Primitive arcs on curves

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Abstract

We introduce the notion of *primitive arc* of a curve defined over a field k and study criterions for the existence of such objects in terms of the geometry of the curve. We prove that this notion provides a criterion which determines when the normalization of an irreductible curve singularity (X, x) induces an isomorphism between the formal neighborhoods of the associated arc schemes at the constant arc x and its lifting $\bar{x}$ to the normalization $\bar{X}$. We also show that the existence of a primitive arc at $x \in X$ is equivalent to the smoothness of the analytically irreducible curve X at x. In this end, we interpret this notion in terms of the formal deformations of the constant arc x in the associated arc scheme.

1 Introduction

1.1 Let k be a field. A *test-ring* A (or $(A, \mathfrak{m}_A)$) is a local k-algebra, whose maximal ideal $\mathfrak{m}_A$ is nilpotent with residue field $A/\mathfrak{m}_A \cong k$. A *primitive arc* γ of a k-curve X at $x \in X(k)$ is a primitive k-parametrization $\mathcal{O}_{X,x} \to k[[T]]$ (see definition 3.1), which satisfies the following property: For every test-ring $(A, \mathfrak{m}_A)$, for every commutative diagram of morphisms of local k-algebras

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "\gamma_A"] \arrow[d, "x"'] & A[[T]] \arrow[d, "r_A"] \\
k \arrow[r, hook] & k[[T]],
\end{tikzcd} \tag{1}$$

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where $r_A\colon A[[T]] \to k[[T]]$ is the continuous morphism of complete local k-algebras defined by $T \mapsto T$ with kernel $\mathfrak{m}_A$, there exists a unique power series $p_A \in \mathfrak{m}_A[[T]]$ which induces a continuous morphism of complete local k-algebras $p_A^\sharp\colon k[[T]] \to A[[T]]$ that verifies the formula $\gamma_A = p_A^\sharp \circ \gamma$. If it exists, a primitive arc is unique (up to isomorphism).

1.2 The basic subject of this article could be summarized by the following question:

Question 1.1. *Which class of pointed k-curves admits primitive arcs?*

This article provides a complete answer to question 1.1 for analytically irreducible curves. Precisely, we establish various criterions for the existence of primitive arcs on k-curves. In this way, the existence of primitive arcs can be interpreted as an original criterion of local smoothness (for k-curves) in terms of the associated arc schemes, or as a criterion for determining when the normalization morphism induces an isomorphism at the level of the involved arc schemes. If X is a k-curve and $x \in X(k)$, recall that the point x can be viewed as a *constant* arc of the associated arc scheme $\mathscr{L}_\infty(X)$, and we denote by $\mathscr{L}_\infty(X)_x$ the formal neighborhood of the arc x in $\mathscr{L}_\infty(X)$, i.e., the formal k-scheme $\mathrm{Spf}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(X),x}})$.

Theorem 1.1. *Let k be an algebraically closed field. Let X be a k-curve which is unibranch at $x \in X(k)$. Then the following assertions are equivalent:*

1. *The k-curve X is smooth at x;*
2. *There exists a primitive arc γ at x on X;*
3. *The formal k-scheme $\mathscr{L}_\infty(X)_x$ is isomorphic to* $\mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$;
4. *The normalization $\pi : \bar{X} \to X$ induces, at the level of formal neighborhoods of the associated arc schemes, an isomorphism of formal k-schemes*

$$\widehat{\mathscr{L}_\infty(\pi)_x}\colon \mathscr{L}_\infty(\bar{X})_{\bar{x}} \to \mathscr{L}_\infty(X)_x,$$

   where $\bar{x} \in \bar{X}(k)$ is the lifting of x;

5. *The morphism $(\widehat{\mathscr{L}_\infty(\pi)_x})^\sharp\colon \widehat{\mathcal{O}_{\mathscr{L}_\infty(X),x}} \to \widehat{\mathcal{O}_{\mathscr{L}_\infty(\bar{X}),\bar{x}}}$ is surjective.*

