On the simplicity of induced modules for reductive Lie algebras*

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1 Introduction

Let G be a reductive algebraic group defined over an algebraically closed field $\mathbf{F}$ of positive characteristic p, and let $\mathfrak{g}$ be the Lie algebra of G. In [3, 5.1], Friedlander and Parshall asked to find necessary and sufficient conditions for the simplicity of a $\mathfrak{g}$-module with p-character $\chi \in \mathfrak{g}^*$ that is induced from a simple module for a parabolic subalgebra of $\mathfrak{g}$. This question has been answered (by V. Kac) when $\mathfrak{g}$ is of type A_2 (see [3, Example 3.6] and [9]), and also when $\mathfrak{g}$ is of type A_3 (see [11]). When $\mathfrak{g}$ is of type A_n, B_n, C_n or D_n and when χ is of standard Levi form, the question is partially answered in [12], in which a sufficient condition is given for the simplicity of above-mentioned $\mathfrak{g}$-modules. In this paper we study the simplicity of these induced $\mathfrak{g}$-modules under certain assumptions on $\mathfrak{g}$ and χ.

Following [7, 6.3] we make the following hypotheses:

(H1) The derived group DG of G is simply connected;

(H2) The prime p is good for $\mathfrak{g}$;

(H3) There exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$.

Let T be a maximal torus of G, let $\mathfrak{h} = \mathrm{Lie}(T)$, and let Φ be the root system of G. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a base of Φ and let Φ^+ be the set of positive roots relative to Π. For each $\alpha \in \Phi^+$ let $\mathfrak{g}_\alpha$ denote the corresponding root space of $\mathfrak{g}$.

*Research supported by China National Science Foundation No. 11471333

Received by the editors in April 2015 - In revised form in December 2015.

Communicated by S. Caenepeel.

2010 *Mathematics Subject Classification* : 17B10; 17B45; 17B50.

Key words and phrases : reductive Lie algebras, parabolic subalgebras, induced modules, simple modules.

According to [7, 6.1] we have $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$, where

$$\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}.$$

Fix a proper subset I of Π and put $\Phi_I = \mathbb{Z}I \cap \Phi$ and $\Phi_I^+ = \Phi_I \cap \Phi^+$. Define $\tilde{\mathfrak{g}}_I = \mathfrak{h} + \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$, as well as

$$\mathfrak{u} = \sum_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha, \quad \mathfrak{u}' = \sum_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_{-\alpha}.$$

Then $\mathfrak{p}_I = \tilde{\mathfrak{g}}_I + \mathfrak{u}$ and $\mathfrak{p}_I' = \tilde{\mathfrak{g}}_I + \mathfrak{u}'$ are parabolic subalgebras of $\mathfrak{g}$, each with Levi factor $\tilde{\mathfrak{g}}_I$ [7, 10.6]. Throughout the paper we assume that $\chi(\mathfrak{n}^+) = 0$. This is done without loss of generality due to [7, Lemma 6.6]. Our method requires the additional assumption that $\chi(\mathfrak{u}') = 0$, which we make throughout,

For any restricted Lie subalgebra L of $\mathfrak{g}$, we denote by $u_\chi(L)$ the *χ-reduced enveloping algebra* of L, where we continue to use χ for the restriction of χ to L ([14, 5.3]). If $\chi = 0$, then $u_\chi(L)$ is referred to as the *restricted enveloping algebra* of L, and denoted more simply by $u(L)$. Let $L_I^\chi(\lambda)$ be a simple $u_\chi(\mathfrak{p}_I)$-module generated by a maximal vector v_λ of weight $\lambda \in \mathfrak{h}^*$. Define the induced $u_\chi(\mathfrak{g})$-module

$$Z_I^\chi(\lambda) = u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{p}_I)} L_I^\chi(\lambda).$$

The main result of the present paper is Theorem 3.7, which gives a necessary and sufficient condition for $Z_I^\chi(\lambda)$ to be simple; we show that $Z_I^\chi(\lambda)$ is simple if and only if λ is not a zero of a certain polynomial $R_\mathfrak{g}^I(\lambda)$. Under our assumption on χ, Theorem 3.7 answers the open question [3, 5.1].

The paper is organized as follows. In Section 2 we introduce the concept of an *extended α-string* for any simple root α in an irreducible root system. Then we investigate extended α-strings for all irreducible root systems (see Proposition 2.1). Using results from Section 2, we prove the main theorem in Section 3, which says that the simplicity of the induced module $Z_I^\chi(\lambda)$ is completely determined by a polynomial $R_\mathfrak{g}^I(\lambda)$. In Section 4 we establish the explicit expression of the polynomial $R_\mathfrak{g}^I(\lambda)$ (see Theorem 4.4). We also use this result to rederive the Kac-Weisfeiler theorem (see [10, Theorem 2] and [2, Theorem 8.5]).

2 α-strings in a root system

Let Π and Φ^+ be as above. Without loss of generality we assume that Φ is irreducible. For each $\alpha \in \Pi$ and $\beta \in \Phi^+ \setminus \alpha$, we denote the α-string through β by $\mathcal{S}_\alpha\beta$. Define an order on the set $\mathcal{S}_\alpha\beta$ by

$$\beta + q\alpha \prec \beta + (q-1)\alpha \prec \cdots \prec \beta \prec \beta - \alpha \prec \cdots \prec \beta - r\alpha,$$

where q (resp. r) is the largest non-negative integer such that $\beta + q\alpha$ (resp. $\beta - r\alpha$) in Φ^+. By [5, 9.4], the length of the string is at most 4. We say that the α-string through β is *isolated* if $r = q = 0$. Note that if $\mathcal{S}_\alpha\beta$ is non-isolated, we have

$\mathcal{S}_\alpha\beta = \mathcal{S}_\alpha\beta'$ for any $\beta' \in \mathcal{S}_\alpha\beta$. To avoid repetitions, we assume in the following that $\beta + \alpha \notin \Phi^+$.

We call the set $((\mathbb{N} \setminus 0)\beta + \mathbb{Z}\alpha) \cap \Phi^+$ the *extended α-string through β*, denoted $\tilde{\mathcal{S}}_\alpha\beta$. Define an order on the extended α-string by

$$l\beta + m\alpha \prec l'\beta + m'\alpha \quad \text{if } l > l' \text{ or } l = l' \text{ but } m > m'.$$

Proposition 2.1. *Assume that Φ is irreducible. Let $\alpha \in \Pi$, and let $\beta \in \Phi^+ \setminus \alpha$ with $\mathcal{S}_\alpha\beta$ non-isolated.*

(1) If Φ is not of type G_2, then we have either $\mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha\}$ or $\mathcal{S}_\alpha\beta = \{\beta, \beta-\alpha, \beta-2\alpha\}$, and either $\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta$ or $\tilde{\mathcal{S}}_\alpha\beta = \{2\beta - \alpha\} \cup \mathcal{S}_\alpha\beta$.

(2) If Φ is of type G_2, then we have either $\mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha\}$ or $\mathcal{S}_\alpha\beta = \{\beta, \beta-\alpha, \beta-2\alpha, \beta-3\alpha\}$, and either $\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta$ or $\tilde{\mathcal{S}}_\alpha\beta = \Phi^+ \setminus \alpha$.

Proof. (1) Set

$$\Phi_{\alpha,\beta} = (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi, \quad \Phi^+_{\alpha,\beta} = (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^+, \quad \Phi^-_{\alpha,\beta} = (\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^-.$$

Then clearly $\Phi_{\alpha,\beta} = \Phi^+_{\alpha,\beta} \cup \Phi^-_{\alpha,\beta}$ is a subsystem of rank 2. In addition, $\alpha \in \Phi^+_{\alpha,\beta}$ is also a simple root. By assumption, the subsystem $\Phi_{\alpha,\beta}$ can only be of type A_2 or B_2.

