A criterion of strong density of operator subalgebras

Antoni Wawrzyńczyk

Abstract

The purpose of the note is to show the usefulness of a simple criterion of density of a subalgebra $\mathcal{A} \subset L(X)$ in the strong operator topology for arbitrary real or complex locally convex vector space X. After proving the criterion we observe its efficiency obtaining short proofs of three important known density theorems plus a new one.

1 Introduction

Let X be a locally convex real or complex vector space. Denote by $L(X)$ the space of linear continuous operators on X.

The strong operator topology on $L(X)$ is the topology of pointwise convergence of nets. A base of the neighbourhoods of zero in this topology can be parametrized by systems $(\mathcal{V}, x_1, \ldots, x_k)$, where $\mathcal{V}$ is a convex neighbourhood of zero in X and $x_1, \ldots, x_k \in X$. It is given by the collection of sets

$$\mathcal{U}(\mathcal{V}, x_1, \ldots, x_k) = \{A \in L(X) : \, Ax_1, \ldots, Ax_k \in \mathcal{V}\}.$$

An operator $A \in L(X)$ belongs to the closure of a subset $\mathcal{S} \subset L(X)$ in the strong operator theory (denoted $\overline{\mathcal{S}}$) if and only if it can be approximated by elements of $\mathcal{S}$ on every finite-dimensional subspace of X.

Let $\mathcal{A}$ be a subalgebra of $L(X)$. We say that $x \in X$ is cyclic for $\mathcal{A}$ if $\mathcal{A}x = \{Ax : A \in \mathcal{A}\}$ is dense in X. The action of $\mathcal{A}$ is irreducible if every non-zero $x \in X$ is cyclic for $\mathcal{A}$.

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Let us provide the dual space X' with the weak* - topology. Denote by $\mathcal{A}^*$ the subalgebra of $L(X')$ of all operators adjoint to the elements of $\mathcal{A}$.

For $v \in X$ and $\varphi \in X'$ we denote by $\varphi \otimes v$ the rank one operator defined by $\varphi \otimes v(x) = \varphi(x)v$. Every rank one operator in X is of this form.

The principal result of the paper (Theorem 2.1) states that $\overline{\mathcal{A}} = L(X)$ if and only if there exist $v \in X$ and $\varphi \in X'$ which are cyclic for $\mathcal{A}$ and $\mathcal{A}^*$, respectively and $\varphi \otimes v \in \overline{\mathcal{A}}$.

In particular, if the action of $\mathcal{A}$ is irreducible and $\mathcal{A}$ contains a rank one operator, it follows that $\mathcal{A}$ is dense in $L(X)$.

In section 3 we obtain very simple proofs of density theorems known from the papers [Z], [MZ], [BS].

2 The density theorems

Let X be a locally convex vector space.

Theorem 2.1. Let $\mathcal{A}$ be a subalgebra of $L(X)$. Suppose that $v \in X$ is cyclic for $\mathcal{A}$ and $\varphi \in X'$ is cyclic for $\mathcal{A}^*$. If $\varphi \otimes v \in \overline{\mathcal{A}}$ then $\overline{\mathcal{A}} = L(X)$.

Proof. Fix $u \in X$, $f \in X'$ and choose a convex neighbourhood of zero $\mathcal{V}$ in X. For arbitrary k - tuple $x_1, \ldots, x_k \in X$ let $M = \max_{1 \leq i \leq k} |f(x_i|$.

There is $T \in \mathcal{A}$ such that $u - Tv \in \mathcal{V}/2M$ because v is cyclic for $\mathcal{A}$. Choose $\varepsilon > 0$ small enough to satisfy $\varepsilon Tv \in \mathcal{V}/2$.

The form φ is cyclic for $\mathcal{A}^*$, hence there is $S \in \mathcal{A}$ such that

$$|S^*\varphi(x_i) - f(x_i)| = |\varphi(Sx_i) - f(x_i)| < \varepsilon, \; i = 1, \ldots, k.$$

Then:

$$f(x_i)u - \varphi(Sx_i)Tv = f(x_i)(u - Tv) + (f(x_i) - \varphi(Sx_i))Tv.$$

By the choice of the objects, each term in the sum belongs to $\mathcal{V}/2$, hence $f(x_i)u - \varphi(Sx_i)Tv \in \mathcal{V}$.

