Arithmetical pseudo-valuations associated to Dubrovin valuation rings and prime divisors of bounded Krull orders

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Abstract

We look at value functions of primes in simple Artinian rings and associate arithmetical pseudo-valuations to Dubrovin valuation rings which, in the Noetherian case, are $\mathbb{Z}$-valued. This allows a divisor theory for bounded Krull orders.

1 Introduction

Valuations on fields have been studied extensively and they provided deep applications both in number theory and algebraic geometry, via number fields and function fields of varieties. In the classical theory there is a close relation between valuations on the quotient field of some ring and properties relating to the integrally closedness of the ring in its fraction field. Thus we see the appearance of Dedekind domains in number fields and noetherian integrally closed domains in function fields. These rings are special Krull orders and their arithmetical properties may be derived from an arithmetical ideal theory for divisorial ideals (cfr. [9]). In the non-commutative situation, for example in skewfields or simple Artinian rings, there is no good notion of integral closure of some central subring – let alone for an arbitrary subring. A definition of a non-commutative Krull ring, say in a simple Artinian quotient ring, was given by Chamarie in [2], as an

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order with a closedness property with respect to some localization closely related to the minimal (non-zero) prime ideals of the order. On the other hand, a suitable concept generalizing valuation theory of fields to the non-commutative case was introduced by J. Van Geel and the first author (cfr. [8]) by defining so-called primes of algebras. A fractional prime is a subring of a simple artinian ring with a prime ideal satisfying certain conditions (cfr. section 1 for the exact definitions), but one can also associate a value function to any fractional prime . A very interesting alternative is explored e.g. in [8]: instead of a function $v : Q \to \Gamma$ for some totally ordered group Γ, one considers an *arithmetical pseudo-valuation* (we will abbreviate this to apv) $v : \mathcal{F}(R) \to \Gamma$ where $\mathcal{F}(R)$ are the fractional R-ideals of Q (cfr. section 2 for the definition). However, the value functions associated to primes have only been studied in skewfields and arithmetical pseudo-valuations have been studied only for orders containing an order with a commutative semigroup of fractional ideals.

In this paper, we will first consider primes (R, P) where R is a Goldie ring with a simple Artinian quotient ring and such that R is invariant under inner A-automorphisms (this is a property of valuation rings in skewfields). We obtain an apv $v : \mathcal{F}(R) \to \Gamma$ where $\mathcal{F}(R)$ is the partially ordered semigroup of fractional ideals of R and Γ is a totally ordered semigroup. We characterize the rings for which Γ is a group and establish that if Γ is an Archimedean group, R is a *Dubrovin valuation*. Dubrovin valuations are another non-commutative generalization of valuations. They were introduced in [3] and have been studied quite extensively, cfr. e.g. [5] and [4] for more information about Dubrovin valuations.

Next we look at (not necessarily invariant) Noetherian Dubrovin valuation rings and establish an apv $v : \mathcal{F}(R) \to \Gamma$ where Γ is totally ordered such that $P = \{a \in A \mid v(RaR) > 0\}$ and $R = \{a \in A \mid v(RaR) \geq 0\}$. The case of non-Noetherian Dubrovin valuations allows $P = P^2$ and then v cannot exist (since $v(P) = 0$ would follow and then $v(I) = 0$ for all $I \in \mathcal{F}(R)$). The condition $\bigcap P^n = 0$ is enough to characterize Dubrovin valuation rings coming from an apv in the well-described way. We may call these the *valued Dubrovin rings*. Our results establish that Noetherian Dubrovin valuations define a $\mathbb{Z}$-valued apv with all the nice properties and so these are good generalizations of discrete valuation rings in skewfields.

Looking at Krull orders having valued Dubrovin valuations corresponding to their minimal non-zero prime ideals leads to the study of bounded Krull orders. Using the apvs of the valued Dubrovin valuations defining the bounded Krull order we establish in the final section the non-commutative versions of classical approximation results necessary to obtain a divisor theory for bounded Krull orders in simple Artinian rings, which is the aim of this paper. Further arithmetical theory is work in progress.

2 Stable fractional primes

A *prime* in a ring R is a subring R' with a prime ideal $P \subset R'$ such that for any x, y in R, $xR'y \subseteq P$ implies either $x \in P$ or $y \in P$. A prime (P, R') is called *localized* if for any $r \in R \setminus R'$, there exist $x, y \in R' \setminus \{0\}$ with $xry \in R' \setminus P$. A prime (P, R') in a ring

R is called *strict fractional* if for any $r \neq 0$ in R there exist $x, y \in R'$ with $0 \neq xry \in R'$, e.g. any localized prime is strict fractional. We refer the interested reader to [5] for much more information about primes.

Proposition 2.1. *Let (P, R') be a strict fractional prime in a simple Artinian ring A. If R is any semisimple Artinian subring of A and $R \neq A$, then R' is not contained in R.*

Proof. Assume $R' \subseteq R$ and pick $a \in A \setminus R$. Since R is Noetherian, we may choose L maximal for the property that $Lay \subseteq R$ for some $y \in R$. Since R is semisimple Artinian, we have $R = L \oplus U$ where U is a left ideal and $uay \notin R$ for every $u \in U$ (otherwise $(L + Ru)ay \subseteq R$ entails $u \in L$ which is a contradiction). There exist $x', y' \in R'$ with $0 \neq x'uayy' \in R' \subseteq R$. Since L is maximal for the property that $Layy' \subseteq R$, it follows that $x'u \in L$ but $x'u \in U$, so $x'u = 0$ contradicting $x'uayy' \neq 0$. ∎

A ring is said to be a *Goldie ring* is the set of regular elements satisfies the Ore condition and $S^{-1}R$ is a semisimple Artinian ring.

Proposition 2.2. *If (P, R') is a strict fractional prime in a simple Artinian ring and R' is a Goldie ring, then $S^{-1}R' = A$ where S is the set of regular elements in R'.*

Proof. Let $r \in S$, then r is regular in A because if $ru = 0$ then $u \in A \setminus R'$ so there exist $x, y \in R'$ with $0 \neq xuy \in R'$. Since S satisfies the Ore condition, there are $x' \in R'$ and $r' \in S$ with $r'x = x'r$ so $r'xuy = x'ruy = 0$ hence $xuy = 0$ which is a contradiction. Since A is simple Artinian, $r^{-1} \in A$ so S is invertible in A and $R' \hookrightarrow A$ extends to $S^{-1}R' \hookrightarrow A$. Since $S^{-1}R'$ is semisimple Artinian, the preceding proposition implies that $S^{-1}R' = A$. ∎

Remark 2.3. *If R is a Goldie prime ring and I is an essential left ideal of R then I is generated by the regular elements of I. (See [7].)*

Consider a strict fractional prime (P, R) of a simple Artinian ring A with $R = A^P$, i.e.

$$R = \{a \in A \mid aP \subseteq P \text{ and } Pa \subseteq P\}.$$

We always assume that R is Goldie hence a prime ring and an order of A (by proposition 2.2). If P is invariant under inner automorphisms of A, we say that (P, R) is an *invariant prime* of A. For the remainder of this section, we will assume that (P, R) is an invariant prime.

