\mu_i^j) a_i + \omega_j.$$

Clearly $\xi_j \in \mathbf{G}(S) \cap intcon\,(\mathrm{S})$. It is straightforward to see that

$$-\mu_i^j = \left\lfloor -\gamma_i^j \right\rfloor + 1. \tag{2.2}$$

For example, let $S \subseteq \mathbb{N}^3$ and $\omega_2 = \frac{3}{2}a_1 + \frac{5}{7}a_2 + \frac{13}{4}a_3 \in \mathrm{Ap(S,E)}$. Since $\gamma_1^2 = \frac{3}{2}, \gamma_2^2 = \frac{5}{7}, \gamma_3^2 = \frac{13}{4}$, we have $\mu_1^2 = \lfloor -\frac{3}{2} \rfloor + 1, \mu_2^2 = \lfloor -\frac{5}{7} \rfloor + 1, \mu_3^2 = \left\lfloor -\frac{13}{4} \right\rfloor + 1$, and so $\xi_2 = -a_1 - 3a_3 + \omega_2$.

Lemma 1. *Let S be a simplicial affine semigroup. Then $f^* \in \mathrm{F(S)}$ if and only if $f^* + \xi_k \in S$, for every $k = 1, \ldots, t$.*

Proof. Let f^* be a Frobenius vector for S. Since $\xi_k \in \mathbf{G}(S) \cap intcone(S)$, $f^* + \xi_k \in S$. Conversely let $x = \sum_{i=1}^{r} z_i a_i + \omega_l \in \mathbf{G}(S) \cap intcone(S)$. Since $x \in intcone(S)$,

$$z_i + \gamma_i^l > 0 \Rightarrow z_i > -\gamma_i^l \Rightarrow z_i \geq \left\lfloor -\gamma_i^l \right\rfloor + 1 = -\mu_i^l.$$

Thus $x - \xi_l = \sum_{i=1}^{r} (z_i - (- \mu_i^l)) a_i \in S$ and so $f^* + x = \underbrace{f^* + \xi_l}_{\in S} + \underbrace{x - \xi_l}_{\in S} \in S.$ ∎

Theorem 2.1. *Let S be a simplicial affine semigroup. Then $\mathrm{MF(S)} \neq \emptyset$.*

Proof. Let $f = \sum_{i=1}^{r} z_i a_i + \omega_j \in \mathbf{G}(S) \setminus S$. First of all, we observe that there exist $N_1, \ldots, N_t \in \mathbb{N}$ large enough such that

$$\begin{aligned} f + \sum_{k=1}^{t} N_k \xi_k &= \sum_{i=1}^{r} z_i a_i + \omega_j + \sum_{k=1}^{t} N_k \sum_{i=1}^{r} \beta_i^k a_i = \sum_{i=1}^{r} z_i a_i + \omega_j + \sum_{i=1}^{r} \sum_{k=1}^{t} N_k \beta_i^k a_i \\ &= \sum_{i=1}^{r} \left(z_i + \sum_{k=1}^{t} N_k \beta_i^k \right) a_i + \omega_j \in S. \end{aligned}$$

Now, if $f \notin \mathrm{F(S)}$, there exists $k_1 \in \{1, \ldots, r\}$ such that $f_1 = f + \xi_{k_1} \in \mathbf{G}(S) \setminus S$. If $f_1 \notin \mathrm{F(S)}$, there exists $k_2 \in \{1, \ldots, r\}$ such that $f_1 = f + \xi_{k_1} + \xi_{k_2} \in \mathbf{G}(S) \setminus S$. Since this process can be repeated only finitely many times, by Lemma 1, we conclude that $\mathrm{F(S)} \neq \emptyset$. Now we prove that $\mathrm{MF(S)} \neq \emptyset$. Let $f = \sum_{i=1}^{r} z_i a_i + \omega_j \in \mathrm{F(S)}$, $f' = \sum_{i=1}^{r} z'_i a_i + \omega_{j'} \in \mathrm{F(S)}$ and $f \in f' + cone(S)$. Then $z'_i + \gamma_i^{j'} \leq z_i + \gamma_i^j$ for all $i = 1, \ldots, r$, and so $z'_i \leq z_i + \gamma_i^j - \gamma_i^{j'}$. On the other hand, since $\sum_{i=1}^{r} a_i \in \mathbf{G}(S) \cap intcone(S)$, we have $f' + \sum_{i=1}^{r} a_i = \sum_{i=1}^{r} (z'_i + \gamma_i^{j'} + 1) \in S$, and thus $-\gamma_i^{j'} - 1 \leq z'_i$. Hence $-\gamma_i^{j'} - 1 < z'_i \leq z_i + \gamma_i^j - \gamma_i^{j'}$. The finiteness of $\mathrm{Ap(S,E)}$ implies that $\{f^* \in \mathrm{F(S)} \mid f \in f^* + \mathrm{cone(S)}\}$ is a finite set, which proves that $\mathrm{MF(S)} \neq \emptyset$. ∎

