Complex and p-adic branched functions and growth of entire functions

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Abstract

Following a previous paper by Jacqueline Ojeda and the first author, here we examine the number of possible branched values and branched functions for certain p-adic and complex meromorphic functions where numerator and denominator have different kind of growth, either when the denominator is small comparatively to the numerator, or vice-versa, or (for p-adic functions) when the order or the type of growth of the numerator is different from this of the denominator: this implies that one is a small function comparatively to the other. Finally, if a complex meromorphic function $\frac{f}{g}$ admits four perfectly branched small functions, then $T(r, f)$ and $T(r, g)$ are close. If a p-adic meromorphic function $\frac{f}{g}$ admits four branched values, then f and g have close growth. We also show that, given a p-adic meromorphic function f, there exists at most one small function w such that $f - w$ admits finitely many zeros and an entire function admits no such a small function.

1 Introduction

We denote by $\mathbb{E}$ an algebraically closed field of characteristic 0, complete with respect to an absolute value and by $\mathbb{K}$ an algebraically closed field of characteristic 0, complete for an ultrametric absolute value, with residue characteristic $p \geq 0$.

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We denote by $\mathcal{A}(\mathbb{E})$ the $\mathbb{E}$-algebra of analytic functions in $\mathbb{E}$ (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{E})$ the field of meromorphic functions in $\mathbb{E}$ (i.e. the field of fractions of $\mathcal{A}(\mathbb{E})$) and by $\mathbb{E}(x)$ the field of rational functions with coefficients in $\mathbb{E}$.

Given $\alpha \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the closed disk $\{x \in \mathbb{K} : |x - \alpha| \leq R\}$ and by $d(\alpha, R^-)$ the open disk $\{x \in \mathbb{K} : |x - \alpha| < R\}$ contained in $\mathbb{K}$,

Given $\alpha \in \mathbb{K}$ and $R > 0$, we denote by $\mathcal{A}(d(\alpha, R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(\alpha, R^-)$ (i.e. the set of power series with a radius of convergence $\geq R$) and by $\mathcal{M}(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}(d(\alpha, R^-))$. We then denote by $\mathcal{A}_b(d(\alpha, R^-))$ the $\mathbb{K}$-algebra of bounded analytic functions in $d(\alpha, R^-)$ and by $\mathcal{M}_b(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}_b(d(\alpha, R^-))$. And we set $\mathcal{A}_u(d(\alpha, R^-)) = \mathcal{A}(d(\alpha, R^-)) \setminus \mathcal{A}_b(d(\alpha, R^-))$ and $\mathcal{M}_u(d(\alpha, R^-)) = \mathcal{M}(d(\alpha, R^-)) \setminus \mathcal{M}_b(d(\alpha, R^-))$. As in complex functions, a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ is said to be *transcendental* if it is not a rational function. Then, transcendental functions are known to be transcendental on the field $\mathbb{K}(x)$ [8].

In complex functions theory, a notion closely linked to Picard's exceptional values [8], [10], [11] was introduced: the notion of "perfectly branched value" [6]. In [10] the same notion was introduced on $\mathcal{M}(\mathbb{K})$ and on $\mathcal{M}_u(d(a, R^-))$. Let us recall these notions.

Definition: Let $f \in \mathcal{M}(\mathbb{E})$ (resp. $f \in \mathcal{M}_u(d(a, R^-))$). A value $b \in \mathbb{E}$ will be called *a quasi-exceptional value* for f if $f - b$ has finitely many zeros in $\mathbb{E}$ (resp. $(d(a, R^-))$) and *an exceptional value* for f if $f - b$ has no zero in $\mathbb{E}$ (resp. $(d(a, R^-))$) [2]. Similarly, ∞ is called *a quasi-exceptional value* for f if f has finitely many poles and *an exceptional value* for f if f has no pole.

Next, b will be called *a perfectly branched value for f* if all zeros of $f - b$ are of multiple order except finitely many. And b will be called *a totally branched value for f* if all zeros of $f - b$ are of multiple order, without exception. Similarly, if all poles of f are multiple except finitely many, ∞ will be called *a perfectly branched value* and if all poles of f are multiple without exception, ∞ will be called *a totally branched value*.