Remark 2.4. *R is invariant under inner automorphisms of A.*

Proof. Consider $u \in A^*$. For $p \in P$ we have $uRu^{-1}p = uRu^{-1}puu^{-1}$ and $u^{-1}pu \in P$ so $Ru^{-1}pu \subseteq P$ and $uRu^{-1}p \subseteq uPu^{-1} \subseteq P$. Hence $uRu^{-1}P \subseteq P$ which implies $uRu^{-1} \subseteq R$. A similar reasoning gives $PuRu^{-1} \subseteq R$. ∎

In general, by a *fractional R-ideal* of A we mean an R-bimodule $I \subseteq A$ such that I contains a regular element of R and for some $r, s \in R$, $rI \subseteq R$ and $Is \subseteq R$. Observe that we may choose r and s regular since R is an order. We will denote the set of fractional ideals of R by $\mathcal{F}(R)$.

Lemma 2.5. *We have:*

(1) *If u is regular and $uI \subseteq R$ then $Iu \subseteq R$. Also: $uI \subseteq P$ if and only if $Iu \subseteq P$.*

(2) *If $I, J \in \mathcal{F}(R)$, then $IJ \subseteq P$ if and only if $JI \subseteq P$.*

(3) *If $I, J \in \mathcal{F}(R)$ then $IJ \subseteq R$ implies $JI \subseteq R$ and vice versa. Moreover, if $J \nsubseteq P$ then $I \subseteq R$ and if $P \nsupseteq I \subseteq R$ then $J \subseteq R$.*

Proof. (1) If $uI \subseteq R$, then $Iu \subseteq u^{-1}Ru = R$. The other case is similar.

(2) If $IJ \subseteq P$ then, since (P, R) is a prime, either I or J is in P, say $I \subseteq P$. Since I is an ideal it is left essential so it is generated by regular elements. For every regular element $u \in I$ $uJ \subseteq P$ yields $Ju \subseteq u^{-1}Pu = P$, hence $JI \subseteq P$. The case $I \nsubseteq P$ and $J \subseteq P$ is similar.

(3) Suppose $I, J \in \mathcal{F}(R)$ such that $IJ \subseteq R$. If $I, J \subseteq P$ there is nothing to prove since then $IJ \subseteq P$ and $JI \subseteq P$, so assume $J \nsubseteq P$ ($I \nsubseteq P$ is completely similar). From $PIJ \subseteq P$ we obtain then $PI \subseteq P$ since (P, R) is a prime of A, so $I \subseteq A^P = R$. Again, I is generated by regular elements since it is left essential and for $u \in I$ regular $uJ \subseteq R$ gives $Ju \subseteq u^{-1}Ru = R$ hence $JI \subseteq R$. ∎

Corollary 2.6. *P is the unique maximal ideal of R.*

Proof. Consider an ideal $I \nsubseteq P$ and a regular element u of R which is in I but not in P (this exists since I is generated by regular elements). Then $Ru = RuR \neq R$ so $u^{-1} \notin R$. From $Ru^{-1}RuR = R$ with $RuR \nsubseteq P$ we obtain $Ru^{-1}R \subseteq R$ which is a contradiction. ∎

Note that we really showed that every regular element in $R \setminus P$ is invertible in R.

Corollary 2.7. *If $\mathcal{C}(P) = \{x \in R \mid x \mod P$ regular in $R/P\}$ satisfies the Ore condition then it is invertible in R, i.e. $Q_P(R) = R$ or R is local and P is the Jacobson radical of R.*

Proof. If $\mathcal{C}(P)$ is an Ore set in the prime Goldie ring R which is also an order in a simple Artinian ring A, then $\mathcal{C}(P)$ consists of regular elements and since $\mathcal{C}(P) \subseteq R \setminus P$ it consists of invertible elements of R. Consequently, the localization of R at $\mathcal{C}(R)$ is equal to R. It then follows that P is the Jacobson radical of R. ∎

Proposition 2.8. *If $\bigcap P^n = 0$ then $\mathcal{C}(P)$ satisfies the Ore condition.*

Proof. We claim that $1 + P$ consists of units. Indeed, consider $1 + p$ with $p \in P$ and assume it is not regular, then $r(1 + p) = 0$ for some $0 \neq r \in R$. Then $r = -rp$ yields $r \in \bigcap P^n$ hence $r = 0$ which is a contradiction. If $c \in \mathcal{C}(P)$ then $\overline{c}$ is regular in R/P. We have that $P + Rc$ is essential in R since it contains P hence it is generated by regular elements. Since $Ru = RuR$ for regular u, it follows that $P + Rc$ is a two-sided ideal of R, hence $P + Rc = R$ and $\overline{R}\overline{c} = \overline{R}$ i.e. $\overline{c}$ is invertible. Then there is an $\overline{u} \in \overline{R}$ with $\overline{uc} = \overline{1}$ which means $uc \in 1 + P$. If $rc = 0$ then $uru^{-1}uc = 0$ which would contradict the fact that all elements of $1 + P$ are units. Consequently, $\mathcal{C}(P)$ consists of R-regular elements. For every $r \in R$ and $c \in \mathcal{C}(P)$ we have $cr = crc^{-1}c = r'c$ which gives the left Ore condition and also $rc = cc^{-1}rc = cr'$ which gives the right Ore condition. Therefore $\mathcal{C}(P)$ is an Ore set. ∎

Corollary 2.9. *If $\bigcap P^n = 0$ then $\mathcal{C}(P)$ is invertible in R and R is local with Jacobson radical P.*

Corollary 2.10. *R/P is a skewfield.*

Proof. If $\overline{a} \in R/P$ is not invertible then it is not regular (cfr. the proof of proposition 2.8), say $\overline{sa} = 0$. Let $\overline{a} = a \mod P$ and $\overline{s} = s \mod P$, then $sa \in P$ implies $(Rs + P)a \subseteq P$. Furthermore, $Rs + P$ is two-sided and it contains P strictly so $Rs + P = R$. This means that $Ra \subseteq P$ so $\overline{a} = 0$. ∎

Proposition 2.11. *Under assumptions as before, the left R-ideals are totally ordered and every finitely generated left R-ideal is generated by one regular element.*

