

Dirichlet-Ford domains and Double Dirichlet domains*

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Abstract

We continue investigations started by Lakeland on Fuchsian and Kleinian groups which have a Dirichlet fundamental domain that also is a Ford domain in the upper half-space model of hyperbolic 2- and 3-space, or which have a Dirichlet domain with multiple centers. Such domains are called DF-domains and Double Dirichlet domains respectively. Making use of earlier obtained concrete formulas for the bisectors defining the Dirichlet domain of center $i \in \mathbb{H}^2$ or center $j \in \mathbb{H}^3$, we obtain a simple condition on the matrix entries of the side-pairing transformations of the fundamental domain of a Fuchsian or Kleinian group to be a DF-domain. Using the same methods, we also complement a result of Lakeland stating that a cofinite Fuchsian group has a DF domain (or a Dirichlet domain with multiple centers) if and only if it is an index 2 subgroup of the discrete group G of reflections in a hyperbolic polygon.

1 Introduction

Fundamental domains in hyperbolic spaces, or spaces of constant curvature in general, have been studied for a long time. Together with Poincaré's Polyhedron

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Theorem, they are often used to find presentations of discrete groups. Fundamental domains also are of use in the construction of discrete groups. Special attention goes to fundamental domains in hyperbolic 2- and 3-space, as they are strongly related to discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$.

A major difficulty one encounters is the effective construction of such a domain. The two most known and used constructions in hyperbolic space are the Ford and Dirichlet fundamental domains. The Ford fundamental domain of a group Γ is defined in terms of the isometric spheres of the elements of Γ . The Dirichlet domain of Γ is based on the bisectors of some chosen center and its images by Γ . It is well-known that in the ball model of hyperbolic 2- or 3-space, the Dirichlet fundamental domain and the Ford fundamental domain of some discrete group Γ are the same (see for example [4, Theorem 9.5.2]). In the upper half-space model however, this is, in general, not the case. Hence an interesting topic is to study when the Dirichlet and the Ford fundamental domain coincide also in the upper half-space model. This is what we call a DF domain, i.e. a fundamental domain in \mathbb{H}^2 or \mathbb{H}^3 that is a Dirichlet and a Ford domain at the same time.

One major problem in the construction of a Dirichlet fundamental domain is the choice of an adequate center. In general changing the center, changes the shape of the fundamental domain completely, as is nicely shown by Martin Deraux's animation [2]. This is different for the hyperbolic reflection groups. Their fundamental domain is canonical and the choice of the center plays no role at all. The fundamental domain is the same for every chosen center, see [18, Exercise 7.1.1]. So one may wonder which other Dirichlet fundamental domains have multiple centers. We call them double Dirichlet domains. In some sense characterizing a group acting discontinuously on hyperbolic space by having a double Dirichlet domain comes down to study 'how close' the group is to a hyperbolic reflection group.

One of the most common examples of a discontinuous action on hyperbolic space is the action of $\mathrm{PSL}_2(\mathbb{Z})$ on hyperbolic 2-space \mathbb{H}^2 . The probably best known fundamental domain for this action is the triangle in the upper half-plane model with vertices ∞ , $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$. This is in fact both a Ford domain and a Dirichlet domain with center ti for every $t > 0$. So this is a first example of a DF-domain and a double Dirichlet domain.

In this paper we continue the investigations initiated by Lakeland in [14] on DF-domains and double Dirichlet domains. The main reason in [14] to study these domains is answering a question raised by Agol-Belolipetsky-Storm-Whyte in [3]: the existence of a maximal arithmetic hyperbolic reflection group which is not congruence. The author constructs a non-congruence arithmetic group Γ_{ref} and, by using the theory of DF-domains, he proves that Γ_{ref} is a maximal reflection group.

The main theorem of [14, Theorem 5.3] states that a finitely generated, finite coarea Fuchsian group Γ admits a DF-domain if and only if Γ is an index 2 subgroup of a reflection group. It also is proved that a Kleinian group Γ has a generating set consisting of elements whose traces are real ([14, Theorem 6.3].) We give a new and independent criterion for the result of [14] that also applies to Kleinian groups. Note that all the groups we are working with are non-cocompact. See

also Remark 2.1. Our criterion is of algebraic nature and easily can be checked once a set of side-pairing transformations is given:

Theorem 1.1. *Let Γ be a non-cocompact cofinite discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, acting on \mathbb{H}^2 , respectively \mathbb{H}^3 , and $P_0 = i$, respectively $P_0 = j$. Suppose that the stabilizer of P_0 in Γ is trivial. Then, Γ admits a DF domain \mathcal{F} with center P_0 if and only if for every side-pairing transformation $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})$ of \mathcal{F} we have that $d = \bar{a}$.*

Moreover, if Γ is a cofinite Fuchsian group, then $\tilde{\Gamma} = \langle \sigma, \Gamma \rangle$ is a reflection group and $\hat{\Gamma} = \langle \tau, \Gamma \rangle$ is a Coxeter group, where σ is the reflection in the imaginary axis and τ is the linear operator represented by the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and both groups contain Γ as a subgroup of index two.

For such groups Γ , as an immediate consequence, one obtains, in the Kleinian case, that the traces of the generating elements are real (see Corollary 4.4). Moreover, as an application, we get most of the results on DF and double Dirichlet domains obtained in [14]. Also, some of the proofs of [14] can be simplified using Theorem 1.1.