For arbitrary rank one operator $f \otimes u$ and arbitrary basic neighbourhood of zero $\mathcal{U}(\mathcal{V}, x_1, \ldots, x_k)$ in $L(X)$ we have found $S, T \in \mathcal{A}$ such that

$$f \otimes u - T(\varphi \otimes v)S \in \mathcal{U}(\mathcal{V}, x_1, \ldots, x_k).$$

Since $\varphi \otimes v \in \overline{\mathcal{A}}$ by assumption, we obtain that $f \otimes u \in \overline{\mathcal{A}}$. The algebra $\overline{\mathcal{A}}$ contains all rank one operators, so it is equal to $L(X)$. ∎

Corollary 2.2. Suppose that the action of a subalgebra $\mathcal{A} \subset L(X)$ on X is irreducible. Then $\mathcal{A}$ is strongly dense in $L(X)$ if and only if $\overline{\mathcal{A}}$ contains a rank one operator.

Proof. If the action of $\mathcal{A}$ on X is irreducible then the action of $\mathcal{A}^*$ on X' is irreducible as well. Every non-zero vector $v \in X$ is cyclic for $\mathcal{A}$ and every non-zero functional $\varphi \in X'$ is cyclic for $\mathcal{A}^*$. The proof follows by Theorem 2.1. ∎

Corollary 2.3. Let H be a Hilbert space and let $\mathcal{A} \subset L(H)$ be a self-adjoint subalgebra. If the action of $\mathcal{A}$ on H is cyclic and the orthogonal projection on the 1-dimensional space spanned by the cyclic vector belongs to $\overline{\mathcal{A}}$, then $\overline{\mathcal{A}} = L(X)$.

3 Applications

We present the simplified proofs of three important results concerning the generation of the space $L(X)$. We use the author's constructions of subalgebras $\mathcal{A} \subset L(X)$ and then we apply Theorem 2.1 or Corollary 2.2 for proving the density of $\mathcal{A}$ in $L(X)$.

1. Generation of $L(H)$ by two commuting C^* - subalgebras.

Let $(H, (\cdot|\cdot))$ be a complex Hilbert space. In the paper [BS] the authors construct two C^*-commutative subalgebras of the space $L(H)$ which generate $L(H)$ in the strong operator topology. The construction uses a simple but very interesting fact that an arbitrary set Z admits a structure of an abelian group. If $\{e_i\}_{i\in Z}$ is an orthonormal basis in H and Z is provided with the structure of an Abelian group, we can define a family of shift operators in H by the formula

$$S_i(e_j) = e_{i+j}.$$

The algebra $\mathcal{A}_1$ generated by the operators S_i $i \in Z$ is self-adjoint.

Let $P_0(x) = (x|e_0)e_0$, where 0 means the neutral element of the group Z. The operator P_0 is the orthogonal projection on the line generated by e_0. The space $\mathcal{A}_2 = \mathbb{C}P_0$ is a 1-dimensional self-adjoint subalgebra of $L(H)$.

The main result of [BS] states that the subalgebra $\mathcal{A}$ generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ is dense in $L(H)$ in the strong operator topology.

It follows immediately by Corollary 2.3.

2. $L(X)$ is strongly generated by two subalgebras of square zero.

An algebra $\mathcal{A}$ is of square zero if $AB = 0$ for every A, $B \in \mathcal{A}$. In [Z] it was proved that for every Banach space X there exist two subalgebras $\mathcal{A}_i$, $i = 1, 2$ with square zero which generate $L(X)$. The construction applied works perfectly for all locally convex spaces, so we prove the theorem supposing that X is a real or complex locally vector space. We denote by $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$.

Let us fix $x_0 \in X$ and $f_0 \in X'$ in such a way that $f_0(x_0) = 1$.

Define two algebras consisting of rank one operators:

$\mathcal{A}_1 = \{f \otimes x_0 : f \in X', f(x_0) = 0\}$ and $\mathcal{A}_2 = \{f_0 \otimes z : z \in X, f_0(z) = 0\}$. A simple calculus shows that $\mathcal{A}_i$, $i = 1, 2$ are subalgebras with zero square. Let $\mathcal{A}$ be the algebra generated by elements of $\mathcal{A}_1$ and $\mathcal{A}_2$. By Corollary 2.2 we only need to prove that the action of $\mathcal{A}$ on X is (algebraically) irreducible.

The space X decomposes in the direct sum $X = \ker f_0 + \mathbb{K}x_0$. Let $x \notin \ker f_0$. Then $(f \otimes x_0 + f_0 \otimes z)x = f(x)x_0 + f_0(x)z$. For arbitrary $v = \lambda x_0 + z \in X$, $z \in \ker f_0$ we can find $f \in X'$ annihilating x_0 and such that $f(x) = \lambda$. Then $v = (f \otimes x_0 + f_0 \otimes \left(\frac{z}{f_0(x)}\right))(x)$ belongs to the orbit of x under $\mathcal{A}$. In particular the orbit of x_0 under $\mathcal{A}$ coincides with X.