Proof. By remarks 2.3 and 2.4, left R-ideals are R-ideals. Suppose $xy \in P$ with either x or y regular in A. We suppose without loss of generality that x is regular, so it is invertible in A. We find $xRy = xRx^{-1}xy = Rxy \subseteq P$ so since (R,P) is prime, x or y must be in P. Consider now x regular (hence invertible) in $A \setminus R$. Since $x \notin R$, there must be a $p \in P$ with $xp \notin P$ (or $px \notin P$ in which case we argue similarly). Then we have $Rx^{-1}xp \subseteq P$ so $x^{-1} \in P$ since it is A-regular. Consider now a finitely generated left ideal I in R. By [7], it is generated by R-regular elements so it is generated by a finite number of R-regular elements say $I = Ru_1 + \cdots + Ru_n$. Since R is Goldie, every R-regular element is A-regular, so by the preceding statements either $u_1u_2^{-1}$ or $u_2u_1^{-1}$ must be in R. Suppose the latter (again, in the other case we argue similarly), then $Ru_2 = Ru_2u_1^{-1}u_1 \subseteq Ru_1$ which means that $Ru_1 + Ru_2 = Ru_1$. By induction we find that every finitely generated left ideal is principal and in fact even principal for a regular element. This in turn implies that the finitely generated left ideals are totally ordered by inclusion. Suppose now that I and J are left R-ideals with $J \nsubseteq I$. There must be a regular $x \in J \setminus I$ and for every $y \in I$ we have either $yx^{-1} \in R$ which would imply $y \in xR \subseteq J$ or $xy^{-1} \in R$ but this is contradictory since it means $x \in Ry \subseteq I$. ∎

3 Arithmetical pseudo-valuations for invariant primes

An *arithmetical pseudo-valuation* on R as before is a function $v : \mathcal{F}(R) \to \Gamma$ for some partially ordered semigroup Γ such that:

(APV1) $v(IJ) = v(I) + v(J)$;

(APV2) $v(I + J) \geq \min\{v(I), v(J)\}$;

(APV3) $v(R) = 0$;

(APV4) $I \subseteq J$ implies $v(I) \geq v(J)$.

For more information about arithmetical pseudo-valuations, we refer to [5] and [8]. In this section, we will assume that (R,P) is an invariant prime.

Theorem 3.1. *For any (R,P) there is an arithmetical pseudo-valuation $v : \mathcal{F}(R) \to \Gamma$, where Γ is a totally ordered semigroup, such that $P = \{a \in A \mid v(RaR) > 0\}$ and $R = \{a \in A \mid v(RaR) \geq 0\}$.*

Proof. Observe that for any $I, J \in \mathcal{F}(R)$ we have $IJ \in \mathcal{F}(R)$ and $I + J \in \mathcal{F}(R)$, moreover for every $a \in A$ we have $RaR \in \mathcal{F}(R)$. Indeed, if $a \in A$ then there is a regular $u \in R$ such that $ua \in R$ since R is an order, then $RuaR = RuRaR \subseteq RaR$ and as an R-ideal, $RuaR$ contains a regular element of R. If I and J are in $\mathcal{F}(R)$ then IJ contains a regular element and if $uI \subseteq R$ and $vJ \subseteq R$ for regular u and v then $Jv \subseteq R$ so $uIJv \subseteq R$ whence $vuIJ \subseteq vRv^{-1} = R$ with vu regular. For $I + J$ we have $vu(I + J) \subseteq R + vuJ$ with $vuJ = vuv^{-1}vJ \subseteq R$ since $vuv^{-1} \in R$.

For any $I \in \mathcal{F}(R)$ we define $v(I) = (P : I) = \{a \in A \mid aI \subseteq P\}$ and since $RaRI \subseteq P$ if and only if $IRaR \subseteq P$ this is also equal to $v(I) = \{a \in A \mid Ia \subseteq P\}$. Note that $v(I) \neq \{0\}$ because $uI \subseteq R$ for some regular $u \in R$, hence $0 \neq Pu \subseteq v(I)$. We also have $v(R) = P$. Put $\Gamma = \{v(I) \mid I \in \mathcal{F}(R)\}$ and define a partial order $\leq$ by

$$v(I) \leq v(J) \quad \Leftrightarrow \quad v(I) \subseteq v(J).$$

Note that if $I \subseteq J$ then $v(I) \geq v(J)$. We claim that Γ is in fact totally ordered. Indeed, if $I, J \in \mathcal{F}(R)$ such that $v(I) \not\subseteq v(J)$ and $v(J) \not\subseteq v(I)$ then there is an $a \in A$ with $aI \subseteq P$ but $aJ \not\subseteq P$ and a $b \in A$ with $bJ \subseteq P$ but $bI \not\subseteq P$. Since P is prime, $aJbI \not\subseteq P$ but $RbIaJ \subseteq RbPJ \subseteq RbJ \subseteq P$ yields $RaJbI \subseteq P$ which is a contradiction in view of lemma 2.5.

We can define a (not necessarily commutative) operation $+$ on Γ by putting $v(I) + v(J) = v(IJ)$. The unit for this operation is $v(R)$. We now verify that $+$ is well-defined. Suppose $v(I) = v(I')$ and $v(J) = v(J')$ and consider $x \in v(IJ)$, then $RxRIJ \subseteq P$ so $RxRI \subseteq v(J) = v(J')$ or $RxRIJ' \subseteq P$. By the same lemma as before, $IJ'RxR \subseteq P$ follows hence $J'RxR \subseteq v(I) = v(I')$ i.e. $I'J'RxR \subseteq P$ which implies $x \in v(I'J')$ and consequently $v(IJ) \subseteq v(I'J')$. The other inclusion can be obtained by the same argument if the roles of I, J and I', J' are interchanged.

We now check that this operation is compatible with $\leq$. Take some $v(I) \geq v(J)$ and consider $v(HI)$ and $v(HJ)$. If $q \in v(HJ)$ then $qHJ \subseteq P$ so $qH \subseteq v(J) \subseteq v(I)$ which implies $qHI \subseteq P$ so $q \in v(HI)$. To prove that $\leq$ is also stable under right multiplication, we consider $q \in v(JH)$. Then $qJH \subseteq P$ or equivalently $JHq \subseteq P$. By lemma 2.5 $HqJ \subseteq P$ follows so $Hq \subseteq v(J) \subseteq v(I)$ hence $IHq \subseteq P$ i.e. $q \in v(IH)$.

If $v(I) \leq v(J)$ then $aI \subseteq P$ yields $a(I + J) \subseteq P$ since $aJ \subseteq P$, so $v(I + J) \supseteq v(I) = \min\{v(I), v(J)\}$. Together with the preceding, this implies that v is an arithmetical pseudo-valuation. The only thing left to prove is that

$$R = \{a \in A \mid v(RaR) \geq 0\} \quad \text{and} \quad P = \{a \in A \mid v(RaR) > 0\}.$$