To prove the above theorem, we make use of concrete formulas given in [12]. In [12], the authors develop explicit formulas for the bisectors of i and $\gamma(i)$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ not fixing i , or j and $\gamma(j)$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ not fixing j . These bisectors are indeed necessary to determine the Dirichlet fundamental domain in hyperbolic 2-space \mathbb{H}^2 and hyperbolic 3-space \mathbb{H}^3 respectively. These fundamental domains are then used to tackle the non-trivial problem of describing units in an order of a non-commutative non-split division algebra or of a 2-by-2 matrix ring over a quadratic imaginary extension of the field \mathbb{Q} . As the unit groups of some of these orders may be considered as discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$, fundamental domains and Poincaré's Polyhedron Theorem are of potential use to determine these unit groups. First attempts to this were done by Pita, del Río and Ruiz in [16, 17], where the authors use Ford domains to get presentations of some small subgroups of congruence subgroups of Bianchi groups and by Corrales, Jespers, Leal and del Río in [9], where a presentation for the unit group of a "small" non-commutative division algebra (a quaternion algebra) is given using Dirichlet domains. As an application, one obtains a description of subgroups of finite index in the unit group of an integral group ring of some finite groups. Making use of concrete formulas, the authors obtain in [12] a more general approach than the isolated cases described above.

The outline of the paper is as follows. For the sake of completeness, we record in Section 2 some fundamentals on hyperbolic geometry, fundamental domains and on Fuchsian and Kleinian groups. In Section 3, we recall a result from [12] and develop a necessary proposition to prove our main result and its corollaries. In particular, we give conditions for an isometric sphere, in the upper half-space (-plane) model, to be a bisector. In the last section, we consider DF domains and double Dirichlet domains, prove Theorem 1.1 and show some corollaries.

2 Background

We begin by recalling basic facts on hyperbolic spaces and we fix notation. Standard references are [4, 8, 10, 11, 15, 18]. By \mathbb{H}^2 and \mathbb{H}^3 we denote the upper half-space model of hyperbolic 2- and 3-space. As is common, we identify \mathbb{H}^2 with the subset $\{x + ri \in \mathbb{C} \mid x \in \mathbb{R}, r \in \mathbb{R}^+\}$ of the complex numbers \mathbb{C} and \mathbb{H}^3 with the subset $\{z + rj \in \mathcal{H} \mid z \in \mathbb{C}, r \in \mathbb{R}^+\}$ of the classical (real) quaternion algebra $\mathcal{H} = \mathcal{H}(\frac{-1, -1}{\mathbb{R}})$. Denote by $\text{Iso}^+(\mathbb{H}^i)$ the group of orientation-preserving isometries of \mathbb{H}^i for $i = 2, 3$. It is well known that $\text{Iso}^+(\mathbb{H}^2)$ is isomorphic with $\text{PSL}_2(\mathbb{R})$ and that $\text{Iso}^+(\mathbb{H}^3)$ is isomorphic with $\text{PSL}_2(\mathbb{C})$.

Throughout, we will use the notation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ both for an element of $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$ as well as for its natural image in $\text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$. Moreover, abusing notation, we use the same letter for both the matrix in $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$ and the Möbius transformation acting on \mathbb{H}^2 or \mathbb{H}^3 respectively. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$, we write $a = a(\gamma)$, $b = b(\gamma)$, $c = c(\gamma)$ and $d = d(\gamma)$ when it is necessary to stress the dependence of the entries on the matrix γ .

We now describe the action of $\text{PSL}_2(\mathbb{R})$ and $\text{PSL}_2(\mathbb{C})$ on hyperbolic space. We do this in detail for the 3-dimensional case, the 2-dimensional case being done in a similar way. The action of $\text{PSL}_2(\mathbb{C})$ on \mathbb{H}^3 is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (P) = (aP + b)(cP + d)^{-1},$$

where $(aP + b)(cP + d)^{-1}$ is calculated in \mathcal{H} . Explicitly, if $P = z + rj$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\gamma(P) = \frac{(az + b)(\bar{c}z + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2r^2} + \left(\frac{r}{|cz + d|^2 + |c|^2r^2}\right)j.$$

The hyperbolic distance ρ in \mathbb{H}^3 (or \mathbb{H}^2 respectively) is determined by $\cosh \rho(P, P') = 1 + \frac{d(P, P')^2}{2rr'}$, where d is the Euclidean distance and $P = z + rj$ and $P' = z' + r'j$ are two elements of \mathbb{H}^3 (respectively $P = x + ri$ and $P' = x' + r'i$ are two elements of \mathbb{H}^2).

Finally, recall that a group Γ is said to act discontinuously on a proper metric space X if for every compact subset K of X , $K \cap \gamma(K) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. A well-known theorem states that if X is a proper metric space, then a group Γ acts discontinuously on X if and only if Γ is a discrete subgroup of $\text{Iso}(X)$. For more details on this, see [18, Theorem 5.3.5]. A fundamental domain for a discontinuous group action of a group Γ on a metric space X , is a closed set $\mathcal{F} \subseteq X$ such that the border of \mathcal{F} has measure 0, the union of the images of \mathcal{F} under Γ is the space X and the images of \mathcal{F}° , the interior of \mathcal{F} , by different elements of Γ are pairwise disjoint. We call \mathcal{F} a convex fundamental polyhedron, if \mathcal{F} is a fundamental domain that is a convex polyhedron. The group Γ is said to be cofinite if \mathcal{F} has finite volume. It is well-known that cofiniteness implies

geometrical finiteness which, in dimension 2 and 3, implies that the \mathcal{F} has finitely many sides. For more details, we refer to [7] and [13]. If \mathcal{F} is a fundamental polyhedron for a discontinuous group action Γ , then, for every side S of \mathcal{F} , there exists an element $\gamma_S \in \Gamma$ such that $S = \mathcal{F} \cap \gamma_S(\mathcal{F})$. If for every side S the element γ_S is the reflection in the hyperplane $\langle S \rangle$, then Γ is called a reflection group with respect to F . Reflection groups are particular cases of Coxeter groups. Formally a Coxeter group is a group $\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$ with $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. Note that $m_{ij} = \infty$ is possible and just means that there is no relation of the form $(r_i r_j)^m$ between r_i and r_j . More details on this may be found in [18, Section 7.1].