Definition 3. The simplicial affine semigroup S is called *pure simplicial* if for each $i = 1, \ldots, m$, $a_{r+i} \in intcone(S)$. We abbreviate pure simplicial as P-simplicial.

We can define the following relation on $\mathbf{G}(S)$: for any $a, b \in \mathbf{G}(S), a \leq_S b \Leftrightarrow b - a \in S$. Let $\max(\mathrm{Ap}(S,E)) = \{\eta_1, \eta_2, \ldots, \eta_s\}$ be the set of maximal elements of Ap(S, E) with respect to $\leq_S$ and let $\gamma_i^{\max} = \left\lfloor \max_j(\gamma_i^j) \right\rfloor + 1$, where $\max_j(\gamma_i^j) = \max\{\gamma_i^1, \gamma_i^2, \ldots, \gamma_i^t\}$.

Theorem 2.2. *Let S be a simplicial affine semigroup.*
(1) If S is Cohen-Macaulay, then

$$\mathrm{MF(S)} \subset \left\{ \sum_{i=1}^{r} z_i a_i + \omega \,|{-1} \leq z_i \leq \gamma_i^{\max}, \omega \in \mathrm{Ap}(S,E) \right\}.$$

(2) If S is Cohen-Macaulay and P-simplicial, then

$$\mathrm{MF(S)} \subset \left\{ \sum_{i=1}^{r} z_i a_i + \eta \,|{-1} \leq z_i \leq \gamma_i^{\max}, \eta \in \max\left(\mathrm{Ap}(S,E)\right) \right\}.$$

Proof. (1) Let $f = \sum_{i=1}^r z_i a_i + \omega \in \mathrm{F(S)}$. As $\sum_{i=1}^r a_i \in \mathbf{G}(S) \cap intcone(S)$, so $f + \sum_{i=1}^r a_i = \sum_{i=1}^r (z_i + 1) + \omega \in S$. Hence by Proposition 2, $z_i \geq -1$. Now let $z_l > \gamma_l^{\max}$ for some $l \in \{1, \ldots, r\}$. We show that $f \notin \mathrm{MF(S)}$. Set $f_1 = \sum_{i=1, i\neq l}^r z_i a_i + \gamma_l^{\max} a_l + \omega$. Since $f \in f_1 + cone(S)$, it suffices to prove that $f_1 \in \mathrm{F(S)}$. Let $x = \sum_{i=1}^r z_i' a_i + \omega' \in \mathbf{G}(S) \cap intcone(S)$.

$$\begin{aligned} f_1 + x &= \sum_{i=1,i\neq l}^{r} z_i a_i + \gamma_l^{\max} a_l + \omega + \sum_{i=1,i\neq l}^{r} z_i' a_i + z_l' a_l + \omega' \\ &= \sum_{i=1,i\neq l}^{r} (z_i + z_i') a_i + (\gamma_l^{\max} + z_l') a_l + (\omega + \omega'). \end{aligned}$$

Since $f \in \mathrm{F(S)}$, we have $f + x = \sum_{i=1}^r (z_i + z_i') a_i + \omega + \omega' \in S$. So by Proposition 2, $\sum_{i=1, i\neq l}^r (z_i + z_i') a_i + \omega + \omega' \in S$. Clearly $\gamma_l^{\max} + z_l' > 0$, so $f_1 + x$ is also in S and therefore $f_1 \in \mathrm{F(S)}$.
(2) Let $f = \sum_{i=1}^r z_i a_i + \omega \in \mathrm{F(S)}$. If $\omega \notin \max\left(\mathrm{Ap}(S,E)\right)$, then there exists $\eta \in \max\left(\mathrm{Ap}(S,E)\right)$ such that $\eta - \omega \in S$. Clearly $\eta - \omega \in \mathrm{Ap}(S,E)$, which implies that it belongs to $intcone(S)$, because S is P-simplicial. Since $f \notin S$, by Proposition 2, $f + \eta - \omega = \sum_{i=1}^r z_i a_i + \eta \notin S$. This contradicts $f \in \mathrm{F(S)}$. ∎