Suppose $v(RaR) > 0 = v(R) = P$, then there is some $x \in v(RaR) \setminus P$. Now $xRaR \subseteq P$ gives $a \in P$, so $\{a \in A \mid v(RaR) > 0\} \subseteq P$. If $p \in P$, then $v(RpR) \supseteq R \supsetneq P$, hence $p \in \{a \in A \mid v(RaR) > 0\}$ so $P = \{a \in A \mid v(RaR) > 0\}$. If $a \in A$ is such that $v(RaR) = 0$ and $p \in P$ then $v(RaRRpR) = v(RaR) + v(RpR) = v(RpR) > 0$ so $RaRRpR \subseteq P$ and therefore $ap \in P$ which implies $a \in R$ since $R = A^P$. On the other hand, if $r \in R$ then $PRrR \subseteq P$. Since RrR is generated by regular elements, it follows that $r \in \sum Ru_iR$ for a finite set of regular u_i. Consequently, since Γ is totally ordered, $v(RrR) = v(Ru_iR)$ where $v(Ru_iR)$ has the minimal value among these regular elements. If $v(RrR) < 0$ then $v(P) \le v(PRrR)$ since $PRrR \subseteq P$ and then

$$v(P) \le v(PRrR) = v(P) + v(RrR) \le v(P) \tag{1}$$

since $v(RrR) < 0$. This means that all $\le$ in 1 are actually equalities and in fact $v(P) = v(P) + v(Ru_iR) = v(PRu_iR)$ so if $aPRu_iR \subseteq P$ then also $aP \subseteq P$. By choosing $a = u_i^{-1}$ we find $u_i^{-1}P \subseteq P$. In a similar fashion we find $Pu_i^{-1} \subseteq P$ and consequently $u_i^{-1} \in A^P = R$ so $v(RrR) = v(Ru_iR) = v(R) = 0$. which contradicts $v(RrR) < 0$. Consequently $R = \{a \in A \mid v(RaR) \ge 0\}$. ∎

Proposition 3.2. *With R, P and A as before, Γ is a group if and only if for any fractional R-ideal I there is a nonzero $y \in R$ with $yI \subseteq R$ but $yI \nsubseteq P$.*

Proof. If Γ is a group and $I \in \mathcal{F}(R)$ then for some $J \in \mathcal{F}(R)$ we have $v(I) + v(J) = 0$ i.e. $v(IJ) = v(JI) = v(R)$. Consequently, $IJP \subseteq P \supseteq PIJ$ so $IJ \subseteq A^P = R$. Since $aIJ \subseteq P$ iff $aR \subseteq P$ we have $IJ \nsubseteq P$. Then we can choose a $y \in J$ with $Iy \subseteq R$ but $Iy \nsubseteq P$ which implies $RIRy \subseteq R$ and $RIRy \nsubseteq P$.

Suppose now that there is some y with $yI \subseteq R$ but $yI \nsubseteq P$. For any $x \in v(RyRI)$ we have $RxRRyRI \subseteq P$ which implies $RxR \subseteq P$ and consequently $x \in v(R)$. From $RyRI \subseteq R$ we can deduce $v(R) \subseteq v(RyRI)$ hence $v(R) = v(RyRI)$ which means that $v(RyR)$ is the inverse of $v(I)$. ∎

Note that the second part of the proof of the preceding theorem guarantees that every $v(I)$ is also $v(RaR)$ for some $a \in A$.

Lemma 3.3. *If $\bigcap P^n = 0$ and Γ is a group then R is a Dubrovin valuation. (Cfr. section 4 for a definition of Dubrovin valuations.)*

Proof. Applying corollary 2.9 gives us that R/P is prime Goldie with invertible regular elements, i.e. it is a simple Artinian ring. Consider $q \in A \setminus R$. There exists some $y \in A$ with $RyRRqR \subseteq R$ but $RyRRqR \nsubseteq P$. Then there exists a $z \in RyR$ with $zq \in R \setminus P$ and since $RqRRyR \subseteq R$ but $RqRRyR \nsubseteq P$ we can use a similar construction to find an element z' with $qz' \in R \setminus P$. ∎

Theorem 3.4. *If Γ is a group and $\bigcap P^n = 0$, then R is a valuation ring and A is a skewfield.*

Proof. If $a \in R \setminus P$ then $P + Ra$ is essential, two-sided and contains P so it is equal to R. Then, for some $r \in R$ and $p \in P$, we have $1 = p + ra$. We have already seen (cfr. proof of proposition 2.8) that $1 + P$ consist of units, so ra is a unit hence a is a unit. If $q \in A \setminus R$ there there is some y with $yq \in R \setminus P$, so yq is a unit of R hence q is a unit of A. Finally, if some $p \in P$ were not invertible, then $Ap \subseteq P$ since no element in Ap is a unit. Then we would have $A(RpR) \subseteq P$, but this would contain some regular u which is invertible in A and $Au \subseteq P$ would give a contradiction. This implies that A is a skewfield and R is an invariant Dubrovin valuation on A, so it must be a valuation ring. ∎

Corollary 3.5. *If Γ is an Archimedean group, then R is a valuation ring.*

Proof. In view of the preceding proposition we only have to show that $\bigcap P^n = 0$. Suppose it is not, then $I = \bigcap P^n$ is a nonzero ideal. Pick $0 \neq b \in I$, then $RbR \subseteq I$ is a fractional ideal, hence there exists an ideal $J \in \mathcal{F}(R)$ with $v(J) + v(RbR) = 0$. Then $v(P^n) + v(RbR) \leq 0$ for any n, so $nv(P) + v(RbR) \leq 0$. However, putting $v(RbR) = \gamma$, there must be some n with $nv(p) > -\gamma$ which is a contradiction. ■

Proposition 3.6. *Let R be any order in a simple Artinian ring A and suppose that $v : \mathcal{F}(R) \to \Gamma$ is an apv which takes values in a totally ordered semigroup Γ. Then:*

(1) $P = \{a \in A \mid v(RaR) > 0\}$ defines a prime (P, A^P) with $\{a \in A \mid v(RaR) \geq 0\} \subseteq A^P$.

(2) if $v(I) = \{a \in A \mid aI \subseteq P\}$ and Γ is a group, then $\{a \in A \mid v(RaR) \geq 0\} = A^P$.

Proof. (1) For $a, b \in P$ we have $v(R(a+b)R) \geq \min\{v(RaR), v(RbR)\}$ which is strictly positive, so $a + b \in P$. Clearly, P is an ideal in A^P and $R \subseteq A^P$ since $v(R) = 0$ and we have $v(RrRRpR) = v(RpR) > 0$ for all $r \in R$ and $p \in P$. If $a, a' \in A$ are such that $aA^Pa' \subseteq P$ then $aRa' \subseteq P$ hence $v(RaRa'R) > 0$. From $v(RaR) + v(Ra'R) > 0$ it follows that either $v(RaR) > 0$ or $v(Ra'R) > 0$, i.e. either $a \in P$ or $a' \in P$. If $v(RaR) \geq 0$ for some $a \in A$ then for all $p \in P$ we have $v(RaRpR) = v(RaR) + v(Rp) > 0$ and $v(RpRaR) = v(RpR) + v(RaR) > 0$ so $a \in A^P$.
(2) Consider $a \in A^P$. RaR is invertible in $\mathcal{F}(R)$ so there is some $J \in \mathcal{F}(R)$ for which we have $v(RaR) + v(J)$
$= 0 = v(J) + v(RaR)$, hence $v(JaR) = v(RaJ) = 0$. If $v(RaR) < 0$ then $v(J) > 0$ or in other words $J \subseteq P$. But then $a \in A^P$ would give $RaRJ \subseteq P$ which implies $v(RaRJ) > 0$ which is a contradiction. Therefore $v(RaR) \geq 0$ and $A^P = \{a \in A \mid v(RaR) \geq 0\}$. ■

4 Arithmetical pseudo-valuations on Dubrovin valuations

A Dubrovin valuation on a simple artinian ring Q is a couple (R, M) where R is a subring of Q and M is a prime ideal in R such that:

(1) R/M is simple artinian.