In this paper we work with two different constructions of fundamental polyhedra, known as Dirichlet and Ford fundamental domain. We recall their construction and how they can be used to give a presentation for the considered groups, the so called Poincaré method (for details see for example [4] or [18]). Let Γ be a discrete subgroup of $\text{Iso}^+(\mathbb{H}^3)$. Let Γ_j be the stabilizer in Γ of $j \in \mathbb{H}^3$ and let \mathcal{F}_j be a convex fundamental polyhedron of Γ_j . Put $D_\gamma(j) = \{u \in \mathbb{H}^3 \mid \rho(u, j) \leq \rho(u, \gamma(j))\}$ and set $\tilde{\mathcal{F}} = \bigcap_{\gamma \in \Gamma \setminus \Gamma_j} D_\gamma(j)$. The border $\partial D_\gamma(j) = \{u \in \mathbb{H}^3 \mid \rho(u, j) = \rho(u, \gamma(j))\}$ is the hyperbolic bisector of the geodesic linking j to $\gamma(j)$. This is called a Poincaré bisector. Note that $\tilde{\mathcal{F}}$ is stable under the action of Γ_j . Moreover $\tilde{\mathcal{F}}$ is a convex polyhedron such that $\bigcup_n \gamma_n(\tilde{\mathcal{F}}) = \mathbb{H}^3$, where γ_n are the coset representatives of Γ_j in Γ . By [4, Theorem 9.6.1], the set

$$\mathcal{F} = \mathcal{F}_j \cap \tilde{\mathcal{F}} = \mathcal{F}_j \cap \left(\bigcap_{\gamma \in \Gamma \setminus \Gamma_j} D_\gamma(j) \right)$$

is a fundamental domain of Γ , which we call the Dirichlet fundamental domain with center j . Moreover, it may be shown that \mathcal{F} is a convex polyhedron and if Γ is geometrically finite then a finite set of generators for Γ consists of the elements $\gamma \in \Gamma$ such that $\mathcal{F} \cap \gamma(\mathcal{F})$ is a side of the polyhedron together with Γ_j , i.e. $\Gamma = \langle \Gamma_j, \gamma \mid \gamma(\mathcal{F}) \cap \mathcal{F} \text{ is a side} \rangle$ (see [18, Theorem 6.8.3]). Let us denote the bisector $\partial D_{\gamma^{-1}(j)}$ of the geodesic linking j to $\gamma^{-1}(j)$ by Σ_γ . It is easy to compute that $\gamma(\Sigma_\gamma) = \Sigma_{\gamma^{-1}}$. From this it follows that $\mathcal{F} \cap \gamma(\mathcal{F}) \subseteq \Sigma_{\gamma^{-1}}$. Note that the same construction can be done in \mathbb{H}^2 , replacing the point $j \in \mathbb{H}^3$ by $i \in \mathbb{H}^2$.

Let Γ be a discrete subgroup of $\text{PSL}_2(\mathbb{C})$ and denote by Γ_∞ the stabilizer in Γ of the point ∞ . Denote a convex fundamental polyhedron of Γ_∞ by \mathcal{F}_∞ . For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus \Gamma_\infty$, denote the isometric sphere of γ by ISO_γ . Note that these are the points $P \in \mathbb{H}^3$ such that $|cP + d|^2 = 1$. Denote the set $\{P \in \mathbb{H}^3 \mid |cP + d|^2 \geq 1\}$ by ISO_γ^\geq . By the same reasoning as above, if Γ_∞ contains a parabolic element then

$$\mathcal{F} = \mathcal{F}_\infty \cap \left(\bigcap_{\gamma \in \Gamma \setminus \Gamma_\infty} \text{ISO}_\gamma^\geq \right)$$

is a fundamental domain of Γ called the Ford fundamental domain of Γ . Again, if \mathcal{F} is a polyhedron and if Γ is geometrically finite then $\Gamma = \langle \Gamma_\infty, \gamma \mid$

$\gamma(\mathcal{F}) \cap \mathcal{F}$ is a side \rangle . And also in this case one can easily show that, for every $\gamma \in \Gamma \setminus \Gamma_\infty$, $\gamma(\text{ISO}_\gamma) = \text{ISO}_{\gamma^{-1}}$ and $\mathcal{F} \cap \gamma(\mathcal{F}) \subseteq \text{ISO}_{\gamma^{-1}}$. Again the same construction is possible in \mathbb{H}^2 .

Remark 2.1. *If we talk about Ford domains, we implicitly assume the discrete subgroup Γ to have a parabolic element fixing the point ∞ .*

3 Poincaré bisectors and isometric spheres

For completeness' sake, we first recall in this section the authors' result from [12] that is needed to prove our main result. Based on this result we prove a proposition that will be used later. As before, we develop the theory in dimension 3, but everything can be applied to dimension 2 as well. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C}) \setminus \text{PSU}_2(\mathbb{C})$. Recall that $\gamma \in \text{PSU}_2(\mathbb{C})$ if and only if $\gamma(j) = j$ (see [4, 10]). As the Poincaré bisector can only exist if $\gamma \notin \Gamma_j$, the case $\gamma \in \text{PSU}_2(\mathbb{C})$ is excluded in the following. In the ball model of the hyperbolic space, it is well-known that the isometric sphere of γ equals the bisector of the geodesic segment linking 0 and its image by γ^{-1} . We will not go into more details on this but refer the interested reader to [4, Section 9.5] for dimension 2. In [12, Theorem 3.1], the authors give an independent proof of this in dimension 3 (which is of course adaptable to dimension 2).

In the upper half-space \mathbb{H}^3 , an isometric sphere is not necessarily a Poincaré bisector. As explained in Section 2, we denote the isometric sphere of γ by ISO_γ and the Poincaré bisector of the geodesic linking j to $\gamma^{-1}(j)$ by Σ_γ . This bisector may be a Euclidean sphere or a plane perpendicular to $\partial\mathbb{H}^3$. If it is a Euclidean sphere, we denote its center by P_γ and its radius by R_γ .

The following result gives concrete formulas for the Poincaré bisectors in the upper half-space model.

Proposition 3.1. [12, Proposition 3.2]

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$, with $\gamma \notin \text{PSU}_2(\mathbb{C})$.