If $\Phi_{\alpha,\beta}$ is of type A_2, then we have $\Phi^+_{\alpha,\beta} = \{\alpha, \beta, \beta - \alpha\}$, so that

$$\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha\}.$$

If $\Phi_{\alpha,\beta}$ is of type B_2, then we have

$$\Phi^+_{\alpha,\beta} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$$

with either $\alpha = \alpha_1$ or $\alpha = \alpha_2$. Since $\mathcal{S}_\alpha\beta$ is non-isolated, in the case $\alpha = \alpha_1$, we must have $\beta = \alpha_1 + \alpha_2$. It follows that

$$\mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha\}, \quad \tilde{\mathcal{S}}_\alpha\beta = \{2\beta - \alpha\} \cup \mathcal{S}_\alpha\beta.$$

In case $\alpha = \alpha_2$, we must have $\beta = \alpha_1 + 2\alpha_2$, so that

$$\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha, \beta - 2\alpha\}.$$

(2) We now discuss the case G_2. According to [1, Ch. 6, 4.13], we have

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \quad \Pi = \{\alpha_1, \alpha_2\}.$$

Case 1. $\alpha = \alpha_1$. For $\beta = 3\alpha_1 + 2\alpha_2$, the α-string $\mathcal{S}_\alpha\beta$ is isolated; for $\beta = 3\alpha_1 + \alpha_2$, we have

$$\mathcal{S}_\alpha\beta = \{3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\} = \{\beta, \beta - \alpha, \beta - 2\alpha, \beta - 3\alpha\}$$

and $\tilde{\mathcal{S}}_\alpha\beta = \Phi^+ \setminus \alpha$.

Case 2. $\alpha = \alpha_2$. For $\beta_1 = 3\alpha_1 + 2\alpha_2$, we have

$$\tilde{\mathcal{S}}_\alpha\beta_1 = \mathcal{S}_\alpha\beta_1 = \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2\} = \{\beta_1, \beta_1 - \alpha\};$$

for $\beta_2 = 2\alpha_1 + \alpha_2$, the α-string through it is isolated; for $\beta_3 = \alpha_1 + \alpha_2$, we have $\mathcal{S}_\alpha\beta_3 = \{\alpha_1 + \alpha_2, \alpha_1\} = \{\beta_3, \beta_3 - \alpha\}$ and

$$\tilde{\mathcal{S}}_\alpha\beta_3 = \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1\} = \Phi^+ \setminus \alpha.$$

Note that $\tilde{\mathcal{S}}_\alpha\beta_1 \subseteq \tilde{\mathcal{S}}_\alpha\beta_3$. ∎

Let Φ be irreducible and let $\alpha \in \Pi$. If $\beta_1, \beta_2 \in \Phi^+ \setminus \alpha$ with $\mathcal{S}_\alpha\beta_1$ and $\mathcal{S}_\alpha\beta_2$ both non-isolated, we have from Proposition 2.1 that $\tilde{\mathcal{S}}_\alpha\beta_1 = \tilde{\mathcal{S}}_\alpha\beta_2$ or $\tilde{\mathcal{S}}_\alpha\beta_1 \cap \tilde{\mathcal{S}}_\alpha\beta_2 = \phi$ if Φ is not of type G_2, but we can have $\tilde{\mathcal{S}}_\alpha\beta_1 \subsetneqq \tilde{\mathcal{S}}_\alpha\beta_2$ in the case Φ is of type G_2.

3 Simplicity criterion

In this section, we keep the assumptions as in the introduction. Let

$$\{e_\alpha, h_\beta | \alpha \in \Phi, \beta \in \Pi\}$$

be a Chevalley basis for $\mathfrak{g}' = \mathrm{Lie}(DG)$ such that

$$[e_\alpha, e_\beta] = \pm(r+1)e_{\alpha+\beta}, \quad \text{if} \quad \alpha,\ \beta,\ \alpha+\beta \in \Phi^+,$$

where r is the greatest integer for which $\beta - r\alpha \in \Phi$ (see [5, Theorem 25.2]). From the proof of Proposition 2.1, we see that our assumption on p ensures that $(r+1) \neq 0$.

For $\alpha \in \Phi^+$ put $f_\alpha = -e_\alpha$. Then we have $\mathfrak{g}_\alpha = \mathbf{F}e_\alpha$ and $\mathfrak{g}_{-\alpha} = \mathbf{F}f_\alpha$ for every $\alpha \in \Phi^+$. For each fixed simple root α, let $N_\alpha = \sum_{\beta\in\Phi^+\setminus\alpha} \mathfrak{g}_{-\beta}$. Then N_α is a restricted subalgebra of $\mathfrak{g}$.

Let $u(N_\alpha)$ be the restricted enveloping algebra of N_α. For each $\beta \in \Phi^+ \setminus \alpha$ with $\mathcal{S}_\alpha\beta$ non-isolated, we define $\tilde{f}^\beta \in u(N_\alpha)$ to be the product of f_γ^{p-1}, $\gamma \in \tilde{\mathcal{S}}_\alpha\beta$ in the order given in Section 2. For example, if $\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta = \{\beta, \beta-\alpha\}$, then

$$\tilde{f}^\beta = f_\beta^{p-1} f_{\beta-\alpha}^{p-1} \in u(N_\alpha).$$

Remark: Let $\alpha, \beta \in \Phi^+$ such that $\alpha+\beta \in \Phi^+$ (resp. $\beta-\alpha \in \Phi^+$). Then we have

$$[e_\alpha, e_\beta] = ce_{\alpha+\beta}, \quad [f_\alpha, f_\beta] = -cf_{\alpha+\beta} \text{ (resp. } [e_\alpha, f_\beta] = cf_{\beta-\alpha})$$

for some $c \in \mathbf{F} \setminus 0$. For brevity, we omit the scalar c. This does not affect any of the proofs in this section.

Lemma 3.1. *Let $\alpha \in \Pi$. For each $\beta \in \Phi^+ \setminus \alpha$ with $\mathcal{S}_\alpha\beta$ non-isolated, we have*

$$[e_\alpha, \tilde{f}^\beta] = 0.$$

Proof. We may assume that Φ is irreducible. Suppose that Φ is not of type G_2. By Proposition 2.1, we need only consider the following cases.

Case 1. $\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta = \{\beta, \beta-\alpha\}$. Then we have $\tilde{f}^\beta = f_\beta^{p-1} f_{\beta-\alpha}^{p-1}$. Since

$$[e_\alpha, f_\beta] = f_{\beta-\alpha}, \quad [e_\alpha, f_{\beta-\alpha}] = 0, \quad [f_{\beta-\alpha}, f_\beta] = 0,$$

and $f_{\beta-\alpha}^p = 0$ in $u(N_\alpha)$, the lemma follows.

Case 2. $\tilde{\mathcal{S}}_\alpha\beta = \{2\beta-\alpha, \beta, \beta-\alpha\}$, $\mathcal{S}_\alpha\beta = \{\beta, \beta-\alpha\}$. In this case we have

$$\tilde{f}^\beta = f_{2\beta-\alpha}^{p-1} f_\beta^{p-1} f_{\beta-\alpha}^{p-1}.$$

Since

$$[e_\alpha, f_{2\beta-\alpha}] = 0, \quad [e_\alpha, f_\beta] = f_{\beta-\alpha}, \quad [f_{\beta-\alpha}, f_\beta] = f_{2\beta-\alpha},$$

and $[f_\beta, f_{2\beta-\alpha}] = 0$, we get $[e_\alpha, \tilde{f}^\beta] = 0$.