(2) For every $q \in Q \setminus R$ there exist r and r' in R such that $qr \in R \setminus M$ and $r'q \in R \setminus M$.

Dubrovin valuations are non-commutative generalizations of valuations and occur naturally as extensions of valuations on the centre of simple artinian rings. For example, if R is simple artinian ring which is finite dimensional over its centre, then every valuation on the centre extends to a Dubrovin valuation on R and all such extensions are conjugate to one another. Dubrovin valuations were introduced in [3] and have been studied quite extensively in the last decades by a.o. Wadsworth, Gräter and Marubayashi. Some good references are [4] and [5]. The following theorem (a proof of which can be found in both forementioned books) gives some alternative characterizations of Dubrovin valuations:

Theorem 4.1. *For a subring R of a simple artinian ring Q, the following are equivalent:*

(1) R is a Dubrovin valuation on Q;

(2) R is a local Bezout order in Q, i.e. every finitely generated left (right) ideal is principal;

(3) R is a local semi-hereditary order in Q, i.e. every finitely generated left (right) ideal is projective.

For primes containing an order with commutative semigroup of fractional ideals, Van Geel ([8]) introduced arithmetical pseudo-valuations, but this condition is very strong and reduces the applicability in practice to maximal orders and Dubrovin valuations in finite dimensional central simple algebras. For Dubrovin valuations on infinite dimensional central simple algebras the semigroup $\mathcal{F}(R)$ need not be commutative. The following facts are known about Dubrovin valuations on simple Artinian rings (see once more [5]):

(D1) R is a (left and right) Goldie ring and a prime order of A;

(D2) (P, R) is a localized prime of R;

(D3) $P = J(R)$ and it is the unique maximal ideal of R. Consequently, $1 + P$ consists of units;

(D4) $\mathcal{F}(R)$ is linearly ordered;

(D5) finitely generated R-submodules of A are cyclic.

Notice that we do not assume invariance in this section.

Proposition 4.2. *For a Noetherian Dubrovin valuation R we have for all $I, J \in \mathcal{F}(R)$ that $IJ \subseteq P$ iff $JI \subseteq P$.*

Proof. From $IJ \subseteq P$ it follows that either $I \subseteq P$ or $J \subseteq P$ (by (D2)), assume without loss of generality $I \subseteq P$. By (D4), if $JI \nsubseteq P$ then $P \subsetneq JI$ so we have $R \subseteq JI \subseteq JP \subseteq J$ hence $J = RJ \subseteq JIJ \subseteq JP \subseteq J$ which gives $J = JP$. Since R is an order, there is some regular $u \in R$ with $uJ \subseteq R$ and since R is Noetherian $uJ = \sum a_i R$ for a finite set of a_i's in uJ. Then also $J = \sum u^{-1} a_i R$, so J is a finitely generated R-submodule of A. By Nakayama's lemma J must be zero, which is a contradiction. ∎

Corollary 4.3. *If R is a Noetherian Dubrovin valuation then there is an arithmetical pseudo-valuation*

$$v : \mathcal{F}(R) \to \Gamma : I \mapsto (P : I) = \{a \in A \mid aI \subseteq P\}$$

for some totally ordered group Γ. Furthermore, $P = \{a \in A \mid v(RaR) > 0\}$ and $R = \{a \in A \mid v(RaR) \geq 0\}$.

Proof. Using the preceding proposition instead of 2.5 we can repeat the proof of theorem 3.1. The only thing we need to prove is that the Γ which said theorem provides is a group, so consider $I \in \mathcal{F}(R)$. By a similar argument as in the proof of proposition 4.2 it is finitely generated as a left R-ideal of A. By (D5) it is cyclic, in fact $I = Ru$ for some regular u and thus $RuR = Ru$. Since R is a Dubrovin valuation, there is some $a \in A$ with $ua \in R \setminus P$. Then $Rua \subseteq R$, $Rua \nsubseteq P$ and $Ia \subseteq R$, $Ia \nsubseteq P$. We can similarly find a b such that $bI \subseteq R$ but $bI \nsubseteq P$. We can now repeat the last part of the proof of proposition 3.2 to conclude that Γ is a group. ∎

Remark 4.4. *If R is a Dubrovin valuation where*

$$v : \mathcal{F}(R) \to \Gamma : I \mapsto \{a \in A \mid aI \subseteq P\}$$

is a non-trivial arithmetical pseudo-valuation with values in a totally ordered group, then $IJ \subseteq P$ if and only if $JI \subseteq P$. Indeed, suppose $IJ \subseteq P$ and $JI \nsubseteq P$ then, as in proposition 4.2 we find $JP = J$ but then $v(P) = 0$ which is impossible.

If R is non-Noetherian, then $P = P^2$ is possible in which case no nice apv can exist since otherwise $v(P) = 2v(P)$ which would imply $v(P) = 0$. If we exclude this slightly pathological case, a nice apv does exist.

Proposition 4.5. *Let R be a Dubrovin valuation with $\bigcap P^n = 0$, then there is an apv as before.*

Proof. If $I, J \in \mathcal{F}(R)$ with $IJ \subseteq P$ but $JI \nsubseteq P$. The same argument as in proposition 4.2 leads to $J = JP$ so $J = JP^n$ for any n. There is some regular $u \in R$ with $uJ \subseteq P$ hence $uJ = uJP^n \subseteq P^{n+1}$. But then $uJ = 0$ which implies $J = 0$ and this is a contradiction. Now we can proceed as in corollary 4.3 to find an apv with values in a semigroup.

The only thing we need to prove is that Γ is a group. Lemma 1.5.4 in [5] says that $P = Rp = pR$ for some regular $p \in P$. Since P is principal as a left R-ideal, lemma 1.5.6 in the same source gives $PP^{-1} = R = P^{-1}P$ (here $P^{-1} = \{a \in A \mid PaP \subseteq P\}$). Consider now a fractional R-ideal I. Clearly, $(R : I)I \subseteq R$. Suppose we also have $(R : I)I \subseteq P$, then $P^{-1}(R : I)I \subseteq R$ hence $P^{-1}(R : I) \subseteq (R : I)$ so $P^{-1}(R : I)I \subseteq P$. This means $(R : I)I \subseteq P^2$ and by repeating this process we find $(R : I)I \subseteq P^n$ for any n, but $(R : I)I \subseteq \bigcap P^n = 0$ which is a contradiction. Therefore, $(R : I)I \subseteq R$ but $(R : I)I \nsubseteq P$, so there exists an $a \in (R : I)$ such that $aI \subseteq R$ but $aI \nsubseteq P$. ∎

The following characterizes Noetherian Dubrovin valuation rings within the class of rank one Dubrovin valuations. The result may be known but we found no reference for it in the literature. Recall that the rank of a Dubrovin valuation ring is the maximal length of a chain of Goldie prime ideals in the ring. A rank 1 Dubrovin valuation ring on a simple Artinian A is a maximal subring of A.

Proposition 4.6. *For a Dubrovin valuation R on a simple Artinian ring A the following are equivalent:*

(1) R is Noetherian.