By the previous theorem and Proposition 2, if $f^* = \sum_{i=1}^r z_i a_i + \eta$ is a minimal Frobenius vector of the Cohen-Macaulay P-simplicial semigroup S, then there exists $k \in \{1, \ldots, r\}$ such that $z_k = -1$ and therefore $f_1^* = -a_k + \sum_{i=1, i\neq k}^r \gamma_i^{\max} a_i + \eta$ is a Frobenius vector for S, because $f_1^* \in f^* + cone(S)$.

Remark 1 (Numerical case). Every numerical semigroup S $= \langle a_1, \ldots, a_{m+1} \rangle$ is a P-simplicial and Cohen-Macaulay semigroup. As a consequence of the above theorem $f^*(S) = -a_1 + \max\left(\mathrm{Ap}(S, a_1)\right)$ (see [10, Proposition 2.12]).

As a consequence of the Theorem 2.2 and Lemma 1, we can compute the elements of MF(S) for Cohen-Macaulay simplicial semigroup, because we only have to check if a finite number of elements of $\mathbf{G}(S) \setminus S$ belongs to MF(S).

Theorem 2.3. *Let S be a simplicial affine semigroup and $|\max(\mathrm{Ap}(S,E))| = 1$. Then $f^* = \eta - \sum_{i=1}^{r} a_i$ is a Frobenius vector for S, where $\eta = \max(\mathrm{Ap}(S,E))$.*

Proof. Let $x \in \mathbf{G}(S) \cap intcone(S)$ and $f^* + x = \sum_{i=1}^{r} z_i a_i + \omega \in \mathbf{G}(S)$. So $\eta - \omega + x = \sum_{i=1}^{r}(z_i+1)a_i$. Since $\eta - \omega \in cone(S)$ and $x \in intcone(S)$, so $\eta - \omega + x \in \mathbf{G}(S) \cap intcone(S)$. Hence for all $i = 1,\ldots,r$, $z_i + 1 > 0$, and so $z_i \geq 0$, consequently, $f^* + x \in S$. ∎

Proposition 3. *Let S be a simplicial affine semigroup. The following statements are equivalent.*
- *S is a Gorenstein semigroup;*
- *S is a Cohen-Macaulay semigroup and the set $\mathrm{Ap}(S,E)$ has a unique maximal element.*

Proof. Combining Theorem 4.6 with Theorem 2.8 in [11]. ∎

Corollary 1. *Let S be a Gorenstein simplicial affine semigroup. Then $f^* = \eta - \sum_{i=1}^{r} a_i$ is a Frobenius vector for S where $\eta = \max(\mathrm{Ap}(S,E))$. Moreover, if S is P-simplicial, then it is a minimal Frobenius vector for S and it is unique.*

In [1] (resp. [2]) it is shown that when S is a complete intersection simplicial semigroup (resp. free semigroup) the vector $f^* = \eta - \sum_{i=1}^{r} a_i$ is the only minimal Frobenius vector for S. We recall that a simplicial affine semigroup is said to be free if $|\mathrm{Ap}(S,E)| = n_1 n_2 \cdots n_m$, where $n_i = \min\{\mathrm{k} \in \mathbb{N} \setminus 0 \mid \mathrm{k}\mathrm{a}_{\mathrm{r+i}} \in \langle \mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_{\mathrm{r+i-1}} \rangle\}$, $\mathrm{i} = 1,\ldots \mathrm{m}$. Clearly every simplicial affine semigroup with $m = 1$ is free. Free semigoups are complete intersection and so they are Gorenstein. If S be a free semigoup, then I_S is generated by the set

$$\left\{ x_{r+1}^{n_1} - \prod_{i=1}^{r} x_i^{t_{1i}}, x_{r+2}^{n_2} - \prod_{i=1}^{r+1} x_i^{t_{2i}}, \ldots, x_{r+m}^{n_m} - \prod_{i=1}^{r+m-1} x_i^{t_{mi}} \right\}$$

where $n_i a_{r+i} = \sum_{k=1}^{r+i-1} t_{ik} a_k$ (for more details, please see [12]).