Case 3. $\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta = \{\beta, \beta-\alpha, \beta-2\alpha\}$. In this case we have

$$\tilde{f}^\beta = f_\beta^{p-1} f_{\beta-\alpha}^{p-1} f_{\beta-2\alpha}^{p-1}.$$

Since

$$[e_\alpha, f_\beta] = f_{\beta-\alpha}, \quad [e_\alpha, f_{\beta-\alpha}] = f_{\beta-2\alpha}, \quad [e_\alpha, f_{\beta-2\alpha}] = 0, \quad [f_{\beta-\alpha}, f_\beta] = 0,$$

and $[f_{\beta-2\alpha}, f_{\beta-\alpha}] = 0$, we have $[e_\alpha, \tilde{f}^\beta] = 0$.

Assume Φ is of type G_2. In the case $\alpha = \alpha_1$, $\beta = 3\alpha_1 + \alpha_2$, we have from the proof of Proposition 2.1 that

$$\tilde{f}^\beta = f_{3\alpha_1+2\alpha_2}^{p-1} f_{3\alpha_1+\alpha_2}^{p-1} f_{2\alpha_1+\alpha_2}^{p-1} f_{\alpha_1+\alpha_2}^{p-1} f_{\alpha_2}^{p-1}.$$

Since $[e_{\alpha_1}, f_{3\alpha_1+2\alpha_2}] = 0$, we have

$$\begin{aligned} e_{\alpha_1}\tilde{f}^\beta &= f_{3\alpha_1+2\alpha_2}^{p-1} e_{\alpha_1} f_{3\alpha_1+\alpha_2}^{p-1} f_{2\alpha_1+\alpha_2}^{p-1} f_{\alpha_1+\alpha_2}^{p-1} f_{\alpha_2}^{p-1} \\ &\text{(using } [e_{\alpha_1}, f_{3\alpha_1+\alpha_2}] = f_{2\alpha_1+\alpha_2} \text{ and } [f_{2\alpha_1+\alpha_2}, f_{3\alpha_1+\alpha_2}] = 0) \\ &= f_{3\alpha_1+2\alpha_2}^{p-1} f_{3\alpha_1+\alpha_2}^{p-1} e_{\alpha_1} f_{2\alpha_1+\alpha_2}^{p-1} f_{\alpha_1+\alpha_2}^{p-1} f_{\alpha_2}^{p-1} \\ &\text{(using } [e_{\alpha_1}, f_{2\alpha_1+\alpha_2}] = f_{\alpha_1+\alpha_2}, [f_{\alpha_1+\alpha_2}, f_{2\alpha_1+\alpha_2}] = f_{3\alpha_1+2\alpha_2}, \\ &\text{and the fact that } f_{3\alpha_1+2\alpha_2} \text{ commutes with all } f_\beta, \beta \in \Phi^+) \\ &= f_{3\alpha_1+2\alpha_2}^{p-1} f_{3\alpha_1+\alpha_2}^{p-1} f_{2\alpha_1+\alpha_2}^{p-1} e_{\alpha_1} f_{\alpha_1+\alpha_2}^{p-1} f_{\alpha_2}^{p-1} \\ &\text{(using } [e_{\alpha_1}, f_{\alpha_1+\alpha_2}] = f_{\alpha_2} \text{ and } [f_{\alpha_1+\alpha_2}, f_{\alpha_2}] = 0) \\ &= \tilde{f}^\beta e_{\alpha_1}, \end{aligned}$$

so that $[e_\alpha, \tilde{f}^\beta] = 0$.

Let $\alpha = \alpha_2$. For $\beta_1 = 3\alpha_1 + 2\alpha_2$, we have from the proof of Proposition 2.1 that $\tilde{f}^{\beta_1} = f_{\beta_1}^{p-1} f_{\beta_1-\alpha}^{p-1}$, and hence $[e_\alpha, \tilde{f}^{\beta_1}] = 0$ as above. For $\beta_3 = \alpha_1 + \alpha_2$, we have

$$\tilde{f}^{\beta_3} = f_{3\alpha_1+2\alpha_2}^{p-1} f_{3\alpha_1+\alpha_2}^{p-1} f_{2\alpha_1+\alpha_2}^{p-1} f_{\alpha_1+\alpha_2}^{p-1} f_{\alpha_1}^{p-1}.$$

Since $[e_{\alpha_2}, f_{3\alpha_1+\alpha_2}] = 0$ and $[e_{\alpha_2}, f_{2\alpha_1+\alpha_2}] = 0$, it is easy to see that $[e_\alpha, \tilde{f}^{\beta_3}] = 0$. ∎

Recall from the introduction the notation $\mathfrak{p}_I, \mathfrak{p}_I', \mathfrak{u}, \mathfrak{u}'$, and $\tilde{\mathfrak{g}}_I$. Each simple $u_\chi(\mathfrak{p}_I)$-module is generated by a maximal vector v_λ of weight $\lambda \in \mathfrak{h}^*$, denoted $L_I^\chi(\lambda)$. Define the induced $u_\chi(\mathfrak{g})$-module

$$Z_I^\chi(\lambda) = u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{p}_I)} L_I^\chi(\lambda).$$

By the PBW theorem for the χ-reduced enveloping algebra $u_\chi(\mathfrak{g})$ ([14, Theorem 5.3.1]), we have

$$Z_I^\chi(\lambda) \cong u_\chi(\mathfrak{u}') \otimes_{\mathbf{F}} L_I^\chi(\lambda)$$

as $u_\chi(\mathfrak{u}')$-modules. By the assumption on χ, we have $u_\chi(\mathfrak{u}') = u(\mathfrak{u}')$.

Let $\Phi^+ \setminus \Phi_I^+ = \{\beta_1, \beta_2, \ldots, \beta_k\}$, and let $v_1, \ldots, v_n$ be a basis of $L_I^\chi(\lambda)$. Then $Z_I^\chi(\lambda)$ has a basis

$$f_{\beta_1}^{l_1} f_{\beta_2}^{l_2} \cdots f_{\beta_k}^{l_k} \otimes v_j, \quad 0 \le l_i \le p-1,\ i = 1, \ldots, k,\ j = 1, \ldots, n.$$

Using (H3), we can show that $\mathfrak{u}$ is the nilradical of the parabolic subalgebra $\mathfrak{p}_I$. By [14, Corollary 1.3.8], $L_I^\chi(\lambda)$ is annihilated by $\mathfrak{u}$, and hence is a simple $u_\chi(\tilde{\mathfrak{g}}_I)$-module.

Lemma 3.2. *For any fixed ordering of $\Phi^+ \setminus \Phi_I^+$: $\beta_{i_1}, \ldots, \beta_{i_k}$, there is a nonzero scalar $c \in \mathbf{F}$ such that $f_{\beta_{i_1}}^{p-1} \cdots f_{\beta_{i_k}}^{p-1} = c f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}$ in $u(\mathfrak{u}')$.*

Proof. Since $u(\mathfrak{u}')$ is restricted, it is naturally a T-module under the adjoint representation. There is a PBW type basis for $u(\mathfrak{u}')$ given as:

$$f_{\beta_1}^{l_1} \cdots f_{\beta_{i_k}}^{l_k}, \quad 0 \le l_1, \ldots, l_k \le p-1 \quad ([2, \text{p.}1057]).$$

The T-weight of each element $f_{\beta_1}^{l_1} \cdots f_{\beta_{i_k}}^{l_k}$ is exactly $\sum_{s=1}^k l_s \beta_{i_s}$. Write $f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}$ as a linear combination of the above basis:

$$f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} = \sum c_l f_{\beta_{i_1}}^{l_1} \cdots f_{\beta_{i_k}}^{l_k}.$$

By comparing the T-weights we see that all coefficients c_l must be zero except for the one for $f_{\beta_{i_1}}^{p-1} \cdots f_{\beta_{i_k}}^{p-1}$, which is nonzero, since $f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}$ is an element of another basis for $u(\mathfrak{u}')$. This completes the proof. ∎

Lemma 3.3. *Let $\alpha \in I$, and let $\beta \in \Phi^+ \setminus \Phi_I^+$. If $\mathcal{S}_\alpha\beta$ is non-isolated, then*

$$\tilde{\mathcal{S}}_\alpha\beta \subseteq \Phi^+ \setminus \Phi_I^+.$$

Proof. Since $\mathfrak{u}$ is an ideal of $\mathfrak{p}_I$ with roots $\Phi^+ \setminus \Phi_I^+$, we have $\mathcal{S}_\alpha\beta \subseteq \Phi^+ \setminus \Phi_I^+$.

