Disk-cyclic and Codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces*

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Abstract

Some sufficient conditions under which the tuple of the adjoint of weighted composition operators $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on the Hilbert space $\mathcal{H}$ of analytic functions is disk-cyclic (or codisk-cyclic) were investigated.

1 Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space of analytic functions defined in $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ such that, for each $\lambda \in \mathbb{D}$, the linear functional of point evaluation $e_\lambda(f) = f(\lambda)$ is bounded. The Riesz representation theorem states that $e_\lambda(f) = \langle f, k_\lambda \rangle$ for some $k_\lambda \in \mathcal{H}$, the reproducing kernel of $\mathcal{H}$. The collection of all holomorphic functions (or self-maps) in $\mathbb{D}$ is denoted as $H(\mathbb{D})$ (or $S(\mathbb{D})$). Recently, hypercyclic and supercyclic operators have received considerable attention, especially since they arise in familiar classes of operators, such as weighted shifts [5, 6, 12, 13, 14, 20], composition operators [15], weighted composition operators [4, 11, 16, 19, 21, 22, 23]. For motivation, examples and

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background about linear dynamics, we refer the readers to the excellent books [2] by Bayart and Matheron, [7] by Grosse-Erdmann and Peris Manguillot.

For a backward shift B on $\ell^p(\mathbb{N})$, Rolewicz [18] showed that λB is hypercyclic if and only if $|\lambda| > 1$. Hence λB is not hypercyclic whenever $|\lambda| \leq 1$, one may wonder if there is an operator $\widetilde{T}$ satisfying its disk-orbit is dense in a Hilbert space. Along with this question, a new dynamical property—disk-cyclicity emerged in [1, 8, 9, 10, 17]. To be specific, an operator $\widetilde{T} \in \mathcal{B}(\mathcal{H})$ is *disk-cyclic* (or *codisk-cyclic*), if there exists a vector $x \in \mathcal{H}$ such that $\mathbb{D}Orb(\widetilde{T}, x) = \{\alpha \widetilde{T}^n x : \ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1, \ n \geq 0\}$ (or $\{\beta \widetilde{T}^n x : \ \beta \in \mathbb{C}, |\beta| \geq 1, \ n \geq 0\}$) is norm-dense in $\mathcal{H}$ and x is called a disk-cyclic (or codisk-cyclic) vector for $\widetilde{T}$, where $\widetilde{T}^n$ is obtained by composing $\widetilde{T}$ with itself n times.

An n-tuple of operators acting on $\mathcal{H}$ is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, \cdots, T_n$ on $\mathcal{H}$, and is written as $T = (T_1, T_2, \cdots, T_n)$. For $T = (T_1, T_2, \cdots, T_n)$, we denote $\mathcal{F} = \mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} : k_i \in \mathbb{N}, \ i = 1, 2, \cdots, n\}$, which is the *semigroup* generated by T. Given $x \in \mathcal{H}$, the orbit of x under the tuple T is $Orb(T, x) = \{Sx : S \in \mathcal{F}\}$. Naturally, $T = (T_1, T_2, \cdots, T_n)$ is *disk-cyclic* (or *codisk-cyclic*) if there is a vector $x \in \mathcal{H}$ such that $\mathbb{D}Orb(T, x) = \{\alpha Sx : \ \alpha \in \mathbb{C}, 0 < |\alpha| \leq 1, \ S \in \mathcal{F}\}$ (or $\mathbb{D}^c Orb(T, x) = \{\beta Sx : \ \beta \in \mathbb{C}, |\beta| \geq 1, \ S \in \mathcal{F}\}$) is dense in $\mathcal{H}$ and x is called a disk-cyclic (or codisk-cyclic) vector for the tuple T.

A complex-valued function ω in $\mathbb{D}$ such that $\omega f \in \mathcal{H}$ for every $f \in \mathcal{H}$ is called a *multiplier* of $\mathcal{H}$ and the collection of all multipliers is denoted by $\mathcal{M}(\mathcal{H})$. In [11, p552] Kamali etc. proved $\mathcal{M}(\mathcal{H}) \subseteq H^\infty$. A multiplication operator M_ω on $\mathcal{H}$ is $M_\omega f = \omega f, \ f \in \mathcal{H}$. For $\omega \in M(\mathcal{H})$ and $\varphi \in S(\mathbb{D})$ such that $f \circ \varphi \in \mathcal{H}$ for every $f \in \mathcal{H}$, then the weighted composition operator $C_{\omega,\varphi} : \mathcal{H} \to \mathcal{H}$, $C_{\omega,\varphi}(f)(z) = M_\omega C_\varphi(f)(z) = \omega(z) f(\varphi(z))$ is bounded. Due to the fact $C^*_{\omega,\varphi}(k_\lambda) = \overline{w(\lambda)} k_{\varphi(\lambda)}$ for every $\lambda \in \mathbb{D}$, it yields that $C^{*n}_{\omega,\varphi}(k_\lambda) = \left(\Pi_{j=0}^{n-1} \overline{\omega(\varphi_j(\lambda))}\right) k_{\varphi_n(\lambda)}$. Given $\omega_1, \ \omega_2 \in \mathcal{M}(\mathcal{H})$ and $\varphi_1, \ \varphi_2 \in S(\mathbb{D})$, we obtain C_{ω_1,φ_1} and C_{ω_2,φ_2}. In the current paper, we assume that *the constants and the identity function* $f(z) = z$ *are in* $\mathcal{H}$, and denote $T = (T_1, T_2) = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$. Firstly, fix $A, \ B, \ C, \ D$ the four subsets of $\mathbb{D}$ as below,