1. Σ_γ is a Euclidean sphere if and only if $|a|^2 + |c|^2 \neq 1$. In this case, its center and its radius are respectively given by $P_\gamma = \frac{-(\bar{a}b + \bar{c}d)}{|a|^2 + |c|^2 - 1}$ and $R_\gamma^2 = \frac{1 + |P_\gamma|^2}{|a|^2 + |c|^2}$.
2. Σ_γ is a plane if and only if $|a|^2 + |c|^2 = 1$. In this case $\text{Re}(\bar{v}z) + \frac{|v|^2}{2} = 0, z \in \mathbb{C}$ is a defining equation of Σ_γ , where $v = \bar{a}b + \bar{c}d$.

The next proposition gives some information on the relation between ISO_γ and Σ_γ , for some $\gamma \in \text{PSL}_2(\mathbb{C}) \setminus \text{PSU}_2(\mathbb{C})$ with $c(\gamma) \neq 0$. Again the case $\gamma \in \text{PSU}_2(\mathbb{C})$ is excluded because otherwise Σ_γ does not exist. This proposition will be useful in the study of DF domains and double Dirichlet domains.

Proposition 3.2. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C}) \setminus \text{PSU}_2(\mathbb{C})$.

1. If $c \neq 0$ and $|a|^2 + |c|^2 \neq 1$ then $\text{ISO}_\gamma = \Sigma_\gamma$ if and only if $d = \bar{a}$. In this case, $\text{tr}(\gamma) \in \mathbb{R}$ and if $b \neq 0$, $c = \lambda \bar{b}$, with $\lambda \in \mathbb{R}$.
2. If $c = 0$ and $|a|^2 = 1$, we also have that $a = \bar{d}$, $\text{tr}(\gamma) \in \mathbb{R}$ and $c = \lambda \bar{b}$, with $\lambda \in \mathbb{R}$.

Proof. First note that in 1, the case $c = 0$ is excluded so that ISO_γ exists and, by Proposition 3.1, the condition $|a|^2 + |c|^2 \neq 1$ guarantees that Σ_γ is a sphere. If we denote the center of ISO_γ by \hat{P}_γ , then $|\hat{P}_\gamma - P_\gamma| = \left| -\frac{d}{c} + \frac{\bar{a}b + \bar{c}d}{|a|^2 + |c|^2 - 1} \right| = \frac{|d - \bar{a}|}{|c|(|a|^2 + |c|^2 - 1)}$. Hence $\text{ISO}_\gamma = \Sigma_\gamma$ implies that $d = \bar{a}$ and therefore $bc = |a|^2 - 1 \in \mathbb{R}$. The latter implies that $\text{tr}(\gamma) = a + \bar{a} \in \mathbb{R}$ and that $b = 0$ or $c = \lambda \bar{b}$ for some $\lambda \in \mathbb{R}$. To prove the converse, suppose that $d = \bar{a}$. Then by the above $P_\gamma = \hat{P}_\gamma$. Moreover if $P_\gamma = -\frac{d}{c}$, then, by Proposition 3.1, $R_\gamma = \frac{1}{|c|}$ and hence $\text{ISO}_\gamma = \Sigma_\gamma$. This proves the first item.

In the second item we have that the isometric sphere does not exist and Σ_γ is a plane (and not a sphere). The conditions $c = 0$ and $|a|^2 = 1$ imply that $ad = 1$ and $a\bar{a} = 1$. Hence $d = \bar{a}$ and $\text{tr}(\gamma) \in \mathbb{R}$. As $c = 0$, the equality $c = \lambda \bar{b}$, with $\lambda \in \mathbb{R}$ is trivially true. ■

Remark 3.3. Note that Proposition 3.2 does not treat the cases $c \neq 0$ and $|a|^2 + |c|^2 = 1$ and $c = 0$ and $|a|^2 \neq 1$. In the first case the isometric sphere exists but Σ_γ does not have the form of a sphere. In the second case Σ_γ exist in the form of a sphere, but the isometric sphere does not exist. So in both cases it does not make sense to compare the isometric sphere with Σ_γ .

4 DF Domains and Double Dirichlet Domains

The goal of this section is to prove Theorem 1.1 and give some consequences that reprove and complement some results in [14]. The following definitions are taken from [14].

Definition 4.1. A Dirichlet fundamental domain which is also a Ford domain in \mathbb{H}^n is called a DF-domain. A Dirichlet fundamental domain which has multiple centers is called a double Dirichlet Domain.

Throughout this section, we work in \mathbb{H}^2 and \mathbb{H}^3 and we assume, without loss of generality, that the stabilizer of i , or j respectively, in Γ , is trivial. The latter is possible by conjugating, if needed, the group Γ by an adequate affine subgroup of $\text{PSL}_2(\mathbb{R})$, or $\text{PSL}_2(\mathbb{C})$ respectively. Indeed, denote by \mathcal{A} the subgroup of $\text{PSL}_2(\mathbb{R})$, or $\text{PSL}_2(\mathbb{C})$ consisting of upper triangular matrices. Consider the conjugated group $\tau\Gamma\tau^{-1}$ of Γ for some $\tau \in \mathcal{A}$ and let $P_0 \in \{i, j\}$, according to the space being \mathbb{H}^2 or \mathbb{H}^3 respectively. Then $(\tau\Gamma\tau^{-1})_{P_0} = \tau\Gamma_{\tau^{-1}(P_0)}\tau^{-1}$ and thus if the stabilizer of $\tau^{-1}(P_0)$ is trivial in Γ , the stabilizer of P_0 is trivial in $\tau\Gamma\tau^{-1}$. Let \mathcal{F} be some fundamental domain for Γ . By definition every point in the interior of \mathcal{F}

has trivial stabilizer. As \mathcal{A} acts transitively on the upper half-plane, there exists $\tau \in \mathcal{A}$ such that $\Gamma_{\tau^{-1}(P_0)}$ is trivial. Moreover, if Γ contains a parabolic element of the form $1 \neq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, then the conjugate $\tau\Gamma\tau^{-1}$ also contains such a parabolic element. So, instead of proving the results for Γ , we will prove them for a group conjugated to Γ with trivial stabilizer of P_0 . It is easy to see that if $\tau\Gamma\tau^{-1}$ has a double Dirichlet domain, Γ has a double Dirichlet domain. Similarly if $\tau\Gamma\tau^{-1}$ has a DF domain, Γ has a DF domain.