Suppose that Φ is irreducible and not of type G_2. By Proposition 2.1 we have

$$\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta \quad \text{or} \quad \tilde{\mathcal{S}}_\alpha\beta = \{2\beta - \alpha\} \cup \mathcal{S}_\alpha\beta.$$

The statement clearly holds in the case $\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta$. So we assume

$$\tilde{\mathcal{S}}_\alpha\beta = \{2\beta - \alpha\} \cup \mathcal{S}_\alpha\beta.$$

Since $\mathcal{S}_\alpha\beta \subseteq \Phi^+ \setminus \Phi_I^+$, we have $\beta - \alpha, \beta \in \Phi^+ \setminus \Phi_I^+$; that is, $\beta - \alpha$ and β are roots of $\mathfrak{u}$. It follows that $e_{\beta-\alpha}, e_\beta \in \mathfrak{u}$, and hence, $e_{2\beta-\alpha} \in \mathfrak{u}$. Therefore, $2\beta - \alpha$ is also a root of $\mathfrak{u}$, implying $\tilde{\mathcal{S}}_\alpha\beta \subseteq \Phi^+ \setminus \Phi_I^+$.

Suppose Φ is of type G_2. For $I = \{\alpha_1\}$, let $\alpha = \alpha_1$ and $\beta = 3\alpha_1 + \alpha_2$. Then we have by the proof of Proposition 2.1 that

$$\mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha, \beta - 2\alpha, \beta - 3\alpha\}, \quad \tilde{\mathcal{S}}_\alpha\beta = \{2\beta - 3\alpha\} \cup \mathcal{S}_\alpha\beta.$$

For $I = \{\alpha_2\}$, let $\alpha = \alpha_2$. By the proof of Proposition 2.1 we have

$$\tilde{\mathcal{S}}_\alpha\beta = \mathcal{S}_\alpha\beta = \{\beta, \beta - \alpha\}$$

for $\beta = 3\alpha_1 + 2\alpha_2$, and

$$\tilde{\mathcal{S}}_\alpha\beta = \Phi^+ \setminus \alpha = \Phi^+ \setminus \Phi_I^+$$

for $\beta = \alpha_1 + \alpha_2$. In each of these cases we have $\tilde{\mathcal{S}}_\alpha\beta \subseteq \Phi^+ \setminus \Phi_I^+$.

Suppose Φ is a disjoint union of irreducible subsystems. Then I is a disjoint union of the subsets of simple roots in these subsystems. Let $\alpha \in I$ and let $\beta \in \Phi^+ \setminus \Phi_I^+$ with $\mathcal{S}_\alpha\beta$ non-isolated. Then α, β are in the same irreducible subsystem. Thus, we have by the above discussion that $\tilde{\mathcal{S}}_\alpha\beta \subseteq \Phi^+ \setminus \Phi_I^+$. ■

Let $\alpha \in I$. By the lemma, we see that the set $\Phi^+ \setminus \Phi_I^+$ is a disjoint union of all different $\tilde{\mathcal{S}}_\alpha\beta$ with non-isolated $\mathcal{S}_\alpha\beta$ and the isolated $\mathcal{S}_\alpha\beta = \{\beta\}$. We order the elements in $\Phi^+ \setminus \Phi_I^+$ in such a way that all elements in the same $\tilde{\mathcal{S}}_\alpha\beta$ with $\mathcal{S}_\alpha\beta$ non-isolated are adjacent in the order defined in Section 2, and call it an *α-order*.

Let $\mathfrak{S}$ be a subset of Φ^+. We say that $\mathfrak{S}$ is a *closed subset* if $\alpha + \beta \in \mathfrak{S}$ for any $\alpha, \beta \in \mathfrak{S}$ such that $\alpha + \beta \in \Phi^+$. Therefore, $\Phi^+ \setminus \Phi_I^+$ is a closed subset of Φ^+. We see that $\mathfrak{S}$ is a closed subset of Φ^+ if and only if $\mathfrak{s} =: \sum_{\alpha\in\mathfrak{S}} \mathfrak{g}_{-\alpha}$ is a Lie subalgebra of $\mathfrak{g}$; it is clear that $\mathfrak{s}$ is restricted.

Let $\mathfrak{S}$ be a closed subset of Φ^+. Applying almost verbatim Humphreys's argument in the proof of [4, Lemma 1.4], we get the following result.

Lemma 3.4. *Let $(\alpha_1, \ldots, \alpha_m)$ be any ordering of $\mathfrak{S}$. If $ht(\alpha_k) = h$, assume that all exponents i_j in $f_{\alpha_1}^{i_1} \cdots f_{\alpha_m}^{i_m} \in u(\mathfrak{s})$ for which $ht(\alpha_j) \geq h$ are equal to $p - 1$. Then, if f_{α_k} is inserted anywhere into this expression, the result is* 0.

Lemma 3.5. *Let $\Phi^+ \setminus \Phi_I^+ = \{\beta_1, \ldots, \beta_k\}$. For $f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \in u(\mathfrak{u}')$, we have in $u_\chi(\mathfrak{g})$ that*

$$[e_\alpha, f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}] = 0, \quad [f_\alpha, f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}] = 0$$

for every $\alpha \in \Phi_I^+$.

Proof. By the remark before Lemma 3.1 it suffices to prove the identities for $\alpha \in I$.

For each $\alpha \in I$, we put the set $\Phi^+ \setminus \Phi_I^+$ in a fixed α-order:

$$\beta_{i_1} \prec \cdots \prec \beta_{i_k}.$$

Then $f_{\beta_{i_1}}^{p-1} \cdots f_{\beta_{i_k}}^{p-1}$ is the product of $\tilde{f}^\beta$ for non-isolated $\mathcal{S}_\alpha\beta$ and f_β^{p-1} with $\mathcal{S}_\alpha\beta$ isolated. By Lemma 3.1, e_α commutes with every $\tilde{f}^\beta$ with $\mathcal{S}_\alpha\beta$ non-isolated. It is also clear that $[e_\alpha, f_\beta^{p-1}] = 0$ if $\mathcal{S}_\alpha\beta$ is isolated. Then we have $[e_\alpha, f_{\beta_{i_1}}^{p-1} \cdots f_{\beta_{i_k}}^{p-1}] = 0$, and hence $[e_\alpha, f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}] = 0$ by Lemma 3.2.