Let S be a Cohen-Macaulay P-simplicial semigroup. By Theorem 2.1, there exists at least one minimal Frobenius vector for S. Using the following algorithm we can compute minimal Frobenius vectors of S.

Algoritm : Computing minimal Frobenius vectors of a P-simplicial Cohen-Macaulay semigroup.

Inpute: A P-simplicial Cohen-Macaulay semigroup $S = \langle a_1, \ldots, a_{r+m} \rangle \subset \mathbb{N}^r$.
Output: The set of minimal Frobenius vectors of S.

Steps of the Algorithm:
1. Compute $I_S = \ker \varphi$ for $\varphi : K[x_1,\ldots,x_{r+m}] \to K[S]$, $x_i \mapsto y^{a_i}$ ([6, Theorem 12.24]), ([13, Chapter 12]).
2. Compute a monomial K-basis $\{M_i \,|\, i\}$ of $\frac{K[x_1,\ldots,x_{r+m}]}{<I_S, x_1,\ldots,x_r>}$ and set

$$\mathrm{Ap}(S,E) = \{\omega_1, \omega_2, \ldots, \omega_t\} = \{\deg_S(M_i) \,|\, i\}.$$

3. Using (2.1) and (2.2), compute $\xi_j = \sum_{i=1}^{r}(-\mu_i^j)a_i + \omega_j, j = 1,\ldots,t$.
4. Choose $\eta \in \max(\mathrm{Ap}(S,E))$ and set $A_\eta = \{-a_1 - a_2 - \cdots - a_r + \eta\}$, $\mathrm{MF}_\eta(\mathrm{S}) = \emptyset$.

5. Using Lemma 1, compute $T = \{t \in A_\eta \,|t \in \mathrm{F(S)}\}$. Set $\mathrm{MF}_\eta(\mathrm{S}) = \mathrm{MF}_\eta(\mathrm{S}) \cup T$. Note that if $f + \xi_k \in S, k \in \{1, \ldots, t\}$, and $f \leq_S f'$, then $f' + \xi_k \in S$.
6. Set $A_\eta = \cup_{i=1}^r \left(a_i + (A_\eta \setminus T)\right) \setminus S$. We see that every element of A_η is of the form $c_1a_1 + \cdots + c_ra_r + \eta$.
7. Set $A_\eta = A_\eta \setminus \{c_1a_1 + \cdots + c_ra_r + \eta \in A_\eta \,|c_i > \gamma_i^{\max}$, for some $i\}$ and repeat step 5.
8. The set of minimal Frobenius vectors of S is equal to $\min_{\leq} \bigcup_\eta \mathrm{MF}_\eta(\mathrm{S})$.

Example 1. Let $S = \langle a_1 = (1,5), a_2 = (5,1), a_3 = (2,2), a_4 = (3,3)\rangle$. By Theorem 2.1, S has at least one minimal Frobenius vector. Performing the steps of the above algorithm we compute the set of minimal Frobenius vectors of S.
Step 1. Using *CoCoA* [3], $K[S] \simeq \frac{K[x,y,z,w]}{I_S}$, where $I_S = \langle z^3 - w^2, -xy + w^2\rangle$.
Step 2. The set $\{1, \bar{z}, \bar{z}^2, \bar{w}, \bar{z}\bar{w}, \bar{z}^2\bar{w}\}$ is a monomial K-basis of $\frac{K[x,y,z,w]}{<I_S,x,y>}$. Hence

$$\mathrm{Ap}(S,E) = \{\omega_1 = 0, \omega_2 = a_3, \omega_3 = 2a_3, \omega_4 = a_4, \omega_5 = a_3 + a_4, \omega_6 = 2a_3 + a_4\}.$$

By Proposition 1, the semigroup S is Cohen-Macaulay, because for every $x, y \in \mathrm{Ap}(S,E)$, $x - y$ or $y - x$ is in $intcone(S)$ and since a_1 and a_2 are linearly independent, $x - y \notin \mathbf{G}(a_1, a_2)$.