1.3 The point of view of formal neighborhoods of arc schemes has been introduced in [8] (see also [6]). If V is a variety, the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ parametrizes the *formal deformations* of the arc γ in $\mathscr{L}_\infty(V)$. In [8, 6] (see also [3] for an analog statement in the context of formal geometry), the authors prove a structure theorem for formal neighborhoods of arc schemes at *non-constant* arcs, which are not contained in the singular locus of the involved variety. The interpretation of such a result in terms of singularity theory remains a challenging problem, and works [3, 4, 5] are, to the best of our knowledge, the first steps in this direction. Let us also mention [7, 12] where some properties of formal neighborhoods of arc schemes are also studied in other frameworks.

Contrary to these works, the involved arcs in our statement are *constant*; hence, our result provides information for arcs contained in the singular locus. (Let us note that in general the main theorem of [8, 6] does not hold for singular constant arcs, see [2] for counter-examples). Roughly speaking, the present work (see assertion (3) of theorem 1.1) investigates the study of the smoothness of an analytically irreducible k-curve X at a point x from the point of view of the "deformations" of the constant arc x in the associated arc scheme $\mathscr{L}_\infty(X)$. In this context, the notion of *rigidity* (i.e., situation where there is no non-trivial deformation) corresponds to the existence of a primitive arc.

2 Preliminaries

2.1 Let k be a field. A *k-variety* is a k-scheme of finite type. A *k-curve* is a *reduced* k-variety of dimension 1. We say that a pointed curve (X, x), with $x \in X(k)$, is *unibranch* (or *analytically irreducible*) at x if the ring $\widehat{\mathcal{O}_{X,x}}$ is a domain.

2.2 Let k be a field. Let V be a k-variety. The functor $S \mapsto \mathrm{Hom}_k(S \hat{\otimes}_k k[[T]], V)$ defined from the category of k-schemes to that of sets is representable by a k-scheme $\mathscr{L}_\infty(V)$. (Let us note that this presentation uses a recent non-trivial result due to B. Bhatt, see [1, Theorem 1.1]). If V is an affine k-variety, for every k-algebra A, every element $\gamma_A \in \mathscr{L}_\infty(V)(A)$ coincides with the datum of a morphism of k-algebras $\mathcal{O}(V) \to A[[T]]$.

2.3 Let V be a k-variety and $\gamma \in \mathscr{L}_\infty(V)(k)$. Yoneda's lemma [9, 8.1.4] and the properties of completion formally imply that the formal neighborhood $\mathscr{L}_\infty(V)_\gamma$ of the k-scheme $\mathscr{L}_\infty(V)$ at γ is completely determined by the functor of points

$$A \mapsto \mathrm{Hom}_k^{\mathrm{cpl}}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}, A),$$

when A runs over the category of test-rings, and where the considered morphisms are the continuous morphisms of complete local k-algebras from $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ to A. See [8, 6] or also [3].

3 The proof of theorem 1.1

Definition 3.1. Let k be a field. Let X be a k-curve with $x \in X(k)$. A *primitive k-parametrization of X at x* is a morphism of local k-algebras $\gamma\colon \mathcal{O}_{X,x} \to k[[T]]$, which satisfies the following property: For every morphism $\gamma'\colon \mathcal{O}_{X,x} \to k[[T]]$ of local k-algebras, there exists a power series $p_k \in Tk[[T]]$ such that we have $\gamma' = p_k^\sharp \circ \gamma$.

If k is algebraically closed and X is unibranch at x, the normalization $\pi\colon \bar{X} \to X$ of X provides a primitive k-parametrization of X at x by considering the induced morphism of local k-algebras $\pi_x\colon \mathcal{O}_{X,x} \to \widehat{\mathcal{O}_{\bar{X},\bar{x}}}$.