To prove the second identity, we apply Lemma 3.4. Recall that $\mathfrak{u}$ is an ideal of $\mathfrak{p}_I$ with roots $\Phi^+ \setminus \Phi_I^+$. Then for each $\beta_i \in \Phi^+ \setminus \Phi_I^+$, we have $\alpha + \beta_i \in \Phi^+ \setminus \Phi_I^+$ if $\alpha + \beta_i \in \Phi^+$. If $[f_\alpha, f_{\beta_i}] = 0$ for all i, then it is trivially true that

$$[f_\alpha, f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}] = 0.$$

Assume that $[f_\alpha, f_{\beta_i}] \neq 0$ for some i. Then we have $[f_\alpha, f_{\beta_i}] = f_{\alpha+\beta_i}$. Since $ht(\alpha+\beta_i) > ht(\beta_i)$ for all β_i with $\alpha+\beta_i \in \Phi^+$, we have by Lemma 3.4 that

$$[f_\alpha, f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}] = \sum_{i, \alpha+\beta_i \in \Phi^+} \sum_{s=0}^{p-2} f_{\beta_1}^{p-1} \cdots (f_{\beta_i}^s f_{\alpha+\beta_i} f_{\beta_i}^{p-2-s}) \cdots f_{\beta_k}^{p-1} = 0. \qquad \blacksquare$$

Lemma 3.6. *There is a uniquely determined scalar $R_{\mathfrak{g}}^I(\lambda) \in \mathbf{F}$ such that*

$$e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda = R_{\mathfrak{g}}^I(\lambda) \otimes v_\lambda$$

in $Z_I^\chi(\lambda)$.

Proof. Let $U(\mathfrak{g})$ (resp. $U(\mathfrak{h})$) be the universal enveloping algebra of $\mathfrak{g}$ (resp. $\mathfrak{h}$). Fix an ordering $\alpha_1, \ldots, \alpha_t$ of the positive roots Φ^+. By the PBW theorem for $U(\mathfrak{g})$, we have

$$e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} = f(h) + \sum u_i^- u_i^0 u_i^+,$$

with $f(h), u_i^0 \in U(\mathfrak{h})$ and where each u_i^+ (resp. u_i^-) is of the form

$$e_{\alpha_1}^{l_1} \cdots e_{\alpha_t}^{l_t} (\text{resp.} \quad f_{\alpha_1}^{s_1} \cdots f_{\alpha_t}^{s_t}), \quad l_j, s_j \geq 0,$$

with u_i^+ and u_i^- not both equal to 1. Note that $U(\mathfrak{g})$ is naturally a T-module under the adjoint representation. Let us denote the T-weight of a weight vector $u \in U(\mathfrak{g})$ by $\mathrm{wt}(u)$. Since

$$\mathrm{wt}(e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1}) = 0,$$

we have $\mathrm{wt}(u_i^+) = -\mathrm{wt}(u_i^-) \neq 0$ for every i. It follows that $\sum_{i=1}^t l_i > 0$, for every $u_i^+ = e_{\alpha_1}^{l_1} \cdots e_{\alpha_t}^{l_t}$.

We use for the images of the generators $e_\alpha, f_\alpha, h_\alpha$ in $u_\chi(\mathfrak{g})$ the same notation as before in $U(\mathfrak{g})$. By our assumption we have

$$u_\chi(\mathfrak{g}) = u_\chi(\mathfrak{n}^-) u_\chi(\mathfrak{h}) u(\mathfrak{n}^+).$$

Then we have in $u_\chi(\mathfrak{g})$:

$$(*) \quad e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} = \bar{f}(h) + \sum \bar{u}_i^- \bar{u}_i^0 \bar{u}_i^+,$$

where $\bar{f}(h), \bar{u}_i^0 \in u_\chi(\mathfrak{h}), \bar{u}_i^- \in u_\chi(\mathfrak{n}^-), \bar{u}_i^+ \in u(\mathfrak{n}^+)$.

For each $u_i^+ = e_{\alpha_1}^{l_1} \cdots e_{\alpha_t}^{l_t} \in U(\mathfrak{g})$, if $l_s \geq p$ for some s, then $\bar{u}_i^+ = 0$. On the other hand, if $l_s \leq p-1$ for every s, the $\bar{u}_i^+ = e_{\alpha_1}^{l_1} \cdots e_{\alpha_t}^{l_t} \in u_\chi(\mathfrak{g})$, so $\bar{u}_i^+ \neq 1$ (since $u_i^+ \neq 1$). It follows that $\bar{u}_i^+ v_\lambda = 0$. Applying both sides of $(*)$ to $1 \otimes v_\lambda$, we have in $Z_I^\chi(\lambda)$ that

$$e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda = 1 \otimes \bar{f}(h) v_\lambda = R_{\mathfrak{g}}^I(\lambda) \otimes v_\lambda$$

for some scalar $R_{\mathfrak{g}}^I(\lambda)$. $\blacksquare$

Theorem 3.7. *The $u_\chi(\mathfrak{g})$-module $Z_I^\chi(\lambda)$ is simple if and only if $R_{\mathfrak{g}}^I(\lambda) \neq 0$.*

Proof. Suppose $R_{\mathfrak{g}}^I(\lambda) \neq 0$. Using the PBW theorem for the χ-reduced enveloping algebra $u_\chi(\mathfrak{g})$ ([14, Theorem 5.3.1]) and our assumption that $\chi(\mathfrak{u}') = 0$, we have a natural vector space isomorphism

$$Z_I^\chi(\lambda) \cong u_\chi(\mathfrak{u}') \otimes_{\mathbf{F}} L_I^\chi(\lambda) = u(\mathfrak{u}') \otimes_{\mathbf{F}} L_I^\chi(\lambda).$$

Put the elements in $\Phi^+ \setminus \Phi_I^+$ in the order of ascending heights: $\beta_1, \ldots, \beta_k$. Therefore, $Z_I^\chi(\lambda)$ has as basis the set $\{f_{\beta_1}^{l_1} \cdots f_{\beta_k}^{l_k} \otimes v_j | 0 \leq l_i \leq p-1, 1 \leq j \leq n\}$, where $\{v_j | 1 \leq j \leq n\}$ is a basis for $L_I^\chi(\lambda)$.

Let N be a nonzero submodule of $Z_I^\chi(\lambda)$. There exists a nonzero element $x \in N$, which we can write

$$x = \sum_l c_l f_{\beta_1}^{l_1} \cdots f_{\beta_k}^{l_k} \otimes v_l,$$

where the sum is over all tuples $l = (l_1, \ldots, l_k)$ with $0 \leq l_i \leq p-1$ and where $c_l \in \mathbf{F}$ and $v_l \in L_I^\chi(\lambda)$. By applying appropriate f_{β_i}'s, we get $f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v \in N$ for some nonzero $v \in L_I^\chi(\lambda)$.

It follows from hypothesis (H3) in the introduction that $\mathfrak{u}$ is the nilradical of the parabolic subalgebra $\mathfrak{p}_I$. By [14, Corollary 3.8], $L_I^\chi(\lambda)$ is annihilated by $\mathfrak{u}$, and is hence a simple $u_\chi(\tilde{\mathfrak{g}}_I)$-module. Therefore $u_\chi(\tilde{\mathfrak{g}}_I)v = L_I^\chi(\lambda)$. Then using Lemma 3.5, we have

$$\begin{aligned} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes L_I^\chi(\lambda) &= f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes u_\chi(\tilde{\mathfrak{g}}_I)v \\ &\subseteq u_\chi(\tilde{\mathfrak{g}}_I) f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v \\ &\subseteq N, \end{aligned}$$

so that $f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda \in N$. By Lemma 3.6,

$$R_{\mathfrak{g}}^I(\lambda) \otimes v_\lambda = e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda \in N,$$

and hence $1 \otimes v_\lambda \in N$, implying $N = Z_I^\chi(\lambda)$. We conclude that $Z_I^\chi(\lambda)$ is simple.

Suppose that $Z_I^\chi(\lambda)$ is simple. Recall the definition of the parabolic subalgebra $\mathfrak{p}_I' = \tilde{\mathfrak{g}}_I + \mathfrak{u}'$. Since $f_{\beta_i} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} = 0$ for all i, it follows that

$$L_I^\chi(\lambda)' =: f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes L_I^\chi(\lambda)$$

is a $u_\chi(\mathfrak{p}_I')$-module that is isomorphic to $L_I^\chi(\lambda)$ as vector spaces. The canonical $u_\chi(\mathfrak{g})$-module homomorphism

$$\varphi : u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{p}_I')} L_I^\chi(\lambda)' \longrightarrow Z_I^\chi(\lambda)$$

induced by the embedding $L_I^\chi(\lambda)' \subseteq Z_I^\chi(\lambda)$ is trivially nonzero, and is therefore surjective since $Z_I^\chi(\lambda)$ is simple. Comparing the dimensions we see that φ must be an isomorphism.