$$A = \Big\{z \in \mathbb{D} : \text{ the sequence } \Big\{\Pi_{j=0}^{n-1}\Big(\omega_1 \circ (\varphi_1)_j(z) \cdot \omega_2 \circ (\varphi_2)_j(z)\Big)\Big\}_n \text{ is bounded}\Big\},$$

$$B = \Big\{z \in \mathbb{D} : \lim_{n\to\infty} \Pi_{j=1}^{n}\Big(\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)\Big)^{-1} = 0\Big\},$$

$$C = \Big\{z \in \mathbb{D} : \lim_{n\to\infty} \Pi_{j=0}^{n-1}\Big(\omega_1 \circ (\varphi_1)_j(z) \cdot \omega_2 \circ (\varphi_2)_j(z)\Big) = 0\Big\},$$

$$D = \Big\{z \in \mathbb{D} : \text{ the sequence } \Big\{\Pi_{j=1}^{n}\Big(\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)\Big)^{-1}\Big\}_n \text{ is bounded}\Big\}.$$

In [16, Theorem 3.1], under the assumptions in (2.2)(in section 2) and $M = \sup_{z\in\mathbb{D}} \sup_{n\in\mathbb{Z}} \|k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\| < \infty$. Either the sets A and B or the sets C and

D have limit points in $\mathbb{D}$, then the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is supercyclic on $\mathcal{H}$. Inspired by the above work and the ideas in [16], we will find some sufficient conditions to guarantee the *disk-cyclicity* and *codisk-cyclicity* of the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on $\mathcal{H}$, which are *new and interesting* question for the investigation of weighted composition operators. *Indeed the similar conclusions could be obtained for the n-tuple* $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2}, \cdots, C^*_{\omega_n,\varphi_n})$, *we leave the proof for the readers.* The organization of the paper is as follows. In Section 2, we prepared some preliminary results, and then we showed several sufficient conditions ensuring the disk-cyclicity and codisk-cyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ in Section 3 and Section 4, respectively.

2 Preliminary results

In this section, we cited and proved some lemmas which were needed in the proofs of our main results. Firstly, we cited a necessary and sufficient condition ensuring C_{ω_1,φ_1} and C_{ω_2,φ_2} to commute.

Lemma 2.1. *[19, Lemma 1] If $\omega_1(z)$ and $\omega_2(z)$ are nonzero for all $z \in \mathbb{D}$, then C_{ω_1,φ_1} and C_{ω_2,φ_2} can commute if and only if*

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 \quad \text{and} \quad \omega_1 \cdot (\omega_2 \circ \varphi_1) = \omega_2 \cdot (\omega_1 \circ \varphi_2). \tag{2.1}$$

Remark 2.2. *Furthermore, we will assume that $\omega_1(z)$ and $\omega_2(z)$ are nonzero for all $z \in \mathbb{D}$ and φ_1, φ_2 satisfy*

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1, \quad \omega_1 = \omega_1 \circ \varphi_2 \quad \text{and} \quad \omega_2 = \omega_2 \circ \varphi_1. \tag{2.2}$$

It's easy to verify that C_{ω_1,φ_1} and C_{ω_2,φ_2} can commute under (2.2) and [19, p456] shows there are many maps satisfying (2.2).

Let $T_i = C^*_{\omega_i,\varphi_i}$ for $i = 1, 2$. A straightforward calculation gives that

$$T_i^n k_z = \left(\Pi_{j=0}^{n-1} \overline{(\omega_i \circ (\varphi_i)_j)(z)}\right) k_{(\varphi_i)_n(z)}, \quad i = 1, 2, \quad n \geq 1.$$

Employing (2.2), it turns out that

$$\begin{aligned}
&T_2^n T_1^n k_z \qquad (2.3)\\
&= \left(\Pi_{k=0}^{n-1} \overline{(\omega_2 \circ (\varphi_2)_k)(z)}\right)\left(\Pi_{j=0}^{n-1} \overline{(\omega_1 \circ (\varphi_1)_j \circ (\varphi_2)_n)(z)}\right) k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\\
&= \left(\Pi_{k=0}^{n-1} \overline{(\omega_2 \circ (\varphi_2)_k)(z)}\right)\left(\Pi_{j=0}^{n-1} \overline{(\omega_1 \circ (\varphi_2)_n \circ (\varphi_1)_j)(z)}\right) k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\\
&= \left(\Pi_{k=0}^{n-1} \overline{(\omega_2 \circ (\varphi_2)_k)(z)}\right)\left(\Pi_{j=0}^{n-1} \overline{(\omega_1 \circ (\varphi_1)_j)(z)}\right) k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\\
&= \left[\Pi_{j=0}^{n-1} \left(\overline{(\omega_2 \circ (\varphi_2)_j)(z)} \cdot \overline{(\omega_1 \circ (\varphi_1)_j)(z)}\right)\right] k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\\
&= \left[\Pi_{j=0}^{n-1} \left(\overline{(\omega_1 \circ (\varphi_1)_j)(z)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(z)}\right)\right] k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}.
\end{aligned}$$

As regards to the disk-cyclicity and codisk-cycliclity, the Disk-Cyclicity Criterion ([10, Proposition 2.5]) and the Codisk-Cyclicity Criterion([8, Proposition 5.2.9]) are dispensable tools in finding disk-cyclic and codisk-cyclic operators, which relate with the Supercyclicity Criterion [3]. In order to derive the corresponding criteria for tuples, we firstly presented the equivalent characterization for the disk-cyclic or codisk-cyclic tuple $T = (T_1, T_2)$ on $\mathcal{H}$ paralleling to [8, Theorem 4.24 and Theorem 5.2.4], respectively.