Step 3. Using *CoCoA*, we see that

$$\begin{aligned}
&\omega_1 = 0a_1 + 0a_2 \Rightarrow \xi_1 = a_1 + a_2 = (6,6), \omega_4 = \tfrac{1}{2}a_1 + \tfrac{1}{2}a_2 \Rightarrow \xi_4 = a_4 = (3,3)\\
&\omega_2 = \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2 \Rightarrow \xi_2 = a_3 = (2,2),\ \omega_5 = \tfrac{5}{6}a_1 + \tfrac{5}{6}a_2 \Rightarrow \xi_5 = a_3 + a_4 = (5,5)\\
&\omega_3 = \tfrac{2}{3}a_1 + \tfrac{2}{3}a_2 \Rightarrow \xi_3 = 2a_3 = (4,4),\\
&\omega_6 = \tfrac{7}{6}a_1 + \tfrac{7}{6}a_2 \Rightarrow \xi_6 = -a_1 - a_2 + 2a_3 + a_4 = (1,1).
\end{aligned}$$

Step 4. Set $A_{2a_3+a_4} = \{f^* = -a_1 - a_2 + 2a_3 + a_4 = (1,1)\}$.
Step 5. Since $f^* + \xi_1 = 2a_3 + a_4 \in S$, $f^* + \xi_2 = a_4 \in S$, $f^* + \xi_3 = a_3 + a_4 \in S$, $f^* + \xi_4 = 2a_3 \in S$, $f^* + \xi_5 = 2a_4 \in S$ and $f^* + \xi_6 = a_3 \in S$, so $f^* \in \mathrm{F(S)}$. Set $T = \{f^*\}$.
Step 6. $A_{2a_3+a_4} = \cup_{i=1}^2 (a_i + \underbrace{A_{2a_3+a_4} \setminus T}_{\emptyset}) = \emptyset$.

It follows that $f^* = (1,1)$ is the only minimal Frobenius vector for S. The semigroup S is free, because $|\mathrm{Ap}(S,E)| = 6$ (see Fig 1).

Example 2. Let $S = \langle a_1 = (2,1), a_2 = (1,5), a_3 = (1,1), a_4 = (4,5)\rangle$.
Step 1. Using *CoCoA*, $K[S] \simeq \frac{K[x,y,z,w]}{I_S}$, where $I_S = \langle -z^6 + xw, x^3y - z^3w, x^2yz^3 - w^2\rangle$.
Step 2. The set $\{1, \bar{z}, \bar{z}^2, \bar{z}^3, \bar{z}^4, \bar{z}^5, \bar{w}, \bar{z}\bar{w}, \bar{z}^2\bar{w}\}$ is a monomial K-basis of $\frac{K[x,y,z,w]}{<I_S,x,y>}$. Hence

$$\begin{aligned}
\mathrm{Ap}(S,E) = \{&\omega_1 = 0, \omega_2 = a_3, \omega_3 = 2a_3, \omega_4 = 3a_3, \omega_5 = 4a_3,\\
&\omega_6 = 5a_3, \omega_7 = a_4, \omega_8 = a_3 + a_4, \omega_9 = 2a_3 + a_4\}.
\end{aligned}$$

Since a_1 and a_2 are linearly independent, one obtains that if $x, y \in \mathrm{Ap}(S,E)$ and $x \neq y$, then $x - y \notin \mathbf{G}(a_1, a_2)$. Hence by Proposition 1, S is a Cohen-Macaulay

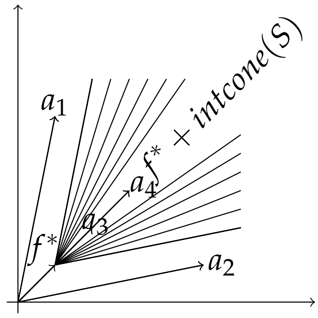


Fig.1

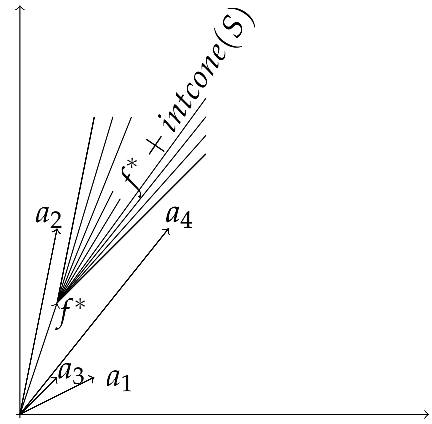


Fig.2

semigroup but not Gorenstein, because $\max(\mathrm{Ap}(S,E)) = \{\eta_1 = 2a_3 + a_4, \eta_2 = 5a_3\}$ (see Proposition 3).
Step 3. Using *CoCoA*, we see that

