Algebrability of some subsets of the disk algebra

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Abstract

We show that the subset of the disk algebra of the functions that are not in some Dales-Davie algebra is algebrable. In other words, the set $\left\{ f \in \mathcal{A}(D) : \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|}{n!} = +\infty \right\}$ is shown to be algebrable.

1 Introduction

In the last ten years, several authors have searched large algebraic structures (linear spaces or algebras) in spaces of functions enjoying a special property. If a vector space V has a subset M such that $M \cup \{0\}$ contains an infinite-dimensional vector space, then M is called *lineable*. If $M \cup \{0\}$ contains a closed infinite-dimensional vector space, then M is called *spaceable*. The origin of the concept of lineability is due to Gurariy [11], see also [12], that showed that there exists an infinite dimensional linear space contained in the set of nowhere differentiable functions on $[0,1]$. Recently, Aron et al in [5] published a book dedicated to this subject. Since Gurariy's definition, it was natural to study subsets in spaces of functions which contains an infinitely generated algebra. Such spaces are called *algebrable*. It is clear that algebrability implies lineability. The concept of algebrability has been also defined by Gurariy and first pointed out in [6], but then has rapidly been investigated by other authors, for example [3, 4, 6, 7, 8]. In [8] Aron and Seoane-Sepúlveda showed that there exists an infinitely generated algebra in the set of everywhere surjective functions on $\mathbb{C}$. In [7], Aron, Pérez-García and Seoane-Sepúlveda showed that the set of continuous functions whose Fourier series expansion diverges is algebrable.

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In 1973, Dales and Davie [10] introduced and studied some algebras of differentiable functions on a given perfect compact subset $X \subset \mathbb{C}$. These algebras were called **Dales-Davie algebras** by Abtahi and Honary in [1], and they denoted them by $\mathcal{D}(X, M)$. If D denotes the open unit disk and $X = \overline{D}$, then $\mathcal{D}(\overline{D}, M)$ is a subalgebra of the disk algebra $\mathcal{A}(D)$. In this article, we study how big the difference $\mathcal{A}(D) \setminus \mathcal{D}(\overline{D}, M)$ is. Indeed, we show that the set $\mathcal{A}(D) \setminus \mathcal{D}(\overline{D}, M)$ contains a closed infinitely generated algebra, that is, it is algebrable and naturally is spaceable.

We refer the interested reader to [5, 9] for a wider range of results in the topic of lineability and algebrability, and to [1, 2, 10, 13] for further informations on the Dales-Davie algebras.

2 Preliminaries

Let $X \subset \mathbb{C}$ be a perfect, compact plane set. A complex valued function $f : X \longrightarrow \mathbb{C}$ is **differentiable at a point** $z_0 \in X$ if the following limit exists:

$$f'(z_0) = \lim \left\{ \frac{f(z) - f(z_0)}{z - z_0} \, : \, z \in X \, , z \to z_0 \right\}.$$

A complex valued function f is **differentiable on** X if it is differentiable at every point of X. Note that, if f is differentiable on X, it is analytic on (int)X. The algebra of functions on X with continuous n-th derivative is denoted by $\mathcal{D}^n(X)$, and $\mathcal{D}^\infty(X)$ denotes the algebra of functions on X with derivative of all orders. For every $f \in \mathcal{D}^\infty(X)$ we denote the n-th derivative of f by $f^{(n)}$, for all $n \in \mathbb{N}$. Also, $\|f^{(n)}\|_X = \sup_{z \in X} |f^{(n)}(z)|$, for all $n \in \mathbb{N}$.

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $M_0 = 1$, and for each $n \geq 1$,

$$\frac{M_n}{M_k \, M_{n-k}} \geq \binom{n}{k} \quad (0 \leq k \leq n).$$

The sequence $M = (M_n)_{n\in\mathbb{N}}$ is called an **algebra sequence** if it satisfies the above conditions.