Theorem 2.3. *Let $T = (T_1, T_2)$ be a 2-tuple of continuous linear operators on $\mathcal{H}$.*
(1) *For the disk-cyclicity, the following statements are equivalent:*
(i) The tuple $T = (T_1, T_2)$ is disk-cyclic.
(ii) For each $x,\ y \in \mathcal{H}$ and each neighborhood W for zero in $\mathcal{H}$, there are $z \in \mathcal{H}$, $k_1, k_2 \in \mathbb{N}$, $\alpha \in \mathbb{C}$ with $0 < |\alpha| \leq 1$ such that $x - z \in W$ and $T_1^{k_1} T_2^{k_2} \alpha z - y \in W$.
(2) *For the codisk-cyclicity, the following statements are equivalent:*
(i') The tuple $T = (T_1, T_2)$ is codisk-cyclic.
(ii') For each $x,\ y \in \mathcal{H}$ and each neighborhood W for zero in $\mathcal{H}$, there are $z \in \mathcal{H}$, $k_1, k_2 \in \mathbb{N}$, $\beta \in \mathbb{C}$ with $|\beta| \geq 1$ such that $x - z \in W$ and $T_1^{k_1} T_2^{k_2} \beta z - y \in W$.

In the following, we verified the Disk-Cyclicity Criterion and the Codisk-cyclicity Criterion for the tuple $T = (T_1, T_2)$ by Theorem 2.3(1) and (2), respectively.

Proposition 2.4. *(Disk-Cyclicity Criterion for tuples) Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $T = (T_1, T_2)$ be a pair of commuting continuous linear mappings on $\mathcal{H}$. If there exist two dense subsets $\mathcal{X}, \mathcal{Y}$ in $\mathcal{H}$, a pair of strictly increasing positive integer sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ and a sequence of mappings S_k (not necessary bounded) such that $S_k(\mathcal{Y}) \subset \mathcal{Y}$ and*
(1) *$T_1^{m_k} T_2^{n_k} S_k y \to y$, $k \to \infty$, and $\lim_{k\to\infty} \|S_k y\| = 0$ for all $y \in \mathcal{Y}$;*
(2) *$\lim_{k\to\infty} \|T_1^{m_k} T_2^{n_k} x\| \|S_k y\| = 0$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$.*
Then we say that $T = (T_1, T_2)$ satisfies the Disk-Cyclicity Criterion. In particular, $T = (T_1, T_2)$ is disk-cyclic.

Proof. Choose z, $w \in \mathcal{H}$ and let W be a neighborhood for zero in $\mathcal{H}$. Without loss of generality, we suppose the diameter of W is 1, that is, $W = \{x \in \mathcal{H},\ \|x\| < 1\}$. By the density of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{H}$, there are $x \in \mathcal{X}, y \in \mathcal{Y}$ such that

$$\|z - x\| < \frac{1}{4} \ \text{ and } \ \|w - y\| < \frac{1}{4}.$$

Denote $u = x + 1/\alpha S_k y \in \mathcal{H}$ for some $k \in \mathbb{N}$ and $0 < \alpha \leq 1$, which are determined later. Using the assumptions in (1) and (2), there exists a positive integer N such that

$$\|T_1^{m_k} T_2^{n_k} S_k y - y\| < \frac{1}{4}, \ \ \|S_k y\| \leq \frac{1}{4} \ \text{ and } \ \|T_1^{m_k} T_2^{n_k} x\| \|S_k y\| < \frac{1}{8},$$

for all $k > N$ and $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

On the one hand, if $\|S_k y\| \neq 0$, fix $\alpha = 4\|S_k y\| \leq 1$ and then

$$\alpha \|T_1^{m_k} T_2^{n_k} x\| = 4\|T_1^{m_k} T_2^{n_k} x\| \|S_k y\| < \frac{1}{2}.$$

Based on the above inequalities, we obtain that

$$\|z-u\| = \|z-x-1/\alpha S_k y\| \leq \|z-x\| + 1/\alpha\|S_k y\| < 1,$$

and

$$\begin{aligned}
&\|\alpha T_1^{m_k} T_2^{n_k} u - w\| = \|\alpha T_1^{m_k} T_2^{n_k}(x + 1/\alpha S_k y) - w\| \\
&\leq \|\alpha T_1^{m_k} T_2^{n_k} x\| + \|T_1^{m_k} T_2^{n_k} S_k y - y\| + \|y - w\| \\
&< \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.
\end{aligned}$$

The above inequalities verify that $z-u \in W$ and $\alpha T_1^{m_k} T_2^{n_k} u - w \in W$ for $u \in \mathcal{H}$, $0 < \alpha \leq 1$ and $m_k, n_k \in \mathbb{N}$ for $k > N$.