Remark 3.1. A primitive k-parametrization may not be a primitive arc. Let X be the affine plane k-curve defined by the datum of the polynomial $F = T_1^3 - T_2^2 \in \mathbf{C}[T_1, T_2]$. Let us consider the primitive k-parametrization γ at the origin $\mathfrak{o}$ in $X = \mathrm{Spec}(\mathbf{C}[T_1, T_2]/\langle T_1^3 - T_2^2\rangle)$ defined by the element $(T^2, T^3) \in \mathbf{C}[[T]]$. Let $A := \mathbf{C}[S]/\langle S^2\rangle$. We observe that the element $\gamma_A \in \mathscr{L}_\infty(X)_{\mathfrak{o}}(A)$ given by $T_1 \mapsto S$, $T_2 \mapsto S$, can not be written under the form $\gamma \circ p_A$. So, γ is not a primitive arc.

Let us mention that implication 4 ⇒ 5 is obvious, and that implication 4 ⇒ 1 also is obvious since we have $\mathscr{L}_\infty(\bar{X})_{\bar{x}} \cong \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$. Let us prove the other implications.

1 ⇒ 2 Since X is smooth at x, there exists an affine open subscheme U of X, which contains x, endowed with an étale morphism of k-schemes $U \to \mathbf{A}_k^1 = \mathrm{Spec}(k[t])$, corresponding to the choice of a local parameter in $\mathcal{O}(U)$ (i.e., a generator t of the maximal ideal $\mathfrak{m}_x$ in the ring $\mathcal{O}_{X,x}$). Up to shrinking X, we may assume that $X = U$. Then, let γ be the arc corresponding to the following morphism of k-schemes:

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, hook] \arrow[rrr, bend left, "\gamma"] & \widehat{\mathcal{O}_{X,x}} \arrow[r, "\sim"] & k[[t]] \arrow[r, "t\mapsto T"] & k[[T]],
\end{tikzcd} \tag{2}$$

obtained by composition via the completion morphism. It gives rise to a primitive k-parametrization of X at x. Then, it is easy to check that the arc γ is primitive, since, in this case, for every test-ring A, and every $\gamma_A \in \mathscr{L}_\infty(X)_x(A)$, we take $p_A = \gamma_A$.

2 ⇒ 4 Let γ be a primitive arc at x on the curve X. Let A be a test-ring. By §2.3, we only have to prove that the map:

$$\pi_A := \widehat{\mathscr{L}_\infty(\pi)}_x(A) \colon \mathscr{L}_\infty(\bar{X})_{\bar{x}}(A) \to \mathscr{L}_\infty(X)_x(A)$$

is a bijection. Let $\gamma_A \in \mathscr{L}_\infty(X)_x(A)$. By assumption, there exists a unique power series $p_A \in \mathfrak{m}_A[[T]]$ such that $\gamma_A = \gamma \circ p_A$ (where we identify p_A and the induced morphisms of k-schemes). Since the morphism π is proper and birational, the valuative criterion of properness implies the existence of a unique non-constant arc $\bar{\gamma} \in \mathscr{L}_\infty(\bar{X})(k)$ such that $\pi \circ \bar{\gamma} = \gamma$. Then, we easily observe that $\bar{\gamma} \circ p_A$ is the unique preimage by π_A of γ_A.

5 ⇒ 1 We assume that the morphism $(\widehat{\mathscr{L}_\infty(\pi)_x})^\sharp$ is surjective. Then, for every test-ring A, the induced map:

$$\pi_A := (\widehat{\mathscr{L}_\infty(\pi)_x})^\sharp(A) \colon \mathrm{Hom}_k^{\mathrm{cpl}}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(\bar{X}),\bar{x}}}, A) \to \mathrm{Hom}_k^{\mathrm{cpl}}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(X),x}}, A)$$

is injective. We are going to show that this property implies the smoothness of X at x. Let us denote by $\mathrm{mult}_x(X)$ the integer defined as follows. If γ is a primitive k-parametrization at x, let us consider the ideal $\gamma(\mathfrak{m}_x)$ in the ring $\widehat{\mathcal{O}_{\bar{X},\bar{x}}} = k[[t]]$ (t is here a generator of the maximal ideal $\mathfrak{m}_{\bar{x}}$ of the ring $\mathcal{O}_{\bar{X},\bar{x}}$). There exists an integer n such that $\gamma(\mathfrak{m}_x) = \langle t^n\rangle$. We then set $n =: \mathrm{mult}_x(X)$. This definition does not depend on the choice of γ. If the k-curve X is singular at x, we have $\mathrm{mult}_x(X) \geq 2$.