For $x \in X$ such that $f_0(x) = 0$ we apply $g \otimes x_0$ where $g(x_0) = 0$ and $g(x) \neq 0$ obtaining $g \otimes x_0(x) = g(x)x_0$ and next we proceed as above.

The algebra $\mathcal{A}$ is strongly dense in $L(X)$ by Corollary 2.2.

3. For separable Banach space X the algebra $L(X)$ is strongly generated by two of its elements.

In the paper [MZ] V. Müller and W. Żelazko proved that for arbitrary separable Banach space there exist two elements of $L(X)$ which generate this algebra.

The construction of the corresponding operators is based on a result of Ovsepian and Pełczyński on the total bounded biorthogonal systems in separable Banach spaces.

Theorem 3.1. [OP], [P] Let X be a separable Banach space. There is a sequence (x_i) in X and a sequence (f_i) in X' such that

1. $f_m(x_n) = \delta_{m,n}$ (the Kronecker symbol) $m, n \in \mathbb{N}$.
2. The linear span of (x_i) is dense in X in the norm topology.
3. The linear span of (f_i) is dense in X' in the weak* - topology.
4. $\|x_i\| = 1$, and $\|f_i\| < 2$ for all i.

The conditions (1)-(4) assure that the operators

$$R = \sum_{i=1}^{\infty} 2^{-i}(f_i \otimes x_{i+1}) \quad \text{and} \quad S = \sum_{i=1}^{\infty} 2^{-i}(f_{i+1} \otimes x_i)$$

are well defined and bounded on X.

Since $Rx_i = 2^{-i}x_{i+1}$, the property (2) implies that the vector x_1 is cyclic for the subalgebra generated by the operator R.

Notice that $S^* = \sum_{i=1}^{\infty} 2^{-i}(x_i \otimes f_{i+1})$, where we identify x_i with the functional on X' defined by $\varphi \to \varphi(x_i)$. Indeed,

$$\begin{aligned} S^*(\varphi)(x) &= \varphi(\sum_{i=1}^{\infty} 2^{-i} f_{i+1}(x) x_i) = \sum_{i=1}^{\infty} 2^{-i} f_{i+1}(x)\varphi(x_i) \\ &= \sum_{i=1}^{\infty} 2^{-i}(x_i \otimes f_{i+1})(\varphi)(x). \end{aligned}$$

In particular we obtain $S^*(f_i) = 2^{-i}f_{i+1}$. By (3) the functional f_1 is cyclic for the algebra spanned by powers of the operator S^*.

By direct computation we get the formula $4SR - RS = f_1 \otimes x_1$. The assumptions of Theorem 2.1 are satisfied, hence the algebra generated by the operators R, S is dense in $L(X)$.

4. If a locally convex space X admits an operator without closed invariant subspaces then $L(X)$ is strongly generated by two elements.

In the previous subsection it was proved that for separable Banach space X the algebra $L(X)$ is generated by two of its elements. The same fact holds for several special spaces without the assumption of the separability.

Suppose that in a locally convex space X there exists $T \in L(X)$ which has no closed invariant subspace. The action of the algebra of polynomials $P(T)$ of this operator is irreducible on X. If S is an arbitrary rank one operator on X then by Corollary 2.2 the operators T and S generate $L(X)$.

References

[BS] R. Berntzen, A. Sołtysiak, *On strong generation of $B(\mathcal{H})$ by two commutative C^*-algebras*, Studia Math. 125 (2) (1997), 175-178.

[MZ] V. Müller, W. Żelazko, $B(X)$ is generated in strong operator topology by two of its elements, Czech. Math. Journ., 39 (114) (1989) 486-489.

[OP] R. I. Ovsiepan, A. Pełczyński, *Existence of a fundamental total and bounded biorthogonal sequence*, Studia Math. 54 (1975), 149-159.

[P] A. Pełczyński, *All separable Banach spaces admit for every $\varepsilon > 0$ fundamental total and bounded by $1 + \varepsilon$ biorthogonal sequences*, Studia Math. 55 (1976), 295-304.

[Z] W. Żelazko, $B(X)$ is generated in strong operator topology by two subalgebras with square zero, Proc. R. Ir. Acad., 88 A (1) (1988) 19-21.

Departamento de Matemáticas,
Universidad Autónoma Metropolitana-Iztapalapa AP 55-534,
09340 México D. F., México
email:awaw@xanum.uam.mx