On the other hand, if $\|S_k y\| = 0$ and since $T_1^{m_k} T_2^{n_k} S_k y \to y,\ k \to \infty$, then

$$\begin{aligned}
\|y\| &\leq \|T_1^{m_k} T_2^{n_k} S_k y - y\| + \|T_1^{m_k} T_2^{n_k} S_k y\| \\
&\leq \|T_1^{m_k} T_2^{n_k} S_k y - y\| + \|T_1^{m_k} T_2^{n_k}\|\|S_k y\| \to 0,\ k \to \infty.
\end{aligned}$$

That is, $y = 0$. Hence $u = x + 1/\alpha S_k y = x$ and $\|w - y\| = \|w\| < 1/4$. It's clear that $z - u = z - x \in W$. After that, we choose $0 < \alpha_0 \leq 1$ small enough, such that $\alpha_0\|T_1^{m_k} T_2^{n_k} x\| < 1/4$. We deduce that

$$\begin{aligned}
&\|\alpha_0 T_1^{m_k} T_2^{n_k} u - w\| = \|\alpha_0 T_1^{m_k} T_2^{n_k} x - w\| \\
&\leq \|\alpha_0 T_1^{m_k} T_2^{n_k} x\| + \|w\| < 1/4 + 1/4 < 1.
\end{aligned}$$

That means $\alpha_0 T_1^{m_k} T_2^{n_k} u - w \in W$ for $0 < \alpha_0 \leq 1$.

In sum, under both cases, employing Theorem 2.3 (1) the tuple $T = (T_1, T_2)$ is disk-cyclic. ■

Proposition 2.5. *(Codisk-Cyclicity Criterion for tuples) Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $T = (T_1, T_2)$ be a pair of commuting continuous linear mappings on $\mathcal{H}$. If there exist two dense subsets $\mathcal{X}, \mathcal{Y}$ in $\mathcal{H}$, a pair of strictly increasing positive integer sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ and a sequence of mappings S_k (not necessary bounded) such that $S_k(\mathcal{Y}) \subset \mathcal{Y}$ and*

(1) $T_1^{m_k} T_2^{n_k} S_k y \to y$, $k \to \infty$ for all $y \in \mathcal{Y}$ and $\lim_{k\to\infty} \|T_1^{m_k} T_2^{n_k} x\| = 0$ for all $x \in \mathcal{X}$.

(2) $\lim_{k\to\infty} \|T_1^{m_k} T_2^{n_k} x\|\|S_k y\| = 0$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

Then we say that $T = (T_1, T_2)$ satisfies the Codisk-Cyclicity Criterion. In particular, $T = (T_1, T_2)$ is codisk-cyclic.

Proof. Let $z,\ w \in \mathcal{H}$ and W be a neighborhood for zero in $\mathcal{H}$. Without loss of generality, we also assume the diameter of W is 1, that is, $W = \{x \in \mathcal{H}, \|x\| \leq 1\}$. By the density of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{H}$, there are $x \in \mathcal{X}, y \in \mathcal{Y}$ such that

$$\|z - x\| < \frac{1}{4} \text{ and } \|w - y\| < \frac{1}{4}.$$

Denote $u = x + 1/\beta S_k y$ for some $k \in \mathbb{N}$ and $\beta \geq 1$, which are determined later. Using the assumptions in (1) and (2), there is a positive integer N such that

$$\|T_1^{m_k} T_2^{n_k} S_k y - y\| < \frac{1}{4},\ \ \|T_1^{m_k} T_2^{n_k} x\| \leq \frac{1}{4} \text{ and } \|T_1^{m_k} T_2^{n_k} x\|\|S_k y\| < \frac{1}{8},$$

for all $k > N$ and $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

On the one hand, if $\|T_1^{m_k}T_2^{n_k}x\| \neq 0$, fix $\beta = (4\|T_1^{m_k}T_2^{n_k}x\|)^{-1} \geq 1$ and then

$$\frac{1}{\beta}\|S_k y\| = 4\|T_1^{m_k}T_2^{n_k}x\|\|S_k y\| < \frac{1}{2}.$$

Based on the above inequalities, we obtain that

$$\|z-u\| = \|z-x-1/\beta S_k y\| \leq \|z-x\| + 1/\beta\|S_k y\| < 1,$$

and

$$\begin{aligned} &\|\beta T_1^{m_k}T_2^{n_k}u - w\| = \|\beta T_1^{m_k}T_2^{n_k}(x+1/\beta S_k y) - w\| \\ &\leq \|\beta T_1^{m_k}T_2^{n_k}x\| + \|T_1^{m_k}T_2^{n_k}S_k y - y\| + \|y-w\| \\ &< \frac{1}{4} + \frac{1}{4} + \frac{1}{4} < 1. \end{aligned}$$

The above inequalities verify that $z-u \in W$ and $\beta T_1^{m_k}T_2^{n_k}u - w \in W$ for $\beta \geq 1$.

On the other hand, if $\|T_1^{m_k}T_2^{n_k}x\| = 0$. Then choose $\beta > 1$ large enough such that $1/\beta\|S_k y\| < 1/4$. Hence

$$\|z-u\| \leq \|z-x\| + \frac{1}{\beta}\|S_k y\| < \frac{1}{4} + \frac{1}{4} < 1.$$

That is, $z-u \in W$. Moreover,

$$\begin{aligned} \|\beta T_1^{m_k}T_2^{n_k}u - w\| &= \|\beta T_1^{m_k}T_2^{n_k}(x+1/\beta S_k y) - w\| \\ &= \|T_1^{m_k}T_2^{n_k}S_k y - w\| \\ &\leq \|T_1^{m_k}T_2^{n_k}S_k y - y\| + \|y-w\| \\ &< \frac{1}{4} + \frac{1}{4} < 1. \end{aligned}$$

That means $\beta T_1^{m_k}T_2^{n_k}u - w \in W$. From Theorem 2.3 (2), we deduce the tuple $T = (T_1, T_2)$ is codisk-cyclic. This ends the proof. ∎

For further use, we cite the definition for conjugacy from [7].