(2) R has rank 1 and $P \neq P^2$.

(3) R has rank 1 and $\bigcap P^n = 0$.

Proof. (1) ⇒ (2) If R is Noetherian then all ideals (and R-ideals of A) are principal, so $P \neq P^2$. Suppose $0 \neq Q$ is another prime ideal in P. Let $P = Rp$, then $Q = Ip$ for some non-trivial ideal I of R. $Q = IP$ yields $I \subseteq Q$ since Q is prime and $P \nsubseteq Q$. Hence $Q = IP \subseteq QP \subseteq Q$ implies $Q = QP$ which implies $Q = 0$ by Nakayama's lemma. (2) ⇒ (3) If $P \neq P^2$, then $\bigcap P^n \neq P$ which, by Lemma 1.5.15 in [5], gives $\bigcap P^n = 0$. (3) ⇒ (1) Since $\bigcap P^n = 0$, $P \neq P^2$. Since R is rank 1, $R = O_l(I) = O_r(I)$ for any R-ideal I. By proposition 1.5.8 in [5], it follows that if I is not principal, then $II^{-1} = P$ and $P = P^2$ which s a contradiction. ■

Example 4.7. The following is perhaps the easiest non-trivial example of a Dubrovin valuation: let $\mathbb{H}$ denote the Hamilton quaternions over $\mathbb{Q}$, let V_p be the p-valuation ring for some prime number p and put $R = V_p + V_p i + V_p j + V_p k$ where $\{1, i, j, k\}$ is the usual basis of $\mathbb{H}$. It can easily be verified that R is indeed a Dubrovin valuation which, if $p \neq 2$, is not a valuation ring. The maximal ideal of R is $M = M_p + M_p i + M_p j + M_p k$ where M_p is the maximal ideal of V_p and M^n is just $M_p^n + M_p^n i + M_p^n j + M_p^n k$. This immediately yields $\bigcap M^n = 0$, so there exists an apv for this Dubrovin valuation; it is simply the map $v : \mathcal{F} \to \mathbb{Z}$ which sends M^n to n. This definition makes sense, since we will show later (cfr. 5.1) that all fractional ideals are of the form M^n.

Example 4.8. Let Q be a simple Artinian ring, let $\sigma \in \mathrm{Aut}(Q)$, and put $Q[X, \sigma]$ the skew polynomial ring over Q. $Q[X, \sigma]$ has a maximal ideal $P = XQ[X, \sigma]$. Put T the localisation of $Q[X, \sigma]$ at P. For any $t = (\sum a_i x^i)(\sum b_i x^i)^{-1}$ in T, we can define $f(t) = a_0 b_0^{-1}$. This gives a map $\phi : T \to Q : t \mapsto f(t)$. It has been shown in [10] that an order R of Q is a Dubrovin valuation ring if and only if $\tilde{R} = \phi^{-1}(R)$ is a Dubrovin valuation ring of T and that $J(\tilde{R}) = J(R) + J(T)$. It is clear that if R is a Dubrovin valuation on Q with $\bigcap J(R)^n = 0$ we also have $\bigcap J(\tilde{R}) = 0$. Therefore an apv exists, but R is not finite dimensional over its centre.

5 Divisors of Bounded Krull orders

Suppose that R is a on order in some simple Artinian ring Q. Put

$$\mathcal{F}_R = \left\{F \mid F \text{ is a left ideal and } (R : Fr^{-1})_r = R \text{ for all } r \in R\right\}.$$

This is a left Gabriel topology on R and we can define the left closure of some left ideal I as $cl_\tau(I) = \{r \in R \mid \forall F \in \mathcal{F}_R : Fr \subseteq I\}$. Similarly, we can define a right Gabriel topology

$$\mathcal{F}'_R = \left\{F \mid F \text{ is a left ideal and } (R : r^{-1}F)_l = R \text{ for all } r \in R\right\}$$

and, for any right ideal J, the right closure $cl_\tau(J)$. R is called τ-Noetherian if it satisfies the ascending chain condition on τ-closed one-sided ideals. If R is a τ-closed maximal order, then we say it is a *Krull*-order.

We consider a prime Noetherian ring R. It is an order in a simple Artinian ring $A = Q(R)$, the classical ring of fractions. If R is a maximal order then the set of divisorial R-ideals of A (cfr. [5]) is a group and since R is Noetherian it is an Abelian group generated by the maximal divisorial ideals and every maximal divisorial ideal is a minimal prime ideal.

Recall that an order is an *Asano order* if every ideal $I \neq 0$ of R is invertible and it is a *Dedekind order* if it is an hereditary Asano order. If R is an Asano order satisfying the ascending chain condition on ideals, then $\mathcal{F}(R)$ is the Abelian group generated by maximal ideals and every maximal ideal is a minimal nonzero prime ideal. Any bounded Noetherian Asano order is a Dedekind order.

A semi-local order R in a simple Artinian A is a Noetherian Asano order if and only if it is a principal ideal ring. If R is a Dubrovin valuation ring of A then R is a maximal order if and only if $\mathrm{rk}(R) = 1$ and R is Asano if and only if it is a principal ideal ring, so a Noetherian Dubrovin valuation ring is a Noetherian maximal order and an Asano order, i.e. a principal ideal ring.

Proposition 5.1. *If R is a Noetherian Dubrovin valuation then the corresponding apv takes values in $\mathbb{Z}$.*

Proof. Since R is a Noetherian Asano order, $\mathcal{F}(R)$ is generated by the maximal ideals of R, but since P is the unique maximal ideal and the value group is necessarily torsion-free, we have $\mathcal{F}(R) = \mathbb{Z}$. ∎

Recall that an order R in a simple Artinian ring A is said to be a Krull order if it is a maximal order and it is τ-Noetherian (see [5], definition 2.2.2 or [2]). A Noetherian order R in a simple Artinian A is a Krull order if and only if it is a maximal order. It is also known (cfr. [2]) that if R is a Krull order, then so is $R[X, \sigma]$.

Theorem 5.2. *Let R be a prime Noetherian ring and an order in $A = Q(R)$. Suppose every minimal nonzero prime ideal is localizable, R_p is a Dubrovin valuation ring for every*

$$p \in X^1(R) = \{\textit{minimal nonzero prime ideals of } R\}$$

and $R = \bigcap R_p$, then R is a bounded Krull order.