$$\begin{aligned}
&\omega_1 = 0a_1 + 0a_2 \Rightarrow \xi_1 = a_1 + a_2 = (3,6), \omega_2 = \tfrac{4}{9}a_1 + \tfrac{1}{9}a_2 \Rightarrow \xi_2 = a_3 = (1,1)\\
&\omega_3 = \tfrac{8}{9}a_1 + \tfrac{2}{9}a_2 \Rightarrow \xi_3 = 2a_3 = (2,2), \omega_4 = \tfrac{12}{9}a_1 + \tfrac{3}{9}a_2 \Rightarrow \xi_4 = -a_1 + 3a_3 = (1,2)\\
&\omega_5 = \tfrac{16}{9}a_1 + \tfrac{4}{9}a_2 \Rightarrow \xi_5 = -a_1 + 4a_3 = (2,3),\\
&\omega_6 = \tfrac{20}{9}a_1 + \tfrac{5}{9}a_2 \Rightarrow \xi_6 = -2a_1 + 5a_3 = (1,3)\\
&\omega_7 = \tfrac{5}{3}a_1 + \tfrac{2}{3}a_2 \Rightarrow \xi_7 = -a_1 + a_4 = (2,4),\\
&\omega_8 = \tfrac{19}{9}a_1 + \tfrac{7}{9}a_2 \Rightarrow \xi_8 = -2a_1 + a_3 + a_4 = (1,4)\\
&\omega_9 = \tfrac{23}{9}a_1 + \tfrac{8}{9}a_2 \Rightarrow \xi_9 = -2a_1 + 2a_3 + a_4 = (2,5).
\end{aligned}$$

Step 4. Set $A_{2a_3+a_4} = \{f_1 = -a_1 - a_2 + 2a_3 + a_4 = (3,1)\}$.
Step 5. We have $f_1 + \xi_1 = 2a_3 + a_4 \in S$, $f_1 + \xi_2 = 2a_1 \in S$, $f_1 + \xi_3 = 2a_1 + a_3 \in S$, $f_1 + \xi_4 = a_1 + 2a_3 \in S$, $f_1 + \xi_5 = a_1 + 3a_3 \in S$,
$f_1 + \xi_6 = 4a_3 \in S$, $f_1 + \xi_7 = 5a_3 \in S$, $f_1 + \xi_8 = a_4 \in S$ and $f_1 + \xi_9 = 2a_1 + a_2 \in S$.
So $f_1 \in \mathrm{F(S)}$ and like in the example above $\mathrm{MF}_{2a_3+a_4} = \{f_1\}$.
Now we use the algorithm for $\eta_2 = 5a_3$. Set $A_{5a_3} = \{f_2 = -a_1 - a_2 + 5a_3 = (2,-1)\}$. As $f_2 + \xi_2 = (3,0) \notin S$, $f_2 \notin \mathrm{F(S)}$. Step 6 yields $A_{5a_3} = \{f_3 = -a_1 + \eta_2 = (3,4), f_4 = -a_2 + \eta_2 = (4,0)\}$. Since $f_3 + \xi_4 = (4,6) \notin S$ and $f_4 + \xi_4 = (5,2) \notin S$, f_3 and f_4 are not in F(S). Using the algorithm, $A_{5a_3} = \{f_5 = -a_1 + a_2 + \eta_2 = (4,9), f_6 = a_1 - a_2 + \eta_2 = (6,1)\}$. As $\gamma_1^{\max} = 3$ and $\gamma_2^{\max} = 1$, we go back to Step 5. The vectors f_5 and f_6 are not in F(S) because $f_5 + \xi_4 = (5,11) \notin S$ and $f_6 + \xi_4 = (7,3) \notin S$. Using steps 6 and 7, $A_{5a_3} = \{f_7 = 2a_1 - a_2 + \eta_2 = (8,2)\}$. As $f_7 + \xi_4 = (9,4) \notin S$, so $f_7 \notin \mathrm{F(S)}$. Set $A_{5a_3} = \{f_8 = 3a_1 - a_2 + \eta_2 = (10,3)\}$. Since $f_8 + \xi_4 = (11,5) \notin S$, so $f_8 \notin \mathrm{F(S)}$. Finally, $A_{5a_3} = \emptyset$ (steps 6 and 7). Hence f_1 is the only minimal Frobenius vector for S (see Fig 2).

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Department of Pure Mathematics,
Faculty of Mathematics and Computer Science,
Amirkabir University of Technology, Tehran, Iran
E-mail addresses: a_mahdavi@aut.ac.ir , frahmati@aut.ac.ir