Definition 2.6. *[7, Definition 1.5] Let $\widetilde{S} : Y \to Y$ and $\widetilde{T} : X \to X$ be two dynamical systems on Banach spaces X and Y. Then $\widetilde{T}$ is called* conjugate *to $\widetilde{S}$ if there exists a homeomorphism $\phi : Y \to X$ such that $\widetilde{T} \circ \phi = \phi \circ \widetilde{S}$.*

Concerning the disk-cyclicity and codisk-cyclicity, the following proposition holds.

Proposition 2.7. *Disk-cyclicity (Codisk-cyclicity) for an operator $\widetilde{T} \in \mathcal{B}(\mathcal{H})$ is preserved under conjugacy.*

3 Disk-cyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$

In this section, we mainly discover some sufficient conditions for the disk-cyclicity of the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on the Hilbert space $\mathcal{H}$. Firstly, we use the sets $A,\ B$ to state our main theorem.

Theorem 3.1. *Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and $\varphi_1(z), \varphi_2(z)$ be two automorphisms in $\mathbb{D}$ satisfying (2.2). Suppose*

$$M = \sup_{z\in\mathbb{D}} \sup_{n\in\mathbb{Z}} \|k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\| < \infty. \tag{3.1}$$

If the sets A and B have limit points in $\mathbb{D}$, then the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is disk-cyclic on $\mathcal{H}$.

Proof. We will use Proposition 2.4 to prove the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is disk-cyclic. Let $S_A = \operatorname{span}\{k_z : \ z \in A\}$ and $S_B = \operatorname{span}\{k_z : \ z \in B\}$. Then $\overline{S_A} = \overline{S_B} = \mathcal{H}$, that is, the sets S_A and S_B are dense in $\mathcal{H}$. For the readers' convenience, we now get down to the details. If $f \in \mathcal{H}$ is orthogonal to k_z for every $z \in S_A$, then $f(z) = \langle f, k_z \rangle$. Since the set A has limit point in $\mathbb{D}$, hence the identity theorem for holomorphic functions implies that f vanishes identically on $\mathcal{H}$. That is, $(S_A)^\perp = \{0\}$. Hence $\overline{S_A} = \mathcal{H}$. Similarly, $\overline{S_B} = \mathcal{H}$.

Let $\mathcal{X} = S_A$ and $\mathcal{Y} = S_B$, which are dense subsets of the Hilbert space $\mathcal{H}$. Since φ_1 and φ_2 are two automorphisms on $\mathbb{D}$, thus φ_1^{-1} and φ_2^{-1} exist on $\mathbb{D}$. Further, (2.2) implies that

$$\varphi_1^{-1} \circ \varphi_2^{-1} = \varphi_2^{-1} \circ \varphi_1^{-1}, \ \ \omega_1 = \omega_1 \circ \varphi_2^{-1} \text{ and } \omega_2 = \omega_2 \circ \varphi_1^{-1}. \tag{3.2}$$

Note $T_i = C^*_{\omega_i,\varphi_i}$ for $i = 1, 2$. We observe from (2.3) that

$$\begin{aligned} T_2^n T_1^n k_z \ &= \ \left[\Pi_{j=0}^{n-1} \left(\overline{(\omega_1 \circ (\varphi_1)_j)(z)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(z)} \right)\right] \\ &\quad \cdot k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}, \ \ n \geq 1. \end{aligned} \tag{3.3}$$

To find the right inverse of $T_2 T_1$, the proof falls into two cases according to the set $G_B = \{k_z : \ z \in B\}$ is linearly independent or not.

Case (i) Assume that G_B is a linearly independent set. Define the operator $S : G_B \to \mathcal{H}$ by

$$Sk_z \ = \ \overline{[(\omega_1 \circ \varphi_1^{-1}(z)) \cdot (\omega_2 \circ \varphi_2^{-1}(z))]^{-1}} k_{\varphi_2^{-1} \circ \varphi_1^{-1}(z)}, \ z \in \mathbb{D}.$$

Employing (3.2), we derive S^n on G_B for all $n \geq 1$,

$$S^n k_z = \Pi_{j=1}^n \overline{[\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)]^{-1}} k_{(\varphi_2)_{-n} \circ (\varphi_1)_{-n}(z)}. \tag{3.4}$$

Since G_B is linearly independent, we extend S by linearity on the set $\mathcal{Y} = S_B = \operatorname{span}\{k_z : \ z \in B\}$. Hence S^n is well-defined on $\mathcal{Y}$ and satisfies $S^n(\mathcal{Y}) \subset \mathcal{Y}$ for all $n \geq 1$. The assumption verifies that

$$\lim_{n\to\infty} \|S^n y\| = 0 \ \text{ for all } y \in \mathcal{Y}. \tag{3.5}$$