The **Dales-Davie algebras** on X are defined by

$$\mathcal{D}(X, M) = \left\{ f \in D^\infty(X) \, : \, \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_X}{M_n} < +\infty \right\}.$$

The norm on $\mathcal{D}(X, M)$ is defined by $\|f\| = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_X}{M_n}$. These algebras were introduced and studied by Dales and Davie in [10], and they have been investigated by Abtahi and Honary in [1, 2, 13].

For each sequence $M = (M_n)_{n\in\mathbb{N}}$ of positive numbers, $\mathcal{D}(X, M)$ is a normed vector space. When $M = (M_n)_{n\in\mathbb{N}}$ is an algebra sequence, then $\mathcal{D}(X, M)$ is a normed algebra.

Let $\mathcal{B}$ be an algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this paper, the **dimension of** $\mathcal{B}$, denoted by $dim(\mathcal{B})$, will always refer to its dimension as a vector space. Let

$S = \{z_i : i \in I\}$ be a subset of an algebra $\mathcal{B}$. The **algebra generated by** S is the set

$$\mathcal{A}(S) = \Big\{ \sum_{j=1}^{k} \alpha_j z_i^j,\ \alpha_j \in \mathbb{K},\ z_i \in S,\ i \in I,\ k \in \mathbb{N} \Big\}.$$

The set S is called a **system of generators of** $\mathcal{A}(S)$. A system of generators S is **minimal** if for every $i_0 \in I$, $z_{i_0} \notin \mathcal{A}(S \setminus \{z_{i_0}\})$. As usual, we denote by $\mathbb{C}[z_1, z_2]$ the ring of all polynomials in two complex variables. We say that two elements a and b of an algebra $\mathcal{B}$ are **algebraically independent** if $P \in \mathbb{C}[z_1, z_2]$ is such that $P(a, b) = 0$ then $P \equiv 0$.

Definition 2.1. *Let A be a subset of an algebra. We say that A is*

- ***(a) lineable** if $A \cup \{0\}$ contains an infinite dimensional vector space.*
- ***(b) spaceable** if $A \cup \{0\}$ contains a closed infinite dimensional vector space.*
- ***(c) algebrable** if there is an algebra $\mathcal{B} \subset A \cup \{0\}$, such that $\mathcal{B}$ has an infinite minimal system of generators.*
- ***(d)** (α, β)-**algebrable** if there is an algebra $\mathcal{B} \subset A \cup \{0\}$, with $\dim(\mathcal{B}) = \alpha$ and $card(S) = \beta$, where α and β are two cardinal numbers, and S is a minimal system of generators of $\mathcal{B}$.*

Other concepts such as *maximal lineability*, *dense-lineability*, *spaceabilily*, etc can be found in [9].

Let D denote the open unit disk of the complex plane, that is, $D = \{z \in \mathbb{C} : |z| < 1\}$. The Banach algebra of continuous functions on $\overline{D}$ and analytic on D with the *sup* norm is denoted by $\mathcal{A}(D)$. As usual we call $\mathcal{A}(D)$ the **disk algebra**. When $X = \overline{D}$ it follows that $\mathcal{D}(\overline{D}, M)$ is a subalgebra of $\mathcal{A}(D)$. However, the difference $\mathcal{A}(D) \setminus \mathcal{D}(\overline{D}, M)$ is not a vector space, hence not an algebra.

In this article, we want to investigate the algebrability of the set $\mathcal{A}(D) \setminus \mathcal{D}(\overline{D}, M)$. We will prove that $\mathcal{A}(D) \setminus \mathcal{D}(\overline{D}, M)$ contains an infinitely generated algebra, for several algebra sequences $M = (M_n)_{n \in \mathbb{N}}$.

For a fixed algebra sequence $(M_n)_{n \in \mathbb{N}}$, we denote

$$\mathcal{H}(M) = \Big\{ f \in \mathcal{A}(D) : \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_X}{M_n} = +\infty \Big\}.$$

When $M_n = n!$, for all $n \in \mathbb{N}$, we will write $\mathcal{H}$ instead of $\mathcal{H}(M)$.