We first give two lemmas on Fuchsian groups.

Lemma 4.2. *The following properties are equivalent for $1 \neq \gamma \in \text{PSL}_2(\mathbb{R})$.*

1. $a(\gamma) = d(\gamma)$.
2. $\gamma = \sigma \circ \sigma_\gamma$, where σ denotes the reflection in the imaginary axes, i.e., $\sigma(z) = -\bar{z}$ and σ_γ is the reflection in Σ_γ .
3. Σ_γ is the bisector of the geodesic linking ti and $\gamma^{-1}(ti)$, for all $t > 0$.
4. There exists $0 < t_0 \neq 1$ such that Σ_γ is the bisector of the geodesic segment linking t_0i and $\gamma^{-1}(t_0i)$.

Proof. We first prove that 1 implies 2. Suppose that $a(\gamma) = d(\gamma)$ and suppose first that $c(\gamma) = 0$. Without loss of generality, we may take $a(\gamma) = d(\gamma) = 1$. By Proposition 3.1, Σ_γ is the line given by the equation $x = -\frac{b}{2}$ and thus $\sigma_\gamma(z) = \sigma(z + b(\gamma)) = \sigma(\gamma(z))$. If $c(\gamma) \neq 0$, by Proposition 3.2, we have that $\Sigma_\gamma = \text{ISO}_\gamma$. Hence the reflection σ_γ in Σ_γ is given by $\sigma_\gamma(z) = P_\gamma - (|c|^2 \sigma(z - P_\gamma))^{-1} = \sigma(\gamma(z))$. In either case, we have that $\gamma = \sigma \circ \sigma_\gamma$.

Suppose now that $\gamma = \sigma \circ \sigma_\gamma$ and let $u \in \Sigma_\gamma$. Then $\rho(u, \gamma^{-1}(ti)) = \rho(u, \sigma_\gamma \circ \sigma(ti)) = \rho(u, \sigma_\gamma(ti)) = \rho(\sigma_\gamma(u), ti) = \rho(u, ti)$ and hence Σ_γ is the bisector of the geodesic linking ti and $\gamma^{-1}(ti)$. This proves that 2 implies 3. Obviously 3 implies 4.

We now prove that 4 item implies 1. Let $u \in \Sigma_\gamma$. Then we have that $\rho(u, t_0i) = \rho(u, \gamma^{-1}(t_0i))$ and hence $\rho(u, t_0i) = \rho(\gamma(u), t_0i)$. Since γ is a Möbius transformation we have that $\text{Im}(\gamma(z)) = |\gamma'(z)|\text{Im}(z)$. Using this and the explicit formula of the hyperbolic distance in the upper half-plane model (see Section 2), we obtain that $|\gamma'(u)||t_0i - u|^2 = |t_0i - \gamma(u)|^2$. It follows that $\text{Re}(u)^2|\gamma'(u)| - \text{Re}(\gamma(u))^2 = (1 - |\gamma'(u)|)t_0^2 + (|\gamma'(u)| - 1)|\gamma'(u)|\text{Im}(u)^2$. We may write this as an equation of the type $\alpha t^2 = \beta$ having $t = t_0$ as a solution. However as $u \in \Sigma_\gamma$, by definition $\rho(u, i) = \rho(u, \gamma^{-1}(i))$ and hence also $t = 1$ is also solution of the given equation. Thus we have that $\alpha = \beta$ and $\alpha(t_0^2 - 1) = 0$. It follows that $\alpha = 0$ and thus $|\gamma'(u)| = 1$, for all $u \in \Sigma_\gamma$, i.e. $\Sigma_\gamma = \text{ISO}_\gamma$. Applying Proposition 3.2, we obtain that $a(\gamma) = d(\gamma)$. ■

Recall that one says that an angle α is a submultiple of an angle β if either there is a positive integer n such that $n\alpha = \beta$ or $\alpha = 0$.

Lemma 4.3. *Let Γ be a cofinite discrete subgroup of $\text{PSL}_2(\mathbb{R})$. Suppose that $i \in \mathbb{H}^2$ has trivial stabilizer and let \mathcal{F} be its Dirichlet fundamental polygon with center i . Let γ_k be the side-pairing transformations of \mathcal{F} for $1 \leq k \leq n$. If, for every $1 \leq k \leq n$,*

$a(\gamma_k) = d(\gamma_k)$, then Γ is the subgroup of the orientation-preserving isometries of a discrete reflection group.