By (2.2), we arrive at $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, $\omega_2 = \omega_2 \circ \varphi_1$ and it yields that

$$\begin{aligned} T_2T_1Sk_z &= T_2T_1\Big(\overline{[(\omega_1 \circ \varphi_1^{-1}(z)) \cdot (\omega_2 \circ \varphi_2^{-1}(z))]^{-1}}k_{\varphi_2^{-1}\circ\varphi_1^{-1}(z)}\Big) \\ &= T_2\Big(\overline{[\omega_2 \circ \varphi_2^{-1} \circ \varphi_1(z)]^{-1}}k_{\varphi_2^{-1}(z)}\Big) \\ &= \overline{\omega_2(z)[\omega_2 \circ \varphi_2^{-1} \circ (\varphi_1 \circ \varphi_2)(z)]^{-1}}k_z \\ &= \overline{\omega_2(z)[\omega_2 \circ \varphi_2^{-1} \circ (\varphi_2 \circ \varphi_1)(z)]^{-1}}k_z \\ &= \overline{\omega_2(z)[\omega_2(\varphi_1(z))]^{-1}}k_z \\ &= \overline{\omega_2(z)[\omega_2(z)]^{-1}}k_z \\ &= k_z. \end{aligned}$$

That is, $T_2T_1S = Id_{\mathcal{Y}}$ and $T_2^nT_1^nS^n$ is the identity on $\mathcal{Y}$ for each $n \geq 1$. Hence

$$T_2^nT_1^nS^ny \to y,\ n \to \infty,\ \text{for every}\ \ y \in \mathcal{Y} = S_B. \tag{3.6}$$

In addition, by the conditions and (3.1), we conclude that

$$\begin{aligned} &\lim_{n\to\infty} \|T_2^nT_1^nk_y\|\|S^nk_z\| \\ &= \lim_{n\to\infty} \|\Big[\Pi_{j=0}^{n-1}\Big(\overline{(\omega_1 \circ (\varphi_1)_j)(y)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(y)}\Big)\Big]k_{(\varphi_1)_n\circ(\varphi_2)_n(y)}\| \\ &\cdot\|\Pi_{j=1}^{n}\overline{[\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)]^{-1}}k_{(\varphi_2)_{-n}\circ(\varphi_1)_{-n}(z)}\| \\ &\leq M^2 \sup_{n\in\mathbb{N}} |\Pi_{j=0}^{n-1}\Big(\overline{(\omega_1 \circ (\varphi_1)_j)(y)} \cdot \overline{(\omega_2 \circ (\varphi_2)_j)(y)}\Big)| \\ &\cdot \lim_{n\to\infty} |\Pi_{j=1}^{n}\overline{[\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)]^{-1}}| \\ &= 0, \ \ \text{for}\ y \in A,\ z \in B. \end{aligned}$$

The above inequalities lead that

$$\lim_{n\to\infty} \|T_2^nT_1^nx\|\|S^ny\| = 0,\ \text{for every}\ x \in \mathcal{X}\ \text{and}\ y \in \mathcal{Y}. \tag{3.7}$$

Depending on (3.5)-(3.7) and Proposition 2.4, the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ fulfils the Disk-Cyclicity Criterion for tuples, hence $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is disk-cyclic on $\mathcal{H}$.

Case (ii). Assume that $G_B = \{k_z :\ z \in B\}$ is not necessarily linearly independent. In this case, we adapt the method used by Godefroy and Shapiro in [5, Theorem 4.5]. Consider a countable dense subset

$$B_1 = \{w_n \in \mathbb{D} :\ n \geq 1\}$$

of B and find a sequence $\{z_n\}$ by mathematical induction. Let $z_1 = w_1$ and denote

$$B_2 = B_1 \setminus \{w \in B_1 :\ k_w \in \text{span}\{k_{z_1}\}\}.$$

Denote the first element of B_2 by z_2 and let

$$B_3 = B_2 \setminus \{w \in B_2 :\ k_w \in \text{span}\{k_{z_1}, k_{z_2}\}\ \}.$$

The infinite dimensionality of $\mathcal{H}$ insures the process never terminates and thus an infinite subset $L = \{z_n \in \mathbb{D} : n \geq 1\}$ of B is obtained. The corresponding set of kernel functions $H_L = \{k_z : z \in L\}$ is linearly independent and is dense in $\mathcal{H}$. Then the operator S can be defined exactly as above, just replacing G_B by H_L. Consequently, the Disk-Cyclicity Criterion for tuples is also true in this case.

So, in both cases, the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is disk-cyclic. This finishes the proof. ∎

Employing Theorem 3.1, we can obtain the disk-cyclicity of the tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ immediately.

Corollary 3.2. *Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$. Denote the sets*

$$\widetilde{A} = \Big\{z \in \mathbb{D} : \text{ the sequence } \{(\omega_1(z)\omega_2(z))^n\}_n \text{ is bounded}\Big\},$$
$$\widetilde{B} = \Big\{z \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{(\omega_1(z)\omega_2(z))^n} = 0\Big\}.$$

If the sets $\widetilde{A}$ and $\widetilde{B}$ have limit points in $\mathbb{D}$, then the tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ is disk-cyclic on $\mathcal{H}$.

Proof. Let $\varphi_1(z) = \varphi_2(z) = z$ in Theorem 3.1. It is evident that $\sup_{z\in\mathbb{D}} \|k_z\| < \infty$ emerged in (3.1) holds. Then the desired result follows from Theorem 3.1. ∎

We show an example to account for Corollary 3.2.

Example 3.3. *Let $w_1(z) = z$ and $w_2(z) = z + 8$. Then*

$$\{x : 0 \leq x < \sqrt{17} - 4\} \subseteq \{z \in \mathbb{D} : \text{the sequence } \{(z(z+8))^n\}_n \text{ is bounded}\},$$
$$\{x : -1 < x < -4 + \sqrt{14}\} \subseteq \{z \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{(z(z+8))^n} = 0\}.$$

The sets $\widetilde{A}$ and $\widetilde{B}$, apparently, have limit points in $\mathbb{D}$. The tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ is disk-cyclic due to Corollary 3.2.