We show in the next Lemma that the set $\mathcal{H}$ is nonempty.

Lemma 2.2. *Let $w = 2e^{i\theta}$, where $0 \le \theta < 2\pi$. Let $f(z) = \dfrac{1}{z - w}$, for all $z \in \overline{D}$. Then $f \in \mathcal{H}$.*

Proof. It is clear that $f \in \mathcal{A}(D)$ and that $\|f\| = |f(e^{i\theta})| = 1$. It is also easy to see that

$$f^{(n)}(z) = \frac{(-1)^n\, n!}{(z - w)^{n+1}} = (-1)^n\, n!\, (f(z))^{n+1},\ \forall z \in \overline{D}.$$

Therefore $\|f^{(n)}\| = n!\,\|f\|^{n+1} = n!$. Then

$$\sum_{n=0}^{\infty} \frac{\|f^{(n)}\|}{n!} = +\infty. \qquad \blacksquare$$

In Section 3, we will show that $\mathcal{H}$ is algebrable. Actually, we show that $\mathcal{H} \cup \{0\}$ contains a closed infinitely generated algebra, and as a consequence it is spaceable. However, it is possible to prove that $\mathcal{H}$ is spaceable with a different technique. Also, we get that $\mathcal{H}$ is $(\aleph_0, 1)$-algebrable. These results will be presented in the sequel.

Proposition 2.3. *The set $\mathcal{H}$ is spaceable.*

Proof: Let $f(z) = \dfrac{1}{z-2}$. Then $f \in \mathcal{H}$ by Lemma 2.2. Let $S = \{f^n : n \in \mathbb{N}, n \geq 1\}$. Firstly, we will show that S is a linearly independent subset of $\mathcal{H} \cup \{0\}$. So let $\beta_1, \beta_2, \cdots, \beta_k \in \mathbb{C}$ and suppose that $g = \sum_{j=1}^{k} \beta_j f^j = 0$. For any k distinct real numbers $x_1, x_2, \cdots, x_k \in D$ we have that $f(x_1), f(x_2), \cdots, f(x_k)$ are also k distinct non-zero real numbers. Then we have the following system:

$$\sum_{j=1}^{k} \beta_j f^j(x_i) = 0, \text{ for } i = 1, \cdots, k.$$

Using a matrix notation, we have

$$\begin{pmatrix} f(x_1) & f(x_1)^2 & \cdots & f(x_1)^k \\ f(x_2) & f(x_2)^2 & \cdots & f(x_2)^k \\ \vdots & \vdots & \cdots & \vdots \\ f(x_k) & f(x_k)^2 & \cdots & f(x_k)^k \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This matrix is a variation of the Vandermonde matrix, and its determinant equals to

$$f(x_1)f(x_2)\cdots f(x_k) \cdot \prod_{1 \leq i < j \leq k} (f(x_j) - f(x_i)).$$

In our setting, this determinant is never zero, and then $\beta_1 = \beta_2 = \cdots = \beta_k = 0$ showing that S is a linearly independent set.

Next, we will show that the vector space generated by S, which will be denoted by $[S]$, is contained in $\mathcal{H} \cup \{0\}$. So let $g \in [S] \setminus \{0\}$. Then there are $\beta_1, \beta_2, \cdots, \beta_k \in \mathbb{C}$, not all zero, such that $g = \sum_{j=1}^{k} \beta_j f^j$. We want to show that $g \in \mathcal{H}$, which means $\sum_{n=0}^{\infty} \dfrac{\|g^{(n)}\|}{n!} = +\infty$. Note that

$$\left(f^j\right)^{(n)} = (-1)^n j \cdot (j+1) \cdots (j+n-1) f^{j+n}.$$

Since $f(1) = -1$, it follows that

$$g^{(n)}(1) = \sum_{j=1}^{k} \beta_j (-1)^n j \cdot (j+1) \cdots (j+n-1)(-1)^{j+n} = \sum_{j=1}^{k} \beta_j (-1)^j \frac{(j+n-1)!}{(j-1)!}.$$