In $\mathbb{C}$ it is known that a transcendental meromorphic function admits at most two quasi-exceptional values and four perfectly branched values [6]. An entire function admits at most one quasi-exceptional value and two perfectly branched values [6]. As explained by K. S. Charak in [6], these numbers of perfectly branched values, respectively four and two, are sharp. The Weierstrass function $\wp$ has 4 totally branched values (considering ∞ as a value) and of course, sine and cosine functions admit two totally branched values: 1 and -1.

On the field $\mathbb{K}$, in [10] it is proven that a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}_u(d(a, R^-))$ has at most 4 perfectly branched values and more precisely, a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ has at most 3 totally branched values. An unbounded analytic function $f \in \mathcal{A}_u(d(a, R^-))$ has at most 2 perfectly branched values. But it is also proven that a transcendental meromorphic function having finitely many poles $f \in \mathcal{M}(\mathbb{K})$ has at most one finite perfectly branched value.

In this paper, we propose to look for additional results on these problems by examining meromorphic functions in the form $\frac{f}{g}$, by comparing the kind of growth of f and g, either through their Nevanlinna characteristic functions or through their order of growth or type of growth. We will also define perfectly branched small functions in order to generalize results obtained on perfectly branched values. However, in the non-Archimedean setting, such a generalization does not work, due to the absence of Yamanoi's Theorem.

Notation: Given $R > 0$ and $f \in \mathcal{A}(d(0, R^-))$, for $r < R$, we put $|f|(r) = \lim_{|x|\to r, |x|\neq r} |f(x)|$. Given $a \in \mathbb{K}$ and $f(x) = \sum_{n=q}^{+\infty} \lambda_n (x-a)^n \in \mathcal{A}(d(a, R^-))$, with $\lambda_q \neq 0$, we put $\omega_a(f) = q$.

The Nevanlinna functions for complex meromorphic functions are well known. We will shortly recall the definition of their equivalent, or so, for p-adic meromorphic functions in the whole p-adic field or inside an open disk [3], [4]. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Let $f \in \mathcal{M}(\mathbb{K})$ (resp, $f \in \mathcal{M}(d(0, R^-))$). Consider any $r > 0$ (resp. $r \in]0, R[$). We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let (a_n), $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, and for each, let s_n be its respective order.

We then set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$.

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f without multiplicity:

$$\overline{Z}(r, f) = \overline{\omega_0}(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|).$$

In the same way, considering the finite sequence (b_n), $1 \leq n \leq \nu(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n, we put

$$N(r, f) = \max(-\omega_0(f), 0) \log r + \sum_{n=1}^{\nu(r)} t_n (\log r - \log |b_n|).$$

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega_0}}(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega_0}}(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$$\overline{N}(r, f) = \overline{\overline{\omega_0}}(f) \log r + \sum_{n=1}^{\nu(r)} (\log r - \log |b_n|).$$

Now we can define the Nevanlinna characteristic function $T(r, f)$ in $]0, +\infty[$ when f belongs to $\mathcal{M}(\mathbb{K})$ (resp. in $]0, R[$ when f belongs to $\mathcal{M}(d(0, R^-))$) as: $T(r, f) = \max(Z(r, f), N(r, f))$ and the function $T(r, f)$ is called *the characteristic function of f*.

Given f and $w \in \mathcal{M}(\mathbb{E})$ (resp. f and $w \in \mathcal{M}(d(0, R^-))$), w is called *a small function with respect to f* if $\lim_{r\to+\infty} \frac{T(r, w)}{T(r, f)} = 0$ (resp. $\lim_{r\to R^-} \frac{T(r, w)}{T(r, f)} = 0$).

Given $f \in \mathcal{M}(\mathbb{E})$ (resp. $f \in \mathcal{M}(d(0,R^-))$), we denote by $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0,R^-))$), the set of functions $w \in \mathcal{M}(\mathbb{E})$ (resp. the set of functions $w \in \mathcal{M}(d(0,R^-))$) which are small functions with respect to f. Similarly, we denote by $\mathcal{A}_f(\mathbb{E})$ (resp. $\mathcal{A}_f(d(0,R^-))$), the set of functions $w \in \mathcal{A}(\mathbb{E})$ (resp. the set of functions $w \in \mathcal{A}(d(0,R^-))$) which are small functions respectively to f.