Proof. First note that, as $a(\gamma_k) = d(\gamma_k)$, we have by Lemma 3.2 that $\Sigma_{\gamma_k} = \text{ISO}_{\gamma_k}$. This means that $\Sigma_{\gamma_k^{-1}}$ has the same radius as Σ_{γ_k} and their centers are the same in absolute value, but have opposite sign, and this for every $1 \leq k \leq n$. Hence \mathcal{F} is symmetric with respect to the imaginary axis Σ . Consider the polygon P , whose sides are Σ and the Σ_{γ_k} 's with $P_{\gamma_k} \geq 0$. We claim that all the dihedral angles of P are submultiples of π . First we prove this statement for the dihedral angles between two sides of \mathcal{F} . For this, consider a vertex $E_k \subseteq \Sigma_{\gamma_k} \cap \Sigma_{\gamma_l}$ for $1 \leq k \neq l \leq n$. By the symmetry of \mathcal{F} , $\{\Sigma_{\gamma_k}, \Sigma_{\gamma_l^{-1}}\}$ is a finite sequence of sides of \mathcal{F} determined by Σ_{γ_k} and E_k , according to the definition of [18, Chapter 6.8]. By [18, Theorem 6.8.7] the dihedral angle at the vertex E_k is a submultiple of π and this is true for every $1 \leq k \leq n$. Let Σ_{γ_0} be the side of \mathcal{F} with $P_{\gamma_0} \geq 0$ and such that $\Sigma_{\gamma_0} \cap \Sigma$ is a vertex of P and consider the angle θ between Σ and Σ_{γ_0} . If $P_{\gamma_0} > 0$, then the side $\Sigma_{\gamma_0^{-1}}$ has the same radius but $P_{\gamma_0^{-1}} < 0$. Hence, if Σ_{γ_0} intersects the imaginary axis Σ , then so does $\Sigma_{\gamma_0^{-1}}$ and θ is half the angle between Σ_{γ_0} and $\Sigma_{\gamma_0^{-1}}$, which is a submultiple of π by the previous. If $P_{\gamma_0} = 0$, then Σ is perpendicular to Σ_{γ_0} . Thus in both cases the angle between Σ and the adjacent side in P is a submultiple of π . This proves the claim. Finally, by [18, Theorem 7.1.3], the group $\tilde{\Gamma} = \langle \sigma, \sigma_{\gamma_k} \mid P_{\gamma_k} \geq 0 \rangle$, where σ_k denotes the reflection in Σ_{γ_k} , is a discrete reflection group with respect to P . The result then follows by Lemma 4.2. ■

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let \mathcal{F} be a DF domain, in \mathbb{H}^2 or \mathbb{H}^3 respectively, for Γ with center $P_0 \in \{i, j\}$. Let Φ_0 be a set of side-pairing transformations. Consider $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Phi_0$ with $c \neq 0$. As \mathcal{F} is a Ford domain, $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) \subseteq \text{ISO}_\gamma$. As \mathcal{F} is also a Dirichlet domain, $\mathcal{F} \cap \gamma^{-1}(\mathcal{F}) \subseteq \Sigma_\gamma$. Thus $\text{ISO}_\gamma = \Sigma_\gamma$ and, as ISO_γ is a sphere, Σ_γ is a sphere and hence $|a|^2 + |c|^2 \neq 1$. Thus, by Proposition 3.2, $d = \bar{a}$. We now consider the case when $c = 0$. Then, as \mathcal{F} is a Ford domain, $\mathcal{F} \cap \gamma(\mathcal{F})$ is a plane coming from the convex fundamental polyhedron of Γ_∞ . As \mathcal{F} is also a Dirichlet domain, Σ_γ is a plane and hence $|a|^2 = |a|^2 + |c|^2 = 1$ and thus the second item of Proposition 3.2 allows to conclude.

We now prove the converse. Let \mathcal{F} be a Dirichlet or Ford fundamental domain and let Φ_0 be a set of side-pairing transformations, such that for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Phi_0$, $d = \bar{a}$. Suppose first that $c \neq 0$, i.e. the isometric sphere associated to γ exist. We claim that $|a|^2 + |c|^2 \neq 1$. By contradiction, suppose the contrary. As $d = \bar{a}$ and $\det(\gamma) = 1$, we have that $b = -\bar{c}$. Hence $\|\gamma\|^2 = 2|a|^2 + 2|c|^2 = 2$. By [4, Theorem 2.5.1], $\gamma \in \text{PSU}_2(\mathbb{C})$ which is in contradiction with the fact that the stabilizer of P_0 is trivial. Hence by Proposition 3.2, $\text{ISO}_\gamma = \Sigma_\gamma$. Suppose now that $c = 0$. Then $\mathcal{F} \cap \gamma(\mathcal{F})$ is a plane coming from the convex fundamental polyhedron of Γ_∞ . The facts that $d = \bar{a}$ and $\det(\gamma) = 1$

imply that $|a|^2 = 1$ and thus Σ_γ is a plane given by the equation $\operatorname{Re}(a\bar{b}z) = -\frac{|b|^2}{2}$. By choosing the fundamental polyhedron of Γ_∞ well, one of its sides coincides with Σ_γ .

To prove the last part of Theorem 1.1, suppose that Γ is Fuchsian. By Lemma 4.3 we have that $\tilde{\Gamma} = \langle \sigma, \Gamma \rangle$ is a reflection group containing Γ as a subgroup of index 2. Consider finally the group $\hat{\Gamma} := \langle \tau, \Gamma \rangle$. It is clear that $\tau^2 = 1$ and by computation $(\tau\gamma)^2 = 1$ for all $\gamma \in \Phi_0$. Moreover, it is easy to compute that $(\tau\gamma)(\tau\gamma')$ has order at least 2. It thus follows that $\hat{\Gamma}$ is a Coxeter group with $[\hat{\Gamma} : \Gamma] = 2$. ■

Note that a presentation of $\tilde{\Gamma}$ and $\hat{\Gamma}$ can be obtained using [10, Theorem II.7.5]. Also this result simplifies a lot the proof of [14, Theorem 3.1], i.e. it easily follows that the orbifold of Γ is a punctured sphere in the Fuchsian case. Moreover, as is shown by the next corollary, [14, Theorem 7.3] follows easily from Theorem 1.1.

Corollary 4.4. *Let $\Gamma < \operatorname{PSL}_2(\mathbb{C})$ be a cofinite discrete group and suppose Γ admits a DF domain \mathcal{F} . Then, for every side-pairing transformation γ , $\operatorname{tr}(\gamma) \in \mathbb{R}$ and the vertical planes bisecting Σ_γ and $\Sigma_{\gamma^{-1}}$ (for $\gamma \notin \Gamma_\infty$) all intersect in a vertical axis.*

Proof. Without loss of generality, we may assume that Γ admits a DF domain with center j (see the beginning of the section). That $\operatorname{tr}(\gamma) \in \mathbb{R}$, for γ a side-pairing transformation, is a direct consequence of Theorem 1.1 or of Proposition 3.2.