Then

$$\frac{g^{(n)}(1)}{n!} = -\beta_1 \frac{1}{0!} + \beta_2 \frac{n+1}{1!} - \beta_3 \frac{(n+1)(n+2)}{2!} + \cdots +$$
$$+ \cdots + (-1)^k \beta_k \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}.$$

If the series $\sum_{n=0}^{\infty} \frac{|g^{(n)}(1)|}{n!}$ converges, then $\lim_{n\to\infty} \frac{|g^{(n)}(1)|}{n!} = 0$. But it happens if, and only if, $\beta_1 = \beta_2 = \cdots = \beta_k = 0$, which is not true. Then it follows that

$$\sum_{n=0}^{\infty} \frac{\|g^{(n)}\|}{n!} \geq \sum_{n=0}^{\infty} \frac{|g^{(n)}(1)|}{n!} = +\infty.$$

So we have proved that $\mathcal{H} \cup \{0\}$ is lineable. Now, since $\mathcal{H}$ with the topology induced by $\mathcal{A}(D)$ is closed, it follows that $\mathcal{H}$ is spaceable. ∎

Corollary 2.4. *The set $\mathcal{H}$ is $(\aleph_0, 1)$-algebrable.*

Proof: Using the same notation of Proposition 2.3, we have that the algebra generated by S equals the vector space generated by S.

3 Algebrability of $\mathcal{H}(M)$

We observe here that in the previous section, when we showed that $\mathcal{H}$ is spaceable, indeed we proved that $\mathcal{H} \cup \{0\}$ contains a closed algebra generated by a single element. Now, using a different technique, we will show that there is a closed infinitely generated algebra contained in $\mathcal{H} \cup \{0\}$, and in particular $\mathcal{H}$ is algebrable.

We will present some auxiliary results in the sequel.

Lemma 3.1. *Let β be a positive real number. Let $n \in \mathbb{N}$ and for each $r \in \mathbb{N}$ such that $1 \leq r \leq n$, consider*

$$a_r = \frac{1}{r!} \sum_{s=1}^{r} \binom{r}{s} (-1)^{r-s+n}\, s \cdot (s+1) \cdots (s+n-1).$$

Then:

(a) *If K is a fixed natural number, $1 \leq r \leq K$ and n is even, then $\lim_{n\to\infty} \beta^r \frac{a_r}{n!} = +\infty$;*

(b) $\lim_{n\to\infty} \lim_{r\to n} \beta^r \frac{a_r}{n!} = 0.$

Proof. Developing the expression for $\frac{a_r}{n!}$, we have that:

$$\frac{a_r}{n!} = \sum_{s=1}^{r} \binom{r}{s} (-1)^{r-s+n} \frac{s \cdot (s+1) \cdots (s+n-1)}{n! r!} =$$
$$\sum_{s=1}^{r} (-1)^{r-s+n} \frac{(s+n-1)!}{(r-s)! s! (s-1)! n!} = \sum_{s=1}^{r} (-1)^{r-s+n} \frac{(n+1)(n+2)\cdots(n+s-1)}{(r-s)! s! (s-1)!}.$$

(a) Note that each term of the summation above is a polynomial function defined in $\mathbb{N}$, with degree $s-1$. Then $\dfrac{a_r}{n!}$ is a polynomial function defined in $\mathbb{N}$, with degree at most $r-1$. If n is even, the coefficient of the highest degree is positive.

Now, for a fixed natural number K, $1 \leq r \leq K$ and n even, it follows that $\lim_{n\to\infty} \beta^r \dfrac{a_r}{n!} = +\infty$.