We can now define perfectly branched small functions. Let $f \in \mathcal{M}(\mathbb{E})$ (resp. $f \in \mathcal{M}(d(0,R^-))$). A function $w \in \mathcal{M}_f(\mathbb{E})$ (resp. $f \in \mathcal{M}_f(d(0,R^-))$) will be called *a perfectly branched small function with respect to f* if all zeros of $f - w$ except finitely many are multiple and w will be called *a totally branched small function with respect to f* if all zeros of $f - w$ are multiple .

Remarks: 1) Given $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0,R^-))$)), we have $T(r,f) = Z(r,f)$.
2) Given $f \in \mathcal{M}(\mathbb{E})$ or $f \in \mathcal{M}(d(0,R^-))$ and $b \in \mathbb{E}$, it is equivalent that b is a perfectly (resp. a totally) branched value for f and that $\frac{1}{b}$ is a perfectly (resp. a totally) branched value for $\frac{1}{f}$.
3) In $\mathcal{M}(d(0,R^-))$, concerning small functions, given any $f \in \mathcal{M}_u(d(0,R^-))$, all functions $h \in \mathcal{M}_b(d(0,R^-))$ belong to $\mathcal{M}_f(d(0,R^-))$.

Theorem 1 is easily proven with help of Nevanlinna-Yamanoi's Theorem on n small functions.

Theorem 1: *Let $f \in \mathcal{M}(\mathbb{C})$ be transcendental. There exist at most four small functions $w_j \in \mathcal{M}_f(\mathbb{C})$, $j = 1, 2, 3, 4$ that are perfectly branched with respect to f. Moreover, if $f \in \mathcal{A}(\mathbb{C})$, then there exist at most two small functions $w_j \in \mathcal{M}_f(\mathbb{C})$, $j = 1, 2$ that are perfectly branched with respect to f.*

Theorem 2 is a serious refinement of Theorem 1.

Theorem 2: *Let $f, g \in \mathcal{A}(\mathbb{C})$ have no common zero and be such that $\limsup_{r\to+\infty} \frac{T(r,f)}{T(r,g)} > 3$ (resp. $\limsup_{r\to+\infty} \frac{T(r,f)}{T(r,g)} > 2$). Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two (resp. three) perfectly branched small functions.*

Example: Set $g(z) = e^z - 1$ and $f(z) = e^{4z} - 2$ and set $\phi(z) = \frac{f(z)}{g(z)}$. Let us estimate $T(r,f)$, $T(r,g)$. Set $h(z) = e^z$. Then

$$2\pi T(r,h) = \int_{-\pi}^{+\pi} \log^+ \left|e^{re^{it}}\right| dt = \int_{-\pi}^{+\pi} \log^+ \left(e^{r\cos(t)} \left|e^{ir\sin(t)}\right|\right) dt$$

$$= \int_{-\pi}^{+\pi} \log^+ (e^{r\cos(t)}) dt = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \log (e^{r\cos(t)}) dt = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} r\cos(t) dt = 2r$$

and hence $T(r,h) = \frac{r}{\pi}$. Therefore, by classical theorems on the Nevanlinna theory, we can derive $T(r,f) = \frac{4r}{\pi} + o(T(r,h))$ and $T(r,g) = \frac{r}{\pi} + o(T(r,h))$.

Consequently, $T(r,f) = 4T(r,g) + o(T(r,h))$. Then, by Theorem 2, ϕ admits at most two perfectly branched small functions.

Similarly, set $\psi(z) = \dfrac{e^{3z} - 2}{e^z - 1}$. Then by Theorem 2, ψ has at most three perfectly branched small functions.