Proof. Since R is Noetherian, every R_p is Noetherian too and since it is a Dubrovin valuation it must be an Asano order hence a principal ideal ring. Since every R_p is a maximal order, so is R. As a Noetherian maximal order, R is a Krull order and by theorem 2.2.16 in [5] every regular element is a non-unit in only finitely many of the R_p's (for p maximal divisorial, i.e. $p \in X^1(R)$). Theorem 2.2.20 in the same source gives the result. ∎

Remark 5.3. *If R is a bounded Krull order then $R = \bigcap_{p \in X^1(R)} R_p$ (cfr. [5] 2.2.18 & 2.2.19). Then every R_p is rank one Dubrovin valuation with $P \neq P^2$ (2.2.16 in the aforementioned source), so by Proposition 4.6 it is Noetherian. This gives a correspondence between $p \in X^1(R)$ and apvs.*

A *divisor* of a bounded Krull order R is an element in the free Abelian group $\bigoplus_{p \in X^1(R)} \mathbb{Z}p$. To any $I \in \mathcal{F}(R)$ we can associate the divisor $\mathrm{div}(I) = \sum v_p(I_p)p$ where $I_p = R_p I$. This definition is justified by the following:

Proposition 5.4. *Suppose R_p is Noetherian. If I is an R-ideal of A, then I_P is an R_P-ideal of A.*

Proof. Let u be regular in R with $uI \subseteq R$, then $RuRI \subseteq R$ and $RuR = Ru'$ for some regular $u' \in R$. Then $R_p u' I$ is the localization of $Ru'I$ and it is an ideal of R_p. $R_p u'$ is the localization of Ru' so it is also an ideal of R_p, hence $R_p u' I = R_p u' R_p I = R_p u' I_p$ is an ideal of R_p. Now $U' I_p \subseteq R_p$, i.e. I_p is an R_p-ideal of A. ■

Observe that since any regular element is a non-unit in only finitely many localizations, $\mathrm{div}(I)$ contains only finitely many non-zero terms. Moreover, $\mathrm{div}(I) \leq \mathrm{div}(J)$ if and only if $v_p(I) \leq v_p(J)$ for all $p \in X^1(R)$. By putting $I^* = \bigcap_{p \in X^1(R)} I_p$ we find $v_p(I) = v_p(I^*)$. An ideal I is called *divisorial* if $I = I^*$. The set of divisorial R-ideal is denoted by $\mathcal{D}(R)$. For bounded Krull orders we can consider $\mathrm{div} : \mathcal{D}(R) \to \mathrm{Div}(R)$ and this is a group morphism of Abelian groups. It is order-reversing in the sense that $I \subseteq J$ yields $|\mathrm{div}(J)|\mathrm{div}(I)$. Further divisor theory requires a version of the approximation theorems.

6 Approximation theorems

Let Q be a simple Artinian ring and let R be an order in Q which is a bounded Krull order. v is an arithmetical pseudo-valuation associated to the localisation of R at a rank 1 prime ideal. By [5] Theorem 2.2.16, these localisations are rank one Dubrovin valuation rings with $J(R_p)^2 \neq J(R_p)$, so by Proposition 4.6 they are Noetherian. In view of 5.4, I_p is an R_p-ideal so it makes sense to define $v(x) = v(RxR_p)$ for an element x of R.

Lemma 6.1. *Let I be a fractional ideal. The set $\{v(x) \mid x \in I\}$ has a minimum which is equal to $v(I)$.*

Proof. It is clear that $v(x) \geq v(I)$ for any $x \in I$, so suppose there is some y with $v(I) \leq v(y) \leq v(x)$ for all $x \in I$. Any $z \in v(y)$ must be in $v(RxR)$, so $yRxR \subseteq P$ for all $x \in I$ which implies $yI \subseteq P$, hence $y \in v(I)$ which means $v(I) = v(y)$. ■

Remark 6.2. *For any fractional ideal I, $v(I) = 0$ for almost all v.*

Proof. For any $a \in Q$ there is some regular $c \in R$ with $ca \in R$. It is known that any regular element of a bounded Krull order is invertible in all but finitely many localisations R_P, so $v(c) = 0$ for almost all v and consequently $v(a) \geq 0$ for almost all v. By the preceding lemma, there is for any v some x_v with $v(I) = \min\{v(x) \mid x \in I\} = v(x_v) = v((Rx_vR)^*)$. Since R is a Krull order, the ascending chain condition holds on divisorial ideals, so $\{(Rx_vR)^* \mid v\}$ has but finitely many elements, $(Rx_{v_1}R)^*, ..., (Rx_{v_n}R)^*$ say. But every divisorial ideal contains a regular element, which is almost always invertible, hence $v(Rx_{v_i}R) = 0$ for all but finitely many v and all $i \in \{1, ..., n\}$ whence $v(I) = 0$ for almost al v. ■

We will denote by (A) the following approximation property:

Let $v_1, ..., v_t$ be a finite number of arithmetical pseudo-valuations associated to rank 1 prime ideals in R, let $n_1, ..., n_t$ be integers, and let $a_1, ..., a_t$ be elements of Q. Then there exists some $x \in Q$ such that $v_i(x - a_i) \geq n_i$ for $i = 1, ..., t$ and $v(x) \geq 0$ for any $v \notin \{v_1, ..., v_t\}$.

Lemma 6.3. *If (A) holds, we can pick x in such a way that $v_i(x - a_i) = n_i$.*

Proof. By (A), we can pick z such that $v_i(z - a_i) > n_i$. Since v is surjective and because of 6.1 we can also find z_i with $v_i(z_i) = n_i$. Then, once more by (A), we can find z' with $v_i(z' - z_i) > n_i$ from which we can deduce

$$v_i(z') = v_i((z' - z_i) + z_i) = n_i$$

and

$$v_i(z + z' - a_i) = n_i.$$

Therefore, $z + z'$ is the x we were looking for. ∎

Lemma 6.4. *Suppose again that (A) holds. For any $v_1, ..., v_t$ and $n_2, ..., n_t \in \mathbb{N}$ there is a regular $c \in R$ with $v_1(c) = 0$, $v_i(c) \geq n_i$ for $i = 2, ..., t$ and $v(c) = 0$ for any $v \notin \{v_1, ..., v_t\}$.*

Proof. We can certainly find some x with $v_1(x) = 0$, $v_i(x) = n_i$ for $i = 2, .., t$ and $v(x) \geq 0$ for any $v \notin \{v_1, ..., v_t\}$. Since $v(x) \geq 0$ for any v, x is in R. Every v_i comes from a Dubrovin valuation that one gets by localizing R at a minimal non-zero prime ideal which we will call P_i. RxR is an ideal with $v_1(RxR) = v_1(x) = 0$, so $RxR \nsubseteq P_1$. Since these ideals are generated by their regular elements, we can find some regular c in $RxR \setminus P_1$. Clearly, $v(c) \geq v(x)$ for any v, but $v(c) = 0$ – otherwise $c \in P_1$ would hold – which means that it satisfies all the conditions from the statement. ∎

Lemma 6.5. *Assume the same setting as before. For every $z \in Q$ and every v there is some regular right invariant r_v with $v(r_v) = v(z)$ and $v'(r_v) \geq v'(z)$ for every other v'.*