(b)

$$\lim_{r\to n}\lim_{s\to r} \beta^r(-1)^{r-s+n}\frac{(s+n-1)!}{(r-s)!s!(s-1)!n!} =$$

$$= \lim_{r\to n} \beta^r(-1)^n \frac{(r+n-1)!}{r!(r-1)!n!} = \beta^n(-1)^n\frac{(2n-1)!}{n!n!(n-1)!}.$$

And by D'Alembert's criterion we have that

$$\lim_{n\to\infty} \beta^n(-1)^n\frac{(2n-1)!}{n!n!(n-1)!} = 0. \qquad \blacksquare$$

In this work we will have to deal with n-th order derivative of composed functions. A formula to this derivative is known as Faá di Bruno formula, but there are several variations of this formula. Here we will use the Hoppe's Formula for the n-th order derivative of composed functions. For better comprehension of the reader we spell out the formula here. For complete information about it we suggest [14].

Theorem 3.2. *[14, Hoppe's Formula] Let f and g be functions with a sufficient number of derivatives, then*

$$(g\circ f)^{(n)}(z) = \sum_{r=1}^{n}\frac{g^{(r)}(f(z))}{r!}\sum_{s=0}^{r}\binom{r}{s}(-f(z))^{r-s}(f^s)^{(n)}(z).$$

In order to show that $\mathcal{H}$ is algebrable, we have to exhibit an infinitely generated algebra contained in $\mathcal{H}$. Since $\mathbb{C}[z_1,z_2]$ contains infinitely generated algebras, we will use $\mathbb{C}[z_1,z_2]$ as a model to construct infinitely generated algebras in $\mathcal{H}$. If g_1 and g_2 are two algebraically independent functions, then the algebra generated by them is isomorphic to $\mathbb{C}[z_1,z_2]$. Therefore $A(\{g_1,g_2\})$ also contains infinitely generated algebras.

Our next result will allow us to apply these remarks in the main result.

Theorem 3.3. *Let $g_1(z) = e^z$ and $g_2(z) = e^{\beta z}$, for all $z \in \mathbb{C}$, where β is an irrational number. Let $P \in \mathbb{C}[z_1,z_2]$ be a non-constant polynomial, let $f(z) = \dfrac{1}{z+2}$ and consider $h = P(g_1,g_2)$. Then $h\circ f \in \mathcal{H}\cup\{0\}$.*

Proof: By Lemma 2.2 we have that $f \in \mathcal{H}$ and $\|f\| = 1$. If we consider $h = P(g_1,g_2)$ then there are $\alpha_{jk} \in \mathbb{C}$ such that

$$h(z) = \sum_{j,k=0}^{m}\alpha_{jk}\,g_1^j\,g_2^k = \sum_{j,k=0}^{m}\alpha_{jk}e^{(j+\beta k)z}.$$

If we denote $\beta_{jk} = j + \beta k$, then we have that

$$h(f(z)) = \sum_{j,k=0}^{m} \alpha_{jk}\, e^{\beta_{jk} f(z)}.$$

We want to show that $\sum_{n=0}^{\infty} \frac{\|(h \circ f)^{(n)}\|}{n!} = +\infty$. By Hoppe's formula (Theorem 3.2), we have that

$$(h \circ f)^{(n)}(z) = \sum_{r=1}^{n} \frac{h^{(r)}(f(z))}{r!} \sum_{s=1}^{r} \binom{r}{s} (-f(z))^{r-s} (f^s)^{(n)}.$$

Since $\left(f^s\right)^{(n)} = (-1)^n s \cdot (s+1) \cdots (s+n-1) f^{s+n}$, it follows that $(h \circ f)^{(n)}(z) =$

$$= \sum_{r=1}^{n} \frac{h^{(r)}(f(z))}{r!} \sum_{s=1}^{r} \binom{r}{s} (-f(z))^{r-s} (-1)^n s \cdot (s+1) \cdots (s+n-1) \big(f(z)\big)^{s+n}.$$

Now

$$(h \circ f)^{(n)}(-1) = \sum_{r=1}^{n} \frac{h^{(r)}(1)}{r!} \sum_{s=1}^{r} \binom{r}{s} (-1)^{r-s+n}\, s \cdot (s+1) \cdots (s+n-1).$$