Corollary 2.1: *Let $f \in \mathcal{A}(\mathbb{C}) \setminus \mathbb{K}[x]$ and let $g \in \mathcal{A}_f(\mathbb{C})$. Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched small functions. Particularly, they admit at most two perfectly branched values.*

Corollary 2.2: *Let $f \in \mathcal{A}(\mathbb{C}) \setminus \mathbb{K}[x]$. Then f has at most two perfectly branched small functions.*

Theorem 3: *Let $f,\ g \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. $f,\ g \in \mathcal{A}_u(d(0,R^-))$) be such that $\limsup\limits_{r\to+\infty} \dfrac{T(r,f)}{T(r,g)} > 2$ (resp. $\limsup\limits_{r\to R^-} \dfrac{T(r,f)}{T(r,g)} > 2$). Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched values.*

Corollary 3.1: *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$, (resp. let $f \in \mathcal{A}_u(d(0,R^-))$) and let $g \in \mathcal{A}_f(\mathbb{K})$, (resp. $g \in \mathcal{A}_f(d(0,R^-))$). Then both $\dfrac{f}{g}$ and $\dfrac{g}{f}$ have at most two perfectly branched values.*

Concerning exceptional small functions in $\mathbb{C}$, the following theorem A is known and generalizes this on exceptional values for complex meromorphic functions [16]:

Theorem A : *Let $f \in \mathcal{M}(\mathbb{C})$ be transcendental. There exist at most two small functions $w_j \in \mathcal{M}_f(\mathbb{C})$, $j = 1,\ 2$ such that $f - w_j$ admits finitely many zeros. Moreover, if $f \in \mathcal{A}(\mathbb{C}) \setminus \mathbb{C}[x]$, then there exists at most one function $w \in \mathcal{A}_f(\mathbb{C})$ such that $f - w$ has finitely many zeros.*

On the field $\mathbb{K}$, we have a better result with p-adic meromorphic functions:

Theorem 4: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. $f \in \mathcal{M}_u(d(0,R^-))$). There exists at most one function $w \in \mathcal{M}_f(\mathbb{K})$, (resp. $w \in \mathcal{M}_f(d(0,R^-))$) such that $f - w$ has finitely many zeros. Moreover, if f belongs to $\mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$, (resp. to $\mathcal{A}_u(d(0,R^-))$), then there exists no function $w \in \mathcal{M}_f(\mathbb{K})$, (resp. $w \in \mathcal{M}_f(d(0,R^-))$) such that $f - w$ has finitely many zeros.*

Theorem B is given in [11] and is an easy consequence of results of [10]:

Theorem B: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have finitely many poles. Then f admits at most one rational function $h \in \mathbb{K}(x)$ which is perfectly branched with respect to f.*

Theorems 1 suggests the following conjecture that we cannot prove due to the absence of a p-adic Yamanoi's theorem:

Conjecture: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $f \in \mathcal{M}_u(d(0,R^-))$). There exists at most four small functions $w \in \mathcal{M}_f(\mathbb{K})$ (resp. $w \in \mathcal{M}_f(d(0,R^-))$) that are perfectly* *branched with respect to f. Moreover, if $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. let $f \in \mathcal{A}_u(d(0,R^-))$) then there exists at most two small functions $w \in \mathcal{A}_f(\mathbb{K})$ (resp. $w \in \mathcal{A}_f(d(0,R^-))$) that are perfectly branched with respect to f.*

The next theorems use the growth order and the growth type for p-adic entire functions. Indeed, in order to obtain some results on branched small functions for p-adic meromorphic functions, since we don't enjoy a Yamanoi-Nevanlinna theorem, we will use another strategy combining the order of growth and the type of growth for entire functions, thanks to the link between the type of growth and the Nevanlinna characteristic function, for an entire function.

In [1] the growth of p-adic entire functions was examined. Here, we will use the compared growth of numerator and denominator of a p-adic meromorphic function in order to examine how many perfectly branched values it can admit.

Definitions and notation: Similarly to the definition known on complex entire functions [15], given $f \in \mathcal{A}(\mathbb{K})$, the superior limit $\limsup\limits_{r\to+\infty} \dfrac{\log(\log(|f|(r)))}{\log(r)}$ is called *the order of growth of f* or *the order of f* in brief and is denoted by $\rho(f)$. We say that f has *finite order* if $\rho(f) < +\infty$.