Now v_λ is nonzero, so $v =: e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} \otimes (f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda)$ is a nonzero element of $u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{p}'_I)} L_I^\chi(\lambda)'$. Therefore,

$$R_\mathfrak{g}^I(\lambda) \otimes v_\lambda = e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda = \varphi(v) \neq 0,$$

implying $R_\mathfrak{g}^I(\lambda) \neq 0$. ■

Let us look at an application of Theorem 3.7. In [3, 5.1], Friedlander and Parshall asked the following question: Can one give necessary and sufficient condition on a simple module for a parabolic subalgebra $\mathfrak{p}_I$ to remain simple upon induction to $\mathfrak{g}$. Clearly under our assumption the question is answered by the theorem.

4 A formula for $R_\mathfrak{g}^I(\lambda)$

In this section we determine $R_\mathfrak{g}^I(\lambda)$ using the polynomial defined by Rudakov ([13]). Recall the notation $\tilde{\mathfrak{g}}_I$ in the introduction and $\mathfrak{g}'$ at the beginning of Section 3. Define $\mathfrak{g}_I = [\tilde{\mathfrak{g}}_I, \tilde{\mathfrak{g}}_I]$. Since $\mathfrak{g}' \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}_I$ by [6, Corollary 10.5], $\mathfrak{g}_I$ is spanned by a subset of the Chevalley basis of $\mathfrak{g}'$. This ensures the application of [13, Proposition 8] to $\mathfrak{g}_I$. For each $\alpha \in \Phi$, we shall write α instead of its derivative $d\alpha$ by abuse of notation.

Let $\chi \in \mathfrak{g}^*$ as given earlier. Then χ can be written as $\chi = \chi_s + \chi_n$, with $\chi_s(\mathfrak{n}^+ + \mathfrak{n}^-) = 0$ and $\chi_n(\mathfrak{h} + \mathfrak{n}^+) = 0$. For each simple $u_{\chi_s}(\mathfrak{p}_I)$-module $L_I^{\chi_s}(\lambda)$ ($\lambda \in \mathfrak{h}^*$), define the induced module

$$Z_I^{\chi_s}(\lambda) = u_{\chi_s}(\mathfrak{g}) \otimes_{u_{\chi_s}(\mathfrak{p}_I)} L_I^{\chi_s}(\lambda).$$

Let $v_\lambda \in L_I^{\chi_s}(\lambda)$ be a maximal vector of weight λ. In a similar way as in the last section we define the scalar $R_\mathfrak{g}^I(\lambda)_s$ by

$$e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} \otimes v_\lambda = R_\mathfrak{g}^I(\lambda)_s \otimes v_\lambda.$$

Lemma 4.1. *$R_\mathfrak{g}^I(\lambda)_s = R_\mathfrak{g}^I(\lambda)$ for any $\lambda \in \mathfrak{h}^*$.*

Proof. From the last section we have in $U(\mathfrak{g})$ that

$$(1) \quad e_{\beta_1}^{p-1} \cdots e_{\beta_k}^{p-1} f_{\beta_1}^{p-1} \cdots f_{\beta_k}^{p-1} = f(h) + \sum u_i^- u_i^0 u_i^+,$$

where each u_i^+ (resp. u_i^-) is in the form $e_{\alpha_1}^{l_1} \cdots e_{\alpha_t}^{l_t}$ (resp. $f_{\alpha_1}^{k_1} \cdots f_{\alpha_t}^{k_t}$) with

$$l_1, \ldots, l_t, k_1, \ldots, k_t \in \mathbb{N}, \quad \sum_{i=1}^t l_i > 0, \quad \sum_{i=1}^t k_i > 0.$$

In view of the PBW type bases for $u_\chi(\mathfrak{g})$ and $u_{\chi_s}(\mathfrak{g})$ (see [14, Theorem 5.3.1]), we have the isomorphisms of vector spaces

$$u_\chi(\mathfrak{g}) \cong u_{\chi_n}(\mathfrak{n}^-) \otimes u_{\chi_s}(\mathfrak{h}) \otimes u(\mathfrak{n}^+), \qquad u_{\chi_s}(\mathfrak{g}) \cong u(\mathfrak{n}^-) \otimes u_{\chi_s}(\mathfrak{h}) \otimes u(\mathfrak{n}^+).$$

Then the images of the elements $f(h), u_i^0, u_i^+$ in (1) are the same in both $u_\chi(\mathfrak{g})$ and $u_{\chi_s}(\mathfrak{g})$. Applying the images of (1) to $1 \otimes v_\lambda$ in $Z_I^\chi(\lambda)$ and $Z_I^{\chi_s}(\lambda)$ respectively, we obtain the same element $1 \otimes \bar{f}(h)v_\lambda \in 1 \otimes u_{\chi_s}(\mathfrak{h})v_\lambda$. It follows that $R_\mathfrak{g}^I(\lambda)_s = R_\mathfrak{g}^I(\lambda)$. ∎

By the lemma, in calculating $R_\mathfrak{g}^I(\lambda)$, we may assume $\chi = \chi_s$. With this assumption, Lemma 3.2 says that any two products $e_{\alpha_1}^{p-1} \cdots e_{\alpha_t}^{p-1} \in u(\mathfrak{n}^+)$ (or $f_{\alpha_1}^{p-1} \cdots f_{\alpha_t}^{p-1} \in u(\mathfrak{n}^-)$) in different orders are equal (up to scalar multiple).

For the Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ of $\mathfrak{g}$, let $\mathbf{F}v_\lambda$ be the 1-dimensional $u_{\chi_s}(\mathfrak{b})$-module with v_λ a maximal vector of weight $\lambda \in \mathfrak{h}^*$. Define the induced $u_{\chi_s}(\mathfrak{g})$-module

$$Z^{\chi_s}(\lambda) = u_{\chi_s}(\mathfrak{g}) \otimes_{u_{\chi_s}(\mathfrak{b})} \mathbf{F}v_\lambda.$$

Put all positive roots in Φ^+ in the order of ascending heights:

$$\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t}.$$

Let $h_\alpha = [e_\alpha, f_\alpha]$ for all $\alpha \in \Phi^+$. Then we have by [13, Proposition 8] that

$$e_{\alpha_{i_1}}^{p-1} \cdots e_{\alpha_{i_t}}^{p-1} f_{\alpha_{i_1}}^{p-1} \cdots f_{\alpha_{i_t}}^{p-1} \otimes v_\lambda = R_\mathfrak{g}(\lambda) \otimes v_\lambda,$$

where $R_\mathfrak{g}(\lambda) = (-1)^t \Pi_{i=1}^t [(\lambda + \rho)(h_{\alpha_i})^{p-1} - 1]$.