If $\gamma \notin \Gamma_\infty$, Σ_γ and $\Sigma_{\gamma^{-1}}$ are Euclidean spheres with center $-\frac{a(\gamma)}{c(\gamma)}$ and $\frac{a(\gamma)}{c(\gamma)}$ respectively. A simple computation then shows that the Euclidean bisector of these two points contains the point 0 and hence the vertical plane bisecting Σ_γ and $\Sigma_{\gamma^{-1}}$ contains the point j . Hence all these vertical planes intersect in a vertical line through j . ■

We now consider when a fundamental domain is a double Dirichlet domain. The next two corollaries of our main Theorem give an alternative way to [14, Section 4] to treat such domains.

Corollary 4.5. *Let Γ be a cofinite Fuchsian group with trivial stabilizer of $i \in \mathbb{H}^2$. Then the following properties are equivalent.*

1. Γ is the subgroup of orientation-preserving isometries of a Fuchsian reflection group containing the reflection in the imaginary axis.
2. Γ has a DF domain with center i .
3. Γ has a Dirichlet fundamental domain \mathcal{F} with center i such that, for every side-pairing transformation γ , $a(\gamma) = d(\gamma)$.
4. Γ has a Dirichlet fundamental domain \mathcal{F} with i and t_0i as centers, for some $1 \neq t_0 > 0$.
5. Γ has a Dirichlet fundamental domain \mathcal{F} such that all the points of the geodesic through i and ti , for $t > 0$, are centers of \mathcal{F} .

Proof. Theorem 1.1 shows that 2 and 3 are equivalent. The equivalence of 3, 4 and 5 is given by Lemma 4.2. Moreover, by Lemma 4.3, 3 implies 1. We show that 1 implies 4. Fix a polygon P for the reflection group such that one of the sides is the imaginary axis Σ and denote the reflection in Σ by σ . Let Σ_k be a side of P . Denote by σ_{γ_k} the reflection in Σ_k and let $\gamma_k = \sigma \circ \sigma_{\gamma_k}$. Then the result follows from Lemma 4.2. ■

The previous result can be generalized, by conjugating the group Γ . We then get the following result, where the Dirichlet fundamental domain has arbitrary center $P \in \mathbb{H}^2$. However, in that case, the third item has to be dropped. We also regrouped parts 4 and 5.

Corollary 4.6. *Let Γ be a cofinite Fuchsian group with trivial stabilizer of $P \in \mathbb{H}^2$. The following properties are equivalent.*

1. Γ is the subgroup of orientation-preserving isometries of a Fuchsian reflection group containing the reflection in the vertical line through P .
2. Γ has a DF domain with center P .
3. Γ has a Dirichlet fundamental domain \mathcal{F} such that all the points of the geodesic through P and $P + i$ are centers of \mathcal{F} .

From Corollary 4.4, it follows that that all examples given in Section VII.3 in the book of Elstrodt, Grunewald and Mennicke [10] are groups whose Ford domain is also a Dirichlet domain. Note that this does not follow immediately from the results of [14].

Hence an interesting question is to analyse when the Bianchi groups have a DF domain. This question can be linked to the following result by Belolipetsky and Mcleod [5, Theorem 2.1]: for the ring of integers \mathcal{O} in $\mathbb{Q}(\sqrt{-d})$ (with d a positive square free integer), the Bianchi group $\text{PSL}_2(\mathcal{O})$ extended by two reflections is a reflection group if and only if $d \leq 19$ and $d \neq 14, 17$. To make a link to Belilopetsky’s and Mcleod’s result, we next show the following lemma.

Lemma 4.7. *Let d be a positive square free integer and let \mathcal{O} be the ring of integers of $\mathbb{Q}(\sqrt{-d})$. Assume $d \neq 1, 3$ and let Γ denote the Bianchi Group $\text{PSL}_2(\mathcal{O})$. Denote by σ_x and σ_y the reflections in the hyperplanes $x = 0$ and $y = 0$ in \mathbb{H}^3 , i.e. for $P = z + rj \in \mathbb{H}^3$, $\sigma_x(P) = -\bar{P}$ and $\sigma_y(P) = \bar{z} + rj$. Suppose that Γ has a DF domain and, moreover, that there is a set of side-pairing transformations γ for Γ that have lower left matrix entry real or purely imaginary. Then $\langle \Gamma, \sigma_x, \sigma_y \rangle$ is a reflection group.*

Proof. First note that the side-pairing transformations of a Dirichlet domain of center j in the case of a Bianchi group are not uniquely determined, as the group has a non-trivial stabilizer of j , namely $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Nevertheless, if the Bianchi group Γ has a DF domain, it is possible to choose the side-pairing transformations in such a way that the isometric sphere equals the bisector, i.e. $d = \bar{a}$ by Proposition 3.2. Moreover, in this lemma we suppose the matrix entry c of each side-pairing transformation to be real or purely imaginary.

Let \mathcal{F} be the Dirichlet domain of Γ as described in [12]. Suppose moreover \mathcal{F} is a DF domain and let $\gamma = \begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix}$ be a side-pairing transformation of \mathcal{F} that does not fix ∞ with $c \in \mathbb{R}$ or $c \in i\mathbb{R}$. We first note that \mathcal{F} is symmetric with respect to the hyperplane $x = 0$. Indeed if $\gamma \in \Gamma \setminus \Gamma_\infty$, then ISO_γ and $\text{ISO}_{\gamma^{-1}}$ have the same radius and $\sigma_x(P_\gamma) = P_{\gamma^{-1}}$. Moreover, if $\begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix} \in \Gamma \setminus \Gamma_\infty$ then also $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & a \end{pmatrix} \in \Gamma \setminus \Gamma_\infty$. Thus we also have symmetry with respect to the hyperplane $y = 0$. Let σ_γ be the reflection in the isometric sphere ISO_γ of γ . Note that ISO_γ has center $-\frac{\bar{a}}{c}$ and radius $\frac{1}{|c|}$. We compute $\gamma\sigma_\gamma(P)$ for $P = z + rj \in \mathbb{H}^3$.