Now, let $f \in \mathcal{A}(\mathbb{K})$ have an order of growth $\alpha < +\infty$. The superior limit $\limsup\limits_{r\to+\infty} \dfrac{\log(|f|(r))}{r^\alpha}$ is called *the type of growth of f* and is denoted by $\sigma(f)$.

Now, it is useful to look at relations between the growth of functions $f,\ g \in \mathcal{A}(\mathbb{K})$ and their characteristic functions.

Theorem 5: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho(f) > \rho(g)$. Then*

$$\liminf_{r\to+\infty} \frac{T(r,g)}{T(r,f)} = 0.$$

By Theorem 3, we can now derive Corollary 5.1:

Corollary 5.1: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho(f) \neq \rho(g)$. Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched values.*

Now, when $\rho(f) = \rho(g)$, we can still give some precision.

Theorem 6: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ and suppose that $\rho(f) = \rho(g) \in]0,+\infty[$ and $\sigma(f) \neq \sigma(g)$. Then both $\frac{f}{g}$ and $\frac{g}{f}$ have at most three perfectly branched values. Moreover, if $2\sigma(g) < \sigma(f)$ or if $2\sigma(f) < \sigma(g)$ then $\frac{f}{g}$ and $\frac{g}{f}$ have at most two perfectly branched values.*

Corollary 6.1: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\frac{f}{g}$ admits four distinct branched values. Then $\rho(f) = \rho(g)$. Moreover, if $\rho(f) \in]0,+\infty[$, then $\sigma(f) = \sigma(g)$.*

2 The proofs

In the proof of Theorem 5, we will use the following Theorems N1 and N2, known as Nevanlinna second main Theorem that holds in complex analysis as in p-adic analysis. In the proof of Theorem 1 we will use Theorem N3. We will also need the following classical lemmas H, J, L, M , P:

Lemma H [9]: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. $f \in \mathcal{M}_u(d(0,R^-))$). Then $\mathcal{M}_f(\mathbb{K})$ is a subfield of $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0,R^-))$ is a subfield of $\mathcal{M}(d(0,R^-))$); $\mathcal{A}_f(\mathbb{K})$ is subalgebra of $\mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}_f(d(0,R^-))$ is a subalgebra of $\mathcal{A}(d(0,R^-))$). Moreover, given $f,g \in \mathcal{M}(\mathbb{K})$ (resp. $f,g \in \mathcal{M}(d(0,R^-))$), then $T(r,\frac{1}{f}) = T(r,f) + O(1)$ and $T(r,fg) \leq T(r,f) + T(r;g) + O(1)$.*

Lemma J [8]: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}(d(0,R^-))$). Then $Z(r,f+g) \leq \max(Z(r,f),Z(r,g)) + O(1)$.*

Lemma L [8]: *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}(d(0,R^-))$). Then $Z(r,f.g) = Z(r,f) + Z(r,g)$.*

Lemma M [8]: *Let $f \in \mathcal{A}(\mathbb{K})$. Then f is a polynomial if and only if there exists $q \in \mathbb{N}$ such that $T(r,f) \leq q\log(r)$.*

Lemma P [8]: *Let $f \in \mathcal{A}(d(0,R^-))$. Then f belongs to $\mathcal{A}_b(d(0,R^-))$ if and only if $T(r,f)$ is bounded when r tends to R. Moreover, if f has finitely many zeros, then f belongs to $\mathcal{A}_b(d(0,R^-))$.*

Notation: As usual, given a function φ defined in $]0,+\infty[$ (resp. in $]0,R[$) we denote by $o(\varphi)$ any function ψ defined in $]0,+\infty[$ (resp. in $]0,R[$) such that $\lim\limits_{r\to+\infty} \frac{\psi(r)}{\varphi(r)} = 0$ (resp. $\lim\limits_{r\to R^-} \frac{\psi(r)}{\varphi(r)} = 0$).

Theorem N1 [12]: *Let $f \in \mathcal{M}(\mathbb{C})$ and let $b_1,...,b_q \in \mathbb{C}$. Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r,f-b_j) + \overline{N}(r,f) + o(T(r,f)).$$

In the p-adic context we know a more precise Nevanlinna Theorem:

Theorem N2 [3], [4], [13] *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-))$) and let $b_1,...,b_q \in \mathbb{K}$. Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r,f-b_j) + \overline{N}(r,f) - \log(r) + O(1)$$

Theorem N3 is given in [16] and provides us with a Nevanlinna theorem on q small functions. Unfortunately, it has no equivalent in p-adic analysis when $q > 3$.