Let $\mathfrak{b}_I = \mathfrak{b} \cap \mathfrak{g}_I$. Then $\mathfrak{b}_I$ is a Borel subalgebra of $\mathfrak{g}_I$. Define the induce $u_{\chi_s}(\mathfrak{g}_I)$-module $u_{\chi_s}(\mathfrak{g}_I) \otimes_{u_{\chi_s}(\mathfrak{b}_I)} \mathbf{F}v_\lambda$, which can be canonically imbedded in $Z^{\chi_s}(\lambda)$. Put the roots in Φ_I^+ in the order of ascending heights: $\alpha_{j_1}, \ldots, \alpha_{j_s}$. Using [13, Proposition 8] for $\mathfrak{g}_I$, we have

$$e_{\alpha_{j_1}}^{p-1} \cdots e_{\alpha_{j_s}}^{p-1} f_{\alpha_{j_1}}^{p-1} \cdots f_{\alpha_{j_s}}^{p-1} \otimes v_\lambda = (-1)^s \Pi_{i=1}^s [(\lambda + \rho_I)(h_{\alpha_{j_i}})^{p-1} - 1] \otimes v_\lambda$$

in $Z^{\chi_s}(\lambda)$, where $\rho_I = \frac{1}{2}\sum_{\alpha \in \Phi_I^+} \alpha$. We denote $(-1)^s \Pi_{i=1}^s [(\lambda + \rho_I)(h_{\alpha_{j_i}})^{p-1} - 1]$ by $R_{\mathfrak{g}_I}(\lambda)$.

As $\lambda \in \mathfrak{h}^*$ varies, each $\lambda(h_\alpha)$ with $\alpha \in \Phi^+$ can be viewed as a (linear) polynomial on $\mathfrak{h}^*$ as follows: For the basis $h_{\alpha_1}, \ldots, h_{\alpha_l}$ of $\mathfrak{h}$, let $h_{\alpha_1}^*, \ldots, h_{\alpha_l}^*$ be a basis of $\mathfrak{h}^*$ such that

$$h_{\alpha_i}^*(h_{\alpha_j}) = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, l.$$

Then each $\lambda \in \mathfrak{h}^*$ can be written as $\lambda = \sum_{i=1}^l x_i h_{\alpha_i}^*$, $x_i \in \mathbf{F}$, so that $\lambda(h_{\alpha_i}) = x_i$ for $i = 1, \ldots, l$. For each $\alpha \in \Phi^+$, using the property of the Chevalley basis ([5, Theorem 25.2(c)]) that h_α is a $\mathbb{Z}$-linear combination of $h_{\alpha_1}, \ldots, h_{\alpha_l}$, say $h_\alpha = \sum_{i=1}^l k_i h_{\alpha_i}$, we get $\lambda(h_\alpha) = \sum_{i=1}^l k_i x_i$. Therefore, $R_\mathfrak{g}^I(\lambda)$, $R_\mathfrak{g}(\lambda)$, and $R_{\mathfrak{g}_I}(\lambda)$ are all polynomials in variables $x_1, \ldots, x_l$.

Lemma 4.2.

$$R_\mathfrak{g}^I(\lambda) R_{\mathfrak{g}_I}(\lambda) = c R_\mathfrak{g}(\lambda), \quad c \in \mathbf{F} \setminus 0.$$

Proof. Put the elements in Φ^+ in the order $\alpha_1, \ldots, \alpha_t$ such that $\alpha_{t-s+1}, \ldots, \alpha_t$ are positive roots of $\mathfrak{g}_I$ in the order of ascending heights, so that

$$\Phi^+ \setminus \Phi_I^+ = \{\alpha_1, \ldots, \alpha_{t-s}\} \text{ (denoted } \{\beta_1, \ldots, \beta_k\} \text{ earlier).}$$

By Lemma 3.2 and analogous conclusions for $u(\mathfrak{u})$, $u(\mathfrak{n}^+)$, and $u(\mathfrak{n}^-)$, there is a nonzero $c \in \mathbf{F}$ such that

$$cR_{\mathfrak{g}}(\lambda) \otimes v_\lambda = e_{\alpha_1}^{p-1} \cdots e_{\alpha_{t-s}}^{p-1} e_{\alpha_{t-s+1}}^{p-1} \cdots e_{\alpha_t}^{p-1} f_{\alpha_1}^{p-1} \cdots f_{\alpha_{t-s}}^{p-1} f_{\alpha_{t-s+1}}^{p-1} \cdots f_{\alpha_t}^{p-1} \otimes v_\lambda.$$

By Lemma 3.5, each e_{α_i}, $t-s < i \leq t$, commutes with $f_{\alpha_1}^{p-1} \cdots f_{\alpha_{t-s}}^{p-1}$, so we get

$$\begin{aligned} cR_{\mathfrak{g}}(\lambda) \otimes v_\lambda &= e_{\alpha_1}^{p-1} \cdots e_{\alpha_{t-s}}^{p-1} f_{\alpha_1}^{p-1} \cdots f_{\alpha_{t-s}}^{p-1} (e_{\alpha_{t-s+1}}^{p-1} \cdots e_{\alpha_t}^{p-1} f_{\alpha_{t-s+1}}^{p-1} \cdots f_{\alpha_t}^{p-1} \otimes v_\lambda) \\ &= e_{\alpha_1}^{p-1} \cdots e_{\alpha_{t-s}}^{p-1} f_{\alpha_1}^{p-1} \cdots f_{\alpha_{t-s}}^{p-1} R_{\mathfrak{g}_I}(\lambda) \otimes v_\lambda \\ &= R_{\mathfrak{g}_I}(\lambda) R_{\mathfrak{g}}^I(\lambda) \otimes v_\lambda. \end{aligned}$$

This completes the proof. ∎

To prove the next theorem, we need to apply (H3). Let $(,)$ be the nondegenerate bilinear form on $\mathfrak{g}$. Define the mapping $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ by $\theta(x) = (-, x)$ for all $x \in \mathfrak{g}$. Let us note that $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is naturally a G-module with the adjoint (resp. coadjoint) action. Then the G-invariance of $(,)$ implies that θ is an isomorphism of G-modules, so that θ is also an isomorphism of $\mathfrak{g}$-modules by [8, 7.11(3)]. Here $\mathfrak{g}$ is a (left) $\mathfrak{g}$-module with the $\mathfrak{g}$-action given by

$$\operatorname{ad} x(y) = [x, y] \quad \text{for} \quad x, y \in \mathfrak{g},$$

whereas the $\mathfrak{g}$-action on $\mathfrak{g}^*$, by [8, 7.11(8)], is defined by

$$(x \cdot f)(y) = f(\operatorname{ad}(-x)(y)) \quad \text{for} \quad x, y \in \mathfrak{g},\ f \in \mathfrak{g}^*.$$

Since θ is a $\mathfrak{g}$-module isomorphism, it follows that

$$\begin{aligned} (-, [x, y]) &= \theta(\operatorname{ad} x(y)) \\ &= (-x) \cdot \theta(y) \\ \text{(using the definition of } \mathfrak{g}\text{-action on } \mathfrak{g}^*) &= ([-, x], y) \end{aligned}$$

for all $x, y \in \mathfrak{g}$; that is, $(,)$ is also $\mathfrak{g}$-invariant.

According to [7, 6.6], the bilinear form on $\mathfrak{g}$ is also non-degenerate on $\mathfrak{h}$. For each $\lambda \in \mathfrak{h}^*$, let $t_\lambda \in \mathfrak{h}$ be such that $\lambda(h) = (h, t_\lambda)$ for all $h \in \mathfrak{h}$. Define the bilinear form $(,)$ on $\mathfrak{h}^*$ by

$$(\lambda, \mu) = (t_\lambda, t_\mu), \quad \lambda, \mu \in \mathfrak{h}^*.$$

Lemma 4.3. *Let W be the Weyl group of G and let $w \in W$. Then*

$$(w\lambda, w\mu) = (\lambda, \mu) \quad \textit{for} \quad \lambda, \mu \in \mathfrak{h}^*.$$

Proof. Let $g \in N_G(T)$ represent w. Then since

$$(h, g^{-1} t_{g\lambda}) = (gh, t_{g\lambda}) = \lambda(h) = (h, t_\lambda)$$

for all $h \in \mathfrak{h}$, so that $g^{-1} t_{g\lambda} = t_\lambda$, it follows that, for $\lambda, \mu \in \mathfrak{h}^*$,

$$\begin{aligned} (g\lambda, g\mu) &= (t_{g\lambda}, t_{g\mu}) \\ &= (g\mu)(t_{g\lambda}) \\ &= \mu(g^{-1} t_{g\lambda}) \\ &= \mu(t_\lambda) \\ &= (\lambda, \mu). \end{aligned}$$

∎

Keep the ordering of the elements of Φ^+ as in the proof of Lemma 4.2. Then we have the following theorem.