Since

$$h^{(r)}(1) = \sum_{j,k=0}^{m} \alpha_{jk}\, e^{\beta_{jk}}\, \beta_{jk}^{r},$$

and

$$a_r = \frac{1}{r!} \sum_{s=1}^{r} \binom{r}{s} (-1)^{r-s+n}\, s \cdot (s+1) \cdots (s+n-1),$$

we have that

$$(h \circ f)^{(n)}(-1) = \sum_{r=1}^{n} \sum_{j,k=0}^{m} a_r \alpha_{jk}\, e^{\beta_{jk}}\, \beta_{jk}^{r}.$$

We claim that $\sum_{n=0}^{\infty} \frac{|(h \circ f)^{(n)}(-1)|}{n!} = +\infty$. In this case, we have that

$$\sum_{n=0}^{\infty} \frac{\|(h \circ f)^{(n)}\|}{n!} > \sum_{n=0}^{\infty} \frac{|(h \circ f)^{(n)}(-1)|}{n!} = +\infty,$$

and the result follows.

To show the claim, and for a better comprehension of the next arguments, we will study the case for $m = 2$, which simplifies the notation. In this case, the expression of H_n is:

$$H_n := \frac{(h \circ f)^{(n)}(-1)}{n!} =$$

$$= \sum_{r=1}^{n} \frac{a_r}{n!} \Big(\alpha_{01} e^{\beta_{01}} \beta_{01}^{r} + \alpha_{02} e^{\beta_{02}} \beta_{02}^{r} + \cdots + \alpha_{22} e^{\beta_{22}} \beta_{22}^{r} \Big) =$$

$$= \frac{a_1}{n!}\big(\alpha_{01}e^{\beta_{01}}\beta_{01} + \alpha_{02}e^{\beta_{02}}\beta_{02} + \cdots + \alpha_{22}e^{\beta_{22}}\beta_{22}\big) +$$
$$+ \frac{a_2}{n!}\big(\alpha_{01}e^{\beta_{01}}\beta_{01}^2 + \alpha_{02}e^{\beta_{02}}\beta_{02}^2 + \cdots + \alpha_{22}e^{\beta_{22}}\beta_{22}^2\big) + \ldots$$
$$+ \frac{a_n}{n!}\big(\alpha_{01}e^{\beta_{01}}\beta_{01}^n + \alpha_{10}e^{\beta_{10}}\beta_{10}^n + \cdots + \alpha_{22}e^{\beta_{22}}\beta_{22}^n\big).$$

If we consider

$$L_r = \sum_{j,k=0}^{2} \alpha_{jk}e^{\beta_{jk}}\beta_{jk}^r, \text{ for all } r \in \mathbb{N},$$

then

$$H_n = \sum_{r=1}^{n} \frac{a_r}{n!} L_r.$$

Observe that $m = 2$ implies only the number of terms of L_r.

We claim that there exists $N \in \mathbb{N}$ such that $L_N \neq 0$. Indeed, suppose that $L_r = 0$, for all $r \in \mathbb{N}$. In particular, we have that $L_1 = L_2 = \cdots = L_8 = 0$. That is:

$$\sum_{j,k=0}^{2} \alpha_{jk}\, e^{\beta_{jk}}\, \beta_{jk}^r = 0, \text{ for each } 1 \leq r \leq 8.$$

That means that the following system holds.

$$\begin{pmatrix} e^{\beta_{01}}\beta_{01} & e^{\beta_{02}}\beta_{02} & \cdots & e^{\beta_{22}}\beta_{22} \\ e^{\beta_{01}}\beta_{01}^2 & e^{\beta_{02}}\beta_{02}^2 & \cdots & e^{\beta_{22}}\beta_{22}^2 \\ \vdots & \vdots & \cdots & \vdots \\ e^{\beta_{01}}\beta_{01}^8 & e^{\beta_{02}}\beta_{02}^8 & \cdots & e^{\beta_{22}}\beta_{22}^8 \end{pmatrix} \cdot \begin{pmatrix} \alpha_{01} \\ \alpha_{02} \\ \vdots \\ \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The determinant of the matrix above is $\prod e^{\beta_{jk}}\, \beta_{jk}\, (\beta_{jk} - \beta_{pq})$, where the product is taken over all $j, k, p, q = 0, 1, 2$, with $(p, q) \neq (j, k)$. But this determinant is zero if, and only if, $\beta_{jk} = \beta_{pq}$, that is $j + \beta k = p + \beta q$. Since β is an irrational number, it follows that $j = p$ and $k = q$, and it is a contradiction. Then $\alpha_{01} = \alpha_{02} = \cdots = \alpha_{22} = 0$, which cannot happen, since P is non-constant.