Theorem N3: *Let $f \in \mathcal{M}(\mathbb{C})$ and let $w_1, ..., w_q \in \mathcal{M}_f(\mathbb{C})$. Let $\epsilon > 0$ be fixed. Then*

$$(q-1-\epsilon)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r, f-w_j) + \overline{N}(r,f) + o(T(r,f)).$$

Now, by Lemma L, we can easily prove the following Lemma Q:

Lemma Q: *Let $f \in \mathcal{M}(\mathbb{C})$ (resp. $f \in \mathcal{M}(\mathbb{K})$, resp.$f \in \mathcal{M}(d(0,R^-))$) and let $g \in \mathcal{M}_f(\mathbb{C})$ (resp. $g \in \mathcal{M}_f(\mathbb{K})$, resp. $g \in \mathcal{M}_f(d(0,R^-))$). Then $T(r, \frac{f}{g}) = T(r,f) + o(T(r,f))$.*

Lemma R is obvious:

Lemma R: *Let $f,\ g \in \mathcal{A}(\mathbb{C})$ (resp. $f,\ g \in \mathcal{A}(\mathbb{K})$, resp. $f,\ g \in \mathcal{A}(d(0,R^-))$) have no common zero. Then $Z(r, \frac{f}{g}) = Z(r,f),\ N(r, \frac{f}{g}) = Z(r,g)$.*

Remark: It is sufficient to prove that the function $\phi = \frac{f}{g}$ has at most 2 perfectly branched values in Theorems 3, 4, 6.

Proof of Theorem 1. Suppose that f admits 5 perfectly branched small functions: $w_j,\ 1 \leq j \leq 5$. Let $\epsilon \in]0, \frac{1}{2}[$. For each $j = 1, ..., 5$, let s_j be the number of zeros of order 1 of $f - w_j$ and let $s = \sum_{j=1}^{5} s_j$. So, we have $\overline{Z}(r, f-w_j) \leq \frac{T(r,f)}{2} + s_j \log(r) + O(1)$. Consequently, by Theorem N3 we have

$$(4-\epsilon)T(r,f) \leq \sum_{j=1}^{5} \overline{Z}(r, f-w_j) + \overline{N}(r,f) + o(T(r,f))$$

therefore

$$(3-\epsilon)T(r,f) \leq \sum_{j=1}^{5} \overline{Z}(r, f-w_j) + o(T(r,f)) \leq \frac{5T(r,f)}{2} + o(T(r,f)).$$

That holds with $\epsilon < \frac{1}{2}$, which leads to a contradiction.

Suppose now that $f \in \mathcal{A}(\mathbb{C})$ and that f admits 3 perfectly branched small functions $w_j,\ j = 1,\ 2,\ 3$. The same reasoning as previously leads to $(2-\epsilon)T(r,f) \leq \sum_{j=1}^{3} \overline{Z}(r, f-w_j) + o(T(r,f))$, therefore

$$(2-\epsilon)T(r,f) \leq \frac{3T(r,f)}{2} + o(T(r,f)),$$

a contradiction when $\epsilon < \frac{1}{2}$. That ends the proof of Theorem 1.

Notation: In Theorem 2, 5, 6 we put $\phi = \frac{f}{g}$.

Proof of Theorem 2: First we notice that $T(r, f) \leq T(r, \phi) + T(r, g) + o(T(r, f))$, hence

$$T(r, \phi) \geq T(r, f) - T(r, g) + o(T(r, f)). \tag{1}$$

Suppose that ϕ has 3 perfectly branched small functions w_j, $j = 1, 2, 3$ and that $\limsup_{r\to+\infty} \frac{T(r, f)}{T(r, g)} > 3$. Applying Theorem N3 to ϕ, for every $\epsilon > 0$ we have

$$(2-\epsilon)T(r, \phi) \leq \sum_{j=1}^{3} \overline{Z}(r, \phi - w_j) + \overline{N}(r, \phi) + o(T(r, \phi)) \leq \frac{3T(r, \phi)}{2} + N(r, \phi) + o(T(r, \phi)),$$