For $a \in \mathbb{D}$, an automorphism $\phi_a(z)$ of $\mathbb{D}$ is defined by

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}. \tag{3.8}$$

There are so many spaces that contain ϕ_a, such as the Hardy space, Bergman space and Dirichlet space. These spaces are called automorphism invariant spaces. As we all know, all holomorphic self-maps of the unit disk $\mathbb{D}$ are dived into classes of *elliptic* and *nonelliptic.* The elliptic map is conjugate to a rotation $z \to \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

If φ_1 and φ_2 are two elliptic disk automorphisms satisfying $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, then their interior fixed points are identical. Indeed, if $\varphi_1(z_1) = z_1 \in \mathbb{D}$ and $\varphi_2(z_2) = z_2 \in \mathbb{D}$, then

$$\varphi_1 \circ \varphi_2(z_2) = \varphi_2 \circ \varphi_1(z_2) \Rightarrow \varphi_1(z_2) = \varphi_2(\varphi_1(z_2)) \Rightarrow \varphi_1(z_2) = z_2 \Rightarrow z_1 = z_2.$$

Remark 3.4. *For general case, if φ_1 and φ_2 have interior fixed points in $\mathbb{D}$ and satisfy $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, then they have the same interior fixed points.*

Theorem 3.5. *Suppose that $\mathcal{H}$ is automorphism invariant. Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two elliptic disk automorphisms with an interior fixed point $a \in \mathbb{D}$ satisfying (2.2). If the sets A and B have limit points in $\mathbb{D}$, then the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is disk-cyclic on $\mathcal{H}$.*

Proof. We divide the proof into two cases.
Case (i) Suppose the interior fixed point $a = 0$. Then there are $\theta_1,\ \theta_2 \in [0, 2\pi]$ such that

$$\varphi_1(z) = e^{i\theta_1} z,\ \varphi_2(z) = e^{i\theta_2} z.$$

It yields that

$$(\varphi_2)_n \circ (\varphi_1)_n(z) = e^{in\theta_1} e^{in\theta_2} z.$$

Hence the iterate $\{(\varphi_2)_n \circ (\varphi_1)_n :\ n \in \mathbb{Z}\} \subseteq z\partial\mathbb{D}$. Since $z\partial\mathbb{D}$ is compact subset of $\mathbb{D}$, thus

$$\Big(f((\varphi_2)_n \circ (\varphi_1)_n) \Big)_{n \in \mathbb{Z}}$$

is a bounded sequence for $f \in \mathcal{H} \bigcap H(\mathbb{D})$. By the uniform boundedness principle, we get that

$$M = \sup_{z \in \mathbb{D}} \sup_{n \in \mathbb{Z}} \|k_{(\varphi_2)_n \circ (\varphi_1)_n}\| < \infty. \tag{3.9}$$

Employing (3.9) and Theorem 3.1, the tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ satisfies the Disk-Cyclicity Criterion for tuples.
Case (ii) If $a \neq 0$ is the interior fixed point of φ_i $(i = 1, 2)$. We notice that $\mathcal{H}$ is automorphism invariant. Let

$$\widetilde{\varphi_1} = \phi_a \circ \varphi_1 \circ \phi_a^{-1},\ \widetilde{\varphi_2} = \phi_a \circ \varphi_2 \circ \phi_a^{-1}$$

be two automorphisms with the interior fixed point zero, and let

$$\widetilde{\omega_1} = \omega_1 \circ \phi_a^{-1},\ \widetilde{\omega_2} = \omega_2 \circ \phi_a^{-1}$$

be two multipliers of $\mathcal{H}$, where ϕ_a is the automorphism provided in (3.8). The disk-cyclicity of the tuple $(C^*_{\widetilde{\omega_1},\widetilde{\varphi_1}}, C^*_{\widetilde{\omega_2},\widetilde{\varphi_2}})$ on $\mathcal{H}$ follows from **Case (i)**, where $C_{\widetilde{\omega_i},\widetilde{\varphi_i}} = C^{-1}_{\phi_a} \circ C_{\omega_1,\varphi_1} \circ C_{\phi_a}$ for $i = 1, 2$. Finally, since C_{ω_i,φ_i} is conjugate $C_{\widetilde{\omega_i},\widetilde{\varphi_i}}$ for $i = 1, 2$, and by Proposition 2.7, we obtain the desired result. This completes the proof. ∎

Example 3.6. *Take two elliptic disk automorphisms $\varphi_1(z) = iz$, $\varphi_2(z) = -iz$ with an interior fixed point $a = 0 \in \mathbb{D}$ and $w_1(z) = z^4$, $w_2(z) = z^4 + 4$. The sets A and B are*

$$A = \Big\{ z \in \mathbb{D} :\ \textit{the sequence } \Big\{ z^{4n}(z^4+4)^n \Big\}_n \textit{ is bounded} \Big\},$$
$$B = \Big\{ z \in \mathbb{D} :\ \lim_{n\to\infty} \frac{1}{z^{4n}(z^4+4)^n} = 0 \Big\}.$$

Since $[0, \frac{1}{2}) \subseteq A$ and $(\frac{1}{\sqrt[4]{2}}, 1) \subseteq B$, hence $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is disk-cyclic on $\mathcal{H}$ from Theorem 3.5.