For the general case m, observe that each L_r is a summation of $(m+1)^2 - 1$ terms, since $\beta_{00} = 0$. Then, following the same arguments, we will have a system of dimension $((m+1)^2 - 1)$, and we will conclude that $\alpha_{jk} = 0$, for all possible jk except for α_{00}. In this case P would be a constant polynomial, which is a contradiction.

Finally, let N with $L_N \neq 0$. Then

$$H_n = \frac{a_1}{n!}L_1 + \frac{a_2}{n!}L_2 + \cdots + \frac{a_N}{n!}L_N + \cdots + \frac{a_n}{n!}L_n, \; n > N.$$

Applying Lemma 3.1, we have that $\lim_{n\to\infty} |H_n| = +\infty$. ∎

Using similar ideas from [8], we can prove the following theorem.

Theorem 3.4. *The set $\mathcal{H}$ is algebrable.*

Proof: Let $g_1(z) = e^z$ and $g_2(z) = e^{\beta z}$, for all $z \in \mathbb{C}$, where β is an irrational number. Since g_1 and g_2 are algebraically independent, we have that $\mathcal{A}(\{g_1, g_2\})$ is isomorphic to $\mathbb{C}[z_1, z_2]$, the algebra of all polynomials on two complex variables. Now, if $S = \{g_1, g_1 g_2, g_1 g_2^2, g_1 g_2^3, \cdots\}$, then S is an infinite minimal system of generators of the algebra $\mathcal{B} = \mathcal{A}(S)$, and $\mathcal{B} \subset \mathcal{A}(\{g_1, g_2\})$.

Let us fix $f(z) = \dfrac{1}{z+2}$. We consider $W = \{g_1 \circ f, (g_1 g_2) \circ f, (g_1 g_2^2) \circ f, (g_1 g_2^3) \circ f, \cdots\}$. The W is a minimal system of generators for $\mathcal{F} = \{h \circ f : h \in \mathcal{B}\}$. By Theorem 3.3, for each $h \in \mathcal{A}(\{g_1, g_2\})$, we have that $h \circ f \in \mathcal{H}$. So $\mathcal{F} = \mathcal{A}(W) \subset \mathcal{H} \cup \{0\}$, and therefore $\mathcal{H}$ is algebrable. ∎

We observe that, since $\mathcal{H}$ is closed in $\mathcal{A}(D)$, it follows that $\mathcal{H} \cup \{0\}$ contains a closed infinitely generated algebra.

In the next corollary, we show that not only $\mathcal{H}$ is algebrable, but actually there is an infinite collection of algebra sequences $(M_n)_{n \in \mathbb{N}}$ such that $\mathcal{H}(M)$ is algebrable.

Corollary 3.5. *Let $(M_n)_{n \in \mathbb{N}}$ be an algebra sequence such that $M_n \leq n!$, for all $n \in \mathbb{N}$. Then $\mathcal{H}(M)$ is algebrable.*

Proof: If $M_n \leq n!$, then $\mathcal{H} \subset \mathcal{H}(M)$. ∎

Remark 3.6. If $0 < \alpha \leq 1$, let $M_n := \alpha^n n!$ for all $n \in \mathbb{N}$. Then $(M_n)_{n \in \mathbb{N}}$ is an algebra sequence such that $M_n \leq n!$, for all $n \in \mathbb{N}$.

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