Clearly, $N(r, \phi) = Z(r, g)$, hence we obtain.

$$(2-\epsilon)T(r, \phi) \leq \frac{3T(r, \phi)}{2} + Z(r, g) + o(T(r, \phi)). \tag{2}$$

Thanks to the hypothesis $\limsup_{r\to+\infty} \frac{T(r, f)}{T(r, g)} > 3$, we can find $\eta > 0$ and a sequence $(r_n)_{n\in\mathbb{N}}$ tending to $+\infty$ such that when n is big enough, we obtain,

$$(3+\eta)T(r_n, g) \leq T(r_n, f). \tag{3}$$

Consequently, by (2),

$$(2-\epsilon)T(r_n, \phi) \leq \frac{3}{2}T(r_n, \phi) + \frac{1}{3+\eta}T(r_n, f) + o(T(r_n, \phi)). \tag{4}$$

On the other hand, by (1) and (3) we have

$$T(r_n, \phi) \geq T(r_n, f) - T(r_n, g) + o(T(r_n, f)) \geq T(r_n, f) - \frac{1}{3+\eta}T(r_n, f) + o(T(r_n, f))$$

and hence,

$$T(r_n, \phi) \geq \frac{2+\eta}{3+\eta}T(r_n, f) + o(T(r_n, f)).$$

But we notice that $o(T(r, \phi)) = o(T(r, f))$. Consequently, by (4) we obtain

$$(2-\epsilon)T(r_n, \phi) \leq \frac{3}{2}T(r_n, \phi) + \Big(\frac{1}{3+\eta}\Big)\Big(\frac{3+\eta}{2+\eta}\Big)T(r_n, \phi) + o(T(r_n, \phi))$$

hence

$$(2-\epsilon)T(r_n, \phi) \leq \Big(\frac{3}{2} + \frac{1}{2+\eta}\Big)T(r_n, \phi) + o(T(r_n, \phi)). \tag{5}$$

Now, since ϵ was chosen arbitrarily, we can choose it small enough so that $2-\epsilon > \dfrac{8+3\eta}{4+2\eta}$ and then we can check that contradicts (5).

Suppose now that ϕ has 4 perfectly branched small functions w_j, $j = 1, 2, 3, 4$ and that $\limsup\limits_{r\to+\infty} \dfrac{T(r,f)}{T(r,g)} > 2$. Applying Theorem N3 to ϕ, for every $\epsilon > 0$ we have

$$(3-\epsilon)T(r,\phi) \leq \sum_{j=1}^{4} \overline{Z}(r,\phi - w_j) + \overline{N}(r,\phi) + o(T(r,\phi)) \leq$$
$$2T(r,\phi) + N(r,\phi) + o(T(r,\phi)),$$

So, similarly as in the previous case, we obtain.

$$(3-\epsilon)T(r,\phi) \leq 2T(r,\phi) + Z(r,g) + o(T(r,\phi)). \tag{6}$$

Now, thanks to the hypothesis $\limsup\limits_{r\to+\infty} \dfrac{T(r,f)}{T(r,g)} > 2$, we can find $\eta > 0$ and a sequence $(r_n)_{n\in\mathbb{N}}$ tending to $+\infty$ such that when n is big enough, we obtain,

$$(2+\eta)T(r_n,g) \leq T(r_n,f). \tag{7}$$

Consequently, by (6),

$$(3-\epsilon)T(r_n,\phi) \leq 2T(r_n,\phi) + \frac{1}{2+\eta}T(r_n,f) + o(T(r_n,\phi)). \tag{8}$$