4 Codisk-cyclicity of the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$

In this section, we turn our attention to study the *codisk-cyclic* tuple $T = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ on $\mathcal{H}$. Since the proofs of the codisk-cyclicity are exactly the same as those in Section 2, thus we omit the details. The main results are based on the sets C and D and Proposition 2.5.

Theorem 4.1. *Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and $\varphi_1(z), \varphi_2(z)$ be two automorphisms in $\mathbb{D}$ satisfying (2.2). Suppose*

$$M = \sup_{z\in\mathbb{D}} \sup_{n\in\mathbb{Z}} \|k_{(\varphi_1)_n \circ (\varphi_2)_n(z)}\| < \infty.$$

If the sets C and D have limit points in $\mathbb{D}$, then the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is codisk-cyclic on $\mathcal{H}$.

In view of Theorem 4.1, the following description for the codisk-cyclic tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ holds.

Corollary 4.2. *Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$. Denote the sets*

$$\widetilde{C} = \left\{ z \in \mathbb{D} : \lim_{n\to\infty} (\omega_1(z)\omega_2(z))^n = 0 \right\},$$
$$\widetilde{D} = \left\{ z \in \mathbb{D} : \text{ the sequence } \left\{ \frac{1}{(\omega_1(z)\omega_2(z))^n} \right\}_n \text{ is bounded} \right\}.$$

If the sets $\widetilde{C}$ and $\widetilde{D}$ have limit points in $\mathbb{D}$, then the tuple $(M^*_{\omega_1}, M^*_{\omega_2})$ is codisk-cyclic on $\mathcal{H}$.

Theorem 4.3. *Suppose that $\mathcal{H}$ is automorphism invariant. Let $\omega_1(z), \omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two elliptic disk automorphisms with an interior fixed point $a \in \mathbb{D}$ satisfying (2.2). If the sets C and D have limit points in $\mathbb{D}$, then the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is codisk-cyclic on $\mathcal{H}$.*

Theorem 4.4. *Suppose that $\mathcal{H}$ is automorphism invariant. Let $\omega_1(z)$, $\omega_2(z)$ be two nonzero complex-valued functions for all $z \in \mathbb{D}$ and φ_1, φ_2 be two elliptic automorphism with an interior fixed point $a \in \mathbb{D}$ satisfying (2.2). Further assume that $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{C}$ satisfy the inequality $|\omega_1(a)\omega_2(a)| < 1$ and there is $0 < \delta < 1$ satisfying $|\omega_1(z)\omega_2(z)| \geq 1$ for all $|z| > 1 - \delta$, then the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is codisk-cyclic on $\mathcal{H}$.*

Proof. As the similar argument used in Theorem 3.5. Since $\mathcal{H}$ is automorphism invariant, we can only prove for the case $a = 0$. Then

$$\varphi_1(z) = e^{i\theta_1}z, \ \varphi_2(z) = e^{i\theta_2}z$$

for some $\theta_1, \theta_2 \in [0, 2\pi]$. Following the ideas in the proof of Theorem 3.5(**Case (i)**), (3.9) holds.

On the one hand, since $|\omega_1(0)\omega_2(0)| < 1$, there is a constant $0 < r < 1$ and a positive number $\widetilde{\delta} \in (0, 1)$ such that

$$|\omega_1(z)\omega_2(z)| < r < 1, \quad \text{whenever} \quad |z| < \widetilde{\delta}.$$

Since $|\varphi_i(z)| = |z|$ for $i = 1, 2$. Thus if $|z| < \widetilde{\delta}$, we have that

$$\left|\Pi_{j=0}^{n-1}\omega_1 \circ (\varphi_1)_j(z) \cdot \omega_2 \circ (\varphi_2)_j(z)\right| < r^n \to 0, \quad n \to \infty.$$

Thus the set $\{z \in \mathbb{D} : \ |z| < \widetilde{\delta}\}$ is a subset of C in Theorem 4.1.

On the other hand, since there is $0 < \delta < 1$ satisfying $|\omega_1(z)\omega_2(z)| \geq 1$ for all $|z| > 1 - \delta$, and $|\varphi_i^{-1}(z)| = |z|$ for $i = 1, 2$, hence if $|z| > 1 - \delta$, we conclude that

$$\left|\Pi_{j=1}^{n}\omega_1 \circ (\varphi_1)_{-j}(z) \cdot \omega_2 \circ (\varphi_2)_{-j}(z)\right|^{-1} \leq 1, \quad \text{for all } n \geq 1.$$

Therefore, the set $\{z \in \mathbb{D} : |z| > 1 - \delta\}$ is a subset of D in Theorem 4.1. Since both $\{z \in \mathbb{D} : \ |z| < \widetilde{\delta}\}$ and $\{z \in \mathbb{D} : |z| > 1 - \delta\}$ have limit points in $\mathbb{D}$, then both C and D have limit points in $\mathbb{D}$. Employing (3.9) and Theorem 4.1, the tuple $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is codisk-cyclic. This completes the proof. ■

Remark 4.5. *Example 3.6 indeed holds for Theorem 4.4. Since* $|w_1(0)w_2(0)| = 0 < 1$ *and there is* $0 < \delta = 1 - \frac{1}{\sqrt[4]{3}} < 1$ *satisfying* $|w_1(z)w_2(z)| = |z|^4|z^4 + 4| \geq |z|^4(4 - 1) = 3|z|^4 \geq 1$ *for all* $|z| > 1 - \delta$*. Hence the tuple* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is codisk-cyclic on* $\mathcal{H}$ *from Theorem 4.4.*

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