$$\begin{aligned} \gamma\sigma_\gamma(P) &= (a\sigma_\gamma(P) + b)(c\sigma_\gamma(P) + \bar{a})^{-1} \\ &= c^{-1}(ac\sigma_\gamma(P) + bc)(c\sigma_\gamma(P) + \bar{a})^{-1} \\ &= c^{-1}\left(a(c\sigma_\gamma(P) + \bar{a}) - (|a|^2 - bc)\right)(c\sigma_\gamma(P) + \bar{a})^{-1} \\ &= \frac{a}{c} - c^{-1} \cdot 1 \cdot \left(c\left(-\frac{\bar{a}}{c} + \frac{P + \frac{\bar{a}}{c}}{|c|^2|P + \frac{\bar{a}}{c}|^2}\right) + \bar{a}\right)^{-1} \\ &= \frac{a}{c} - c^{-1}(\bar{c}\bar{P} + a) \\ &= -c^{-1}\bar{P}\bar{c} \\ &= -\frac{\bar{c}}{c}z + c^{-1}rj\bar{c} \end{aligned}$$

Suppose now that $c \in \mathbb{R}$ or $c \in i\mathbb{R}$. Then $\gamma\sigma_\gamma(P) = -\bar{P}$ or $\gamma\sigma_\gamma(P) = \bar{z} + rj$ and hence $\gamma = \sigma_x\sigma_\gamma$ or $\gamma = \sigma_y\sigma_\gamma$. We now have to distinguish two cases. Suppose first that $d \equiv 1, 2 \pmod 4$. Then, by [12, Lemma 4.9], the vertical hyperplanes of \mathcal{F} are given by $x = \pm\frac{1}{2}$ and $y = \pm\frac{\sqrt{d}}{2}$. Define

$$\tilde{\mathcal{F}} = \mathcal{F} \cap \{z + rj \mid \text{Re}(z) \geq 0, \text{Im}(z) \geq 0\}.$$

Then the hyperplanes and spheres defining the border of $\tilde{\mathcal{F}}$ are the four hyperplanes given by $x = 0$, $x = \frac{1}{2}$, $y = 0$ and $y = \frac{\sqrt{d}}{2}$, the spheres ISO_γ having center in $\{z + rj \mid \text{Re}(z) \geq 0, \text{Im}(z) \geq 0\}$ and the unit sphere having center in 0. This sphere comes from the unique stabilizer of j , given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By the above, $\sigma_x\gamma$ or $\sigma_y\gamma$ is the reflection in ISO_γ if γ does not fix ∞ . If γ fixes infinity then it is easy to see that $\sigma_x\gamma$ or $\sigma_y\gamma$ gives reflection in the vertical hyperplanes. If $\gamma(j) = j$, then $\sigma_x\gamma = \sigma_y\gamma = \gamma$. As \mathcal{F} is symmetric with respect to $x = 0$ and $y = 0$, $\tilde{\mathcal{F}}$ has finite volume. We now show that all dihedral angles are submultiples of π . Similar as in the proof of Lemma 4.3, by the symmetry of \mathcal{F} and [18, Theorem 6.8.7], the dihedral angles of $\tilde{\mathcal{F}}$ are all submultiples of π . Hence, by [18, Theorem 7.1.3], $\langle \Gamma, \sigma_x, \sigma_y \rangle$ is a reflection group with respect to $\tilde{\mathcal{F}}$.

Suppose now that $d \equiv 3 \pmod 4$. Then, again by Lemma 4.3, the vertical hyperplanes of \mathcal{F} form a hexagon. This is not suited for a reflection polyhedron.

Therefore we define $\tilde{\mathcal{F}}$ in the following way.

$$\tilde{\mathcal{F}} = (\mathcal{F} \cup \theta(\mathcal{F})) \cap \{z + rj \mid 0 \leq \operatorname{Re}(z) \leq \frac{1}{2}, 0 \leq \operatorname{Im}(z) \leq \frac{(d+1)\sqrt{d}}{4d}\},$$

where θ is the translation given by $\begin{pmatrix} 1 & \frac{1+\sqrt{-d}}{2} \\ 0 & 1 \end{pmatrix}$. In this way, the vertical planes forming the border of $\tilde{\mathcal{F}}$ form a rectangle. Again, similar as above, the dihedral angles of $\tilde{\mathcal{F}}$ are all submultiples of π and hence $\langle \Gamma, \sigma_x, \sigma_y \rangle$ is a reflection group with respect to $\tilde{\mathcal{F}}$. ■

Using Aurel Page's package KleinianGroups [1] and the algorithm described in [12], we determine the side-pairing transformations for the Dirichlet fundamental domain with center j for the Bianchi groups with $d \leq 19$. This yields the following result.

Lemma 4.8. *Let d and \mathcal{O} be as in Lemma 4.7. The Bianchi group $\operatorname{PSL}_2(\mathcal{O})$ has a DF domain if and only if $d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19\}$.*

Combining Lemma 4.8 and Lemma 4.7, we thus get a part of [5, Theorem 2.1]. Indeed for all the values of d stated in Lemma 4.8, the condition that the matrix entry c is real or purely imaginary is fulfilled and hence for these d the Bianchi group is a subgroup of a reflection group. Note that we excluded $d = 1$ and $d = 3$ from Lemma 4.7 because in that case the stabilizer of j is more complicated. Nevertheless it can be easily verified, that also in these two cases, the Bianchi group $\operatorname{PSL}_2(\mathcal{O})$ is a subgroup of index 4 of a reflection group. This coincides in fact with a much earlier result of Bianchi. In [6], Bianchi showed already that the Bianchi groups extended by two reflections are reflection groups for $d \leq 19$ and $d \neq 14, 17$. For $d = 10$ and $d = 13$, Bianchi states that the group is generated by a so-called improper reflection, see [6, Paragraph 17]. This is reflected in Lemma 4.8, which shows that the Bianchi group does not have a DF domain for these values of d .

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