Theorem 4.4.
$$R^I_{\mathfrak{g}}(\lambda) = c\Pi_{i=1}^{t-s}[(\lambda+\rho)(h_{\alpha_i})^{p-1} - 1]$$
for some nonzero $c \in \mathbf{F}$.

Proof. From above we have

$$R_{\mathfrak{g}}(\lambda) = (-1)^t \Pi_{i=1}^{t}[(\lambda+\rho)(h_{\alpha_i})^{p-1} - 1]$$

and
$$R_{\mathfrak{g}_I}(\lambda) = (-1)^s \Pi_{i=t-s+1}^{t}[(\lambda+\rho_I)(h_{\alpha_i})^{p-1} - 1].$$

Since $R_{\mathfrak{g}}(\lambda)$, $R^I_{\mathfrak{g}}(\lambda)$, and $R_{\mathfrak{g}_I}(\lambda)$ are all elements in the polynomial algebra $\mathbf{F}[x_1, \ldots, x_l]$, which contains no zero divisors, by the cancellation law and Lemma 4.2 it suffices to show that $\rho(h_\alpha) = \rho_I(h_\alpha)$ for all $\alpha \in \Phi^+_I = \{\alpha_{t-s+1}, \ldots, \alpha_t\}$.

For every $\alpha \in \Phi^+_I$, applying the argument for the proof [5, Proposition 8.3(c)] we have, for all $h \in \mathfrak{h}$,
$$\begin{aligned}(h, h_\alpha) &= (h, [e_\alpha, f_\alpha]) \\ &= ([h, e_\alpha], f_\alpha) \\ &= \alpha(h)(e_\alpha, f_\alpha) \\ &= (h, t_\alpha)(e_\alpha, f_\alpha) \\ &= (h, (e_\alpha, f_\alpha)t_\alpha),\end{aligned}$$
so that $h_\alpha = c_\alpha t_\alpha$, in which $c_\alpha =: (e_\alpha, f_\alpha)$ is nonzero since the bilinear form is nondegenerate.

If $\alpha \in I$, then we have

$$\begin{aligned}(\rho - \rho_I)(h_\alpha) &= c_\alpha(\rho - \rho_I)(t_\alpha) \\ &= c_\alpha(t_\alpha, t_{\rho-\rho_I}) \\ &= c_\alpha(\alpha, \rho - \rho_I) \\ \text{(using Lemma 4.3)} &= c_\alpha(s_\alpha(\alpha), s_\alpha(\rho - \rho_I)) \\ &= c_\alpha(-\alpha, \rho - \rho_I) \\ &= -(\rho - \rho_I)(h_\alpha),\end{aligned}$$

implying that $\rho(h_\alpha) = \rho_I(h_\alpha)$. For every $\alpha \in \Phi^+_I$, by the property of the Chevalley basis mentioned before, h_α is a $\mathbb{Z}$-linear combination of $h_{\alpha_i}, \alpha_i \in I$, so we have $\rho(h_\alpha) = \rho_I(h_\alpha)$. This completes the proof. ■

As an application of Theorem 4.4, we give a new proof of the Kac-Weisfeiler theorem (cf. [2, Theorem 8.5]).

Theorem 4.5. *Let $\mathfrak{g} = Lie(G)$ be a restricted Lie algebra of classical type. Keep the assumptions from the introduction. Assume that $\chi(h_\alpha) \neq 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_I$. Then the induced module $Z^\chi_I(\lambda)$ is simple.*

Proof. Recall from the proof of Lemma 4.2 that $\Phi^+ \setminus \Phi_I^+ = \{\alpha_1, \ldots, \alpha_{t-s}\}$. Since $\chi(h_\alpha) \neq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_I^+$, we have

$$(\lambda + \rho)(h_{\alpha_i})^{p-1} - 1 \neq 0 \quad \text{for} \quad i = 1, \ldots, t-s,$$

so that $R_{\mathfrak{g}}^I(\lambda) \neq 0$. Thus, $Z_I^\chi(\lambda)$ is simple. ∎

Assume that $\chi(\mathfrak{g}_{-\alpha}) \neq 0$ for all $\alpha \in I$, $\chi(\mathfrak{g}_{-\alpha}) = 0$ for all $\alpha \in \Phi^+ \setminus I$ and $\chi(\mathfrak{h} + \mathfrak{n}^+) = 0$. Then χ is referred to as having *standard Levi form*. Put $J = \Pi \setminus I$. Define the parabolic subalgebra $\mathfrak{p}_J$ similarly as in the introduction. Then we have $\chi(\mathfrak{p}_J) = 0$. Define the induced $u_\chi(\mathfrak{g})$-module

$$Z_J^\chi(\lambda) = u_\chi(\mathfrak{g}) \otimes_{u(\mathfrak{p}_J)} L_J(\lambda)$$

for $\lambda \in \mathfrak{h}^*$, where $L_J(\lambda)$ is a simple $u(\mathfrak{p}_J)$-module generated by a (unique) maximal vector of weight λ.

Note that in [11, 12] the weight λ is an element in $X(T)$. We identify it with its differential by abuse of notation. Then the sufficient condition that λ is p-regular given in [11, 12] implies that $(\lambda + \rho)(h_\alpha) \in \mathbb{F}_p \setminus 0$ for all $\alpha \in \Phi^+$, and hence, $R_{\mathfrak{g}}^I(\lambda) = 0$. Under this condition $Z_I^\chi(\lambda)$ is not simple by Theorem 3.7 and 4.4.

We now use Theorem 4.4 to show that λ is p-regular is not a necessary condition for the simplicity of $Z_J^\chi(\lambda)$.

Using the notation in Theorem 4.4, let $\lambda \in \mathfrak{h}^*$ satisfy $(\lambda + \rho)(h_{\alpha_i}) = 0$ for $i = 1, \ldots, t-s$, so that the $u_\chi(\mathfrak{g})$-module $Z_I^\chi(\lambda)$ is simple by Theorem 3.7 and 4.4. Let $v_\lambda \in L_J(\lambda)$ be the maximal vector of weight λ. By [2, Theorem 4.2], $u_\chi(\mathfrak{p}_I) \otimes v_\lambda \subseteq Z_J^\chi(\lambda)$ is a simple $u_\chi(\mathfrak{g}_I)$-submodule and hence a simple $u_\chi(\mathfrak{p}_I)$-submodule. This induces a homomorphism κ of $u_\chi(\mathfrak{g})$-modules from $Z_I^\chi(\lambda)$ into $Z_J^\chi(\lambda)$. Since $Z_I^\chi(\lambda)$ is simple, κ is a monomorphism. Furthermore, by comparing dimensions using the fact that $\dim Z_I^\chi(\lambda) = \dim u_\chi(\mathfrak{n}^-)$ and $\dim Z_J^\chi(\lambda) \leq \dim u_\chi(\mathfrak{n}^-)$, we see that κ is also an epimorphism and hence an isomorphism. Therefore, $Z_J^\chi(\lambda)$ is a simple $u_\chi(\mathfrak{g})$-module. But the weight λ is not p-regular.

ACKNOWLEDGMENT

The author would like to thank an anonymous expert for pointing out the simple proof for Proposition 2.1. The author thanks also Randall Holmes and the referee for useful suggestions.

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