Let us assume that x is a singular point of X. Up to shrinking X, we may assume that X is affine, embedded in $\mathbf{A}_k^N = \mathrm{Spec}(k[T_1, \dots, T_N])$, and, up to a translation, we may assume that x is the origin $\mathfrak{o}$ of $\mathbf{A}_k^N$. Let $A := k[U]/\langle U^2 \rangle$. The power series $\varphi_1 = 0 \in A[[T]]$ and $\varphi_2 = UT \in A[[T]]$ define two elements of $\mathscr{L}_\infty(\bar{X})_{\bar{x}}(A)$. It follows from the definition that

$$\mathscr{L}_\infty(\pi)(\varphi_1) = \mathscr{L}_\infty(\pi)(\varphi_2). \tag{3}$$

since $\mathrm{mult}_x(X) > 1$. Indeed, every variable T_i (seen in the ring $\mathcal{O}_{X,x}$) is sent by the morphism of local k-algebras $\pi_x \colon \mathcal{O}_{X,x} \to \widehat{\mathcal{O}_{\bar{X},\bar{x}}}$ to an element in the ideal $\langle t^{\mathrm{mult}_x(X)} \rangle$, where t is a generator of the ideal $\mathfrak{m}_{\bar{x}}$. So, we obtain formula (3) since $\mathscr{L}_\infty(\pi)(\varphi_i)$ corresponds, for every integer $i \in \{1, 2\}$, to the following composition of morphisms of local k-algebras:

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\pi_x"] & \widehat{\mathcal{O}_{\bar{X},\bar{x}}} \arrow[r, "t \mapsto \varphi_i"] & A[[T]] \end{tikzcd} .$$

The injectivity of the map π_A then implies that $\varphi_1 = \varphi_2$. That is a contradiction, which concludes the proof.

3 ⇒ 1 It is sufficient to prove that the ring $\mathcal{O}_{X,x}$ is formally smooth for the $\mathfrak{m}_x$-adic topology thanks to [11, 17.5.3]. By [10, 19.3.3,19.3.6], we observe that, due to our assumption, the ring $\widehat{\mathcal{O}_{\mathscr{L}_\infty(X),x}}$ is formally smooth, and we conclude by the existence of the following diagram of continuous morphisms of local k-algebras:

$$\begin{tikzcd} \widehat{\mathcal{O}_{X,x}} \arrow[r] \arrow[rr, bend left, "\mathrm{Id}"] & \widehat{\mathcal{O}_{\mathscr{L}_\infty(X),x}} \arrow[r] & \widehat{\mathcal{O}_{X,x}}. \end{tikzcd}$$

Example 3.1. *Keep the notation of remark 3.1. In this case, the normalization morphism* $\pi^\sharp \colon \mathcal{O}(X) \to k[T]$ *is defined by* $T_1 \mapsto T^2, T_2 \mapsto T^3$*; hence, every* A*-deformation of the origin in* $\bar{X}$ *is sent to the origin in* X*. We easily conclude that the deformation* (S, S) *of the origin in* X *does not lift to the normalization* $\bar{X}$*.*

Remark 3.2. By base change, we observe that theorem 1.1 can be generalized to every geometrically unibranch integral curve X and any closed point $x \in X$.

Remark 3.3. Keep the notation and assumptions of theorem 1.1. It is not hard to prove that the morphism $(\widehat{\mathscr{L}_\infty(\pi)_x})^\sharp \colon \widehat{\mathcal{O}_{\mathscr{L}_\infty(X),x}} \to \widehat{\mathcal{O}_{\mathscr{L}_\infty(\bar{X}),\bar{x}}}$ is a formal invariant of the curve singularity (X, x). By this way, formal neighborhoods at constant arcs in arc scheme provide *new* formal invariants of curve singularities. It would be interesting to study these invariants with respect to the classical theory of singularities.

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