Proof. Consider the fractional ideal $R_v z R_v$. This is equal to $R_v r'_v R_v$ for some regular r'_v. Clearly $v(z) = v(r'_v)$. Since $r'_v \in R_v z R_v$ there are some $a_i, b_i \in R_v$ with $r'_v = \sum_{i=0}^{n} a_i z b_i$. Since $a_i, b_i \in R_v$, we have $v(a_i), v(b_i) \geq 0$. There are only finitely many v', say $v_2, ..., v_t$, for which there is some i with $v'(a) < 0$ or $v'(b) < 0$. Take c regular with $v(c) = 0$ and $v_j(c) \geq 2\max_{j,i}\left\{-v_j(a_i), -v_i(a_j)\right\}$ for $j = 2, ..., t$. By our choice of c, we have $v'(ca_i) \geq 0$ and $v'(cb_i) \geq 0$ for all v', which implies $ca_i \in R \ni cb_i$. The r_v we're looking for is cr'_v. Indeed: $v(r_v) = v(cr'_v) = v(r'_v) = v(z)$ and for any v' we have $v'(r_v)$. ∎

Lemma 6.6. *In the same context as before, we can find some regular right invariant element r with $v_1(r) = n_1$ and $v_i(r) \geq n_i$.*

Proof. We know we can find some z with $v_1(z) = n_1$ and $v_i(z) \geq 0$ for the other i. By the previous lemma we find the desired r. ∎

We now will consider systems of equations

$$\begin{aligned} y_1 &= x_1 a_{11} + \cdots + x_n a_{1n} + b_1 \\ \vdots \quad &\quad \vdots \\ y_m &= x_1 a_{m1} + \cdots + x_n a_{mn} + b_m \end{aligned}$$

for certain a_{ij} and b_i in Q. A *local solution with respect to R_v* for such a system of equations is a set of elements $x_i \in R_v$ such that all y_j are in R_v as well. A *global solution* is a set of $x_i \in R$ such that all y_j are in R too.

Lemma 6.7. *A system of equations as described above has a global solution if and only if it has local solutions with respect to any R_v.*

Proof. It is quite clear that any global solution immediately entails a local solution with respect to any R_v: if v is such that $v(a_{ij}) \geq 0$ and $v(b_i) \geq 0$, then any n-tuple $x_1, ..., x_n$ with $v(x_i) \geq 0$ for all i is a local solution.

Suppose now that x_i is a local solution with respect to v and suppose that there are x'_i such that $v(x_i - x'_i) + v(a_{ki}) \geq 0$. We have now

$$\begin{aligned} v(\sum (x_i - x'_i)a_{ki}) &\geq \min_i v((x_i - x'_i)a_{ki}) \\ &\geq \min_i (v(x_i - x'_i) + v(a_{ki})) \\ &\geq 0 \end{aligned}$$

which, since $\sum x_i a_{ki} \in R_v$, implies $\sum x'_i a_{ki} \in R_v$. Hence the n-tuple $x'_1, ..., x'_n$ is also a local solution with respect to v.

Suppose now that there is a local solution for every v. It is clear that we can choose the same solution for all but finitely many v's, so we can just consider a finite set of local solutions $x_{i1}, ..., x_{is}$ with respect to the respective pseudo-valuations $v_1, ..., v_s$. Define $n_{ti} = \max_k(a_{ki})$ for any $1 \leq t \leq s$. By the approximation property (A), we can find $x_1, ..., x_n$ with $v_t(x_i - x_{it}) \geq -n_{ti}$ and $v(x_i) \geq 0$ for any $v \notin \{v_1, ..., v_s\}$. These x_i are a local solution for every v, hence a global solution. ∎

Lemma 6.8. *Let $x_1 a_1 + \cdots + x_n a_n = b$ be an equation such that if there is some i with $v(a_i) < 0$ for a certain v, then there is some regular right invariant a_k with $v(a_k) = \min_i v(a_i)$. Then a global solution exists (for this equation) if and only if $v(b) \geq v(a_i)$ for all v.*

Proof. If there is a global solution for the equation at hand, then certainly $v(x_i) \geq 0$ for all v and all i. Consequently,

$$\begin{aligned} v(b) &\geq \min_i v(x_i a_i) \\ &\geq \min_i v(a_i) \end{aligned}$$

which implies that the condition is necessary.

We will now show that it is also sufficient. Suppose $v(b) \geq \min_i v(a_i)$. If the right-hand side is greater than or equal to zero, then $v(b) \geq 0$ so a local solution with respect to v exists by the same argument as in the beginning of the previous lemma. If the right-hand side is smaller than zero, we find $\min_i(v(a_i)) = v(a_k)$ where we can choose a_k to be regular and right invariant. Since a_k is regular and Q is simple Artinian, a_k is invertible in Q. Consider now the equation $x_k = -x_1 a_1 a_k^{-1} - \cdots - x_{k-1} a_{k-1} a_k^{-1} - x_{k+1} a_{k+1} a_k^{-1} - \cdots - x_s a_s a_k^{-1} - b a_k^{-1}$. Since a_k is right-invariant, $R a_k^{-1} a_k R = R a_k^{-1} R a_k R$ so $-v(a_k) = v(a_k^{-1})$. This in turn implies $v(b a_k^{-1}) \geq 0$, but then $x_k = b a_k^{-1}$ and $x_i = 0$ for $i \neq k$ gives a local solution with respect to v. By the previous lemma, a global solution must exist. ∎

Proposition 6.9. *There is a* $1-1$ *correspondence between divisorial ideals of R and divisors.*

Proof. Consider a divisorial ideal I with $v(I) = \gamma_v$. We already know that $\gamma_v = 0$ for all but finitely many v, so let $v_1, ..., v_s$ be the set of arithmetical pseudo-valuations for which $\gamma_{v_i} \neq 0$. Since $v(I) = \min\{v(r) \mid r \in I\}$, there are $z_i \in I$ with $v_i(z_i) = \gamma_i$ and $v(z_i) \geq 0$ for all other v. By a previous lemma, one can also find regular right-R_{P_i}-invariant a_i with $v_i(a_i) = v_i(z_i) = \gamma_i$, $v_j(a_i) \geq \gamma_j$, $v(a_i) \geq 0$ for all other v, and with $a_i \in Rz_iR$. Consider now the equation $x_1a_1 + \cdots + x_sa_s = b$. By our choice of a_i, there is a global solution if and only if $v_i(b) \geq \gamma_i$. Hence the set of global solutions $\{b \mid \forall v : v(b) \geq \gamma_v\}$ is a subset of $I = I^*$. The other inclusion obviously holds too, so divisorial ideals are uniquely determined by their associated divisor.

Suppose we have a divisor δ. Then there is some z with $v(z) \geq \mathrm{ord}_v(\delta)$ if $\mathrm{ord}_v(\delta) \neq 0$ and $v'(z) \geq 0$ otherwise. Moreover, this z can be chosen to be regular and R_{P_u}-right-invariant for some fixed u. Define $V = \{v \mid v(z) \neq 0 \text{ or } \mathrm{ord}_v(\delta) \neq 0\}$. This is a finite set, so there is some x with $v(x) = \mathrm{ord}_v(\delta)$ for all $v \in V$ and $v'(x) \geq 0$ for all other v'. Put I the ideal generated by z and x, then $v(I^*) = \mathrm{ord}_v(\delta)$, but since $v(I^*) = \min\{v(y) \mid y \in I^*\}$ we have $v(I^*) = \mathrm{ord}_v(\delta)$ for all v, which shows that every divisor is associated to some divisorial ideal I^*. ∎

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