On the other hand, by (1) and (7) we have

$$T(r_n,\phi) \geq T(r_n,f) - T(r_n,g) + o(T(r_n,f)) \geq T(r_n,f) - \frac{1}{2+\eta}T(r_n,f) + o(T(r_n,f))$$

and hence, we can derive

$$T(r_n,\phi) \geq \frac{1+\eta}{2+\eta}T(r_n,f) + o(T(r_n,f)).$$

Consequently, by (8) we obtain

$$(3-\epsilon)T(r_n,\phi) \leq 2T(r_n,\phi) + \Big(\frac{1}{2+\eta}\Big)\Big(\frac{2+\eta}{1+\eta}\Big)T(r_n,\phi) + o(T(r_n,\phi))$$

hence

$$(3-\epsilon)T(r_n,\phi) \leq \Big(2 + \frac{1}{1+\eta}\Big)T(r_n,\phi) + o(T(r_n,\phi)). \tag{9}$$

Now, since ϵ was arbitrary, we can choose it small enough so that $1-\epsilon > \dfrac{1}{1+\eta}$ and then we can check that contradicts (9).

Remark: In Theorems 3, when f and g belong to $\mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$, we can write f and g in the form $f = \widetilde{f}.h$ and $g = \widetilde{g}.h$ where $\widetilde{f}$ and $\widetilde{g}$ have no common zero. Then we can check that $\widetilde{f}$ and $\widetilde{g}$ satisfy the hypotheses of Theorem 3, like do f and g, thanks to the equality $T(r,F) + T(r,G) = T(r,FG)$ that holds in $\mathcal{A}(\mathbb{K})$ and in $\mathcal{A}(d(0,R^-))$.

Now, if $f,\ g \in \mathcal{A}_u(d(0,r^-))$, by Lazard's Theorem [14], we can place ourselves in an algebraically closed spherically complete extension to obtain the same conclusion because the Nevanlinna functions are the same in such an extension. Therefore we can assume that f and g have no common zero without loss of generality.

Proof of Theorem 3: As noticed in the above remark, without loss of generality, we can suppose that f and g have no common zeros. Consequently, we have $T(r,\phi) = \max(T(r,f), T(r,g))$.

Now, by hypothesis, there exists $\lambda < \dfrac{1}{2}$ and a sequence $(r_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} r_n = +\infty$ (resp. $\lim\limits_{n\to+\infty} r_n = R$) and such that

$$(1) \qquad T(r_n, g) \le \lambda T(r_n, f)\ \forall n \in \mathbb{N}.$$

Suppose that ϕ has 3 perfectly branched values b_j, $j = 1, 2, 3$. Applying Theorem N2 we have

$$(2) \qquad 2T(r,\phi) \le \sum_{j=1}^{3} \overline{Z}(r,\phi - b_j) + \overline{N}(r,\phi) - \log r + O(1).$$

But here, for each $j = 1, 2, 3$, we notice that $\overline{Z}(r,\phi - b_j) \le \dfrac{Z(r,\phi - b_j)}{2} + q_j \log(r)$ with $q_j \in \mathbb{N}$ and by Lemma J, $Z(r,\phi - b_j) = Z(r, f - b_j g) \le \max(T(r,f), T(r,g)) + O(1)$. But since $T(r_n,f) > T(r_n,g)$, we have $T(r_n,\phi - b_j) \le T(r_n,f) + O(1)$, hence $\overline{Z}(r,\phi - b_j) \le \frac{T(r_n,f)}{2} + q_j \log(r_n) + O(1)$. Now, putting $q = q_1 + q_2 + q_3$, by (2) we obtain

$$2T(r_n,f) \le \frac{3T(r_n,f)}{2} + T(r_n,g) + q\log(r_n) + O(1)$$

hence

$$T(r_n,f) \le 2T(r_n,g) + 2q\log(r_n) + O(1),$$

a contradiction to (1).

Proof of Theorem 4: Suppose that there exist two distinct functions $g_1,\ g_2 \in \mathcal{M}_f(\mathbb{K})$, (resp. $g_1,\ g_2 \in \mathcal{M}_f(d(0,R^-))$) such that $f - g_k$ has finitely many zeros. So, there exist $P_1,\ P_2 \in \mathbb{K}[x]$ and $h_1,\ h_2 \in \mathcal{A}(\mathbb{K})$ (resp. $h_1,\ h_2 \in \mathcal{A}(d(0,R^-))$) such that $f - g_k = \dfrac{P_k}{h_k}$, $k = 1, 2$ and hence we notice that

$$(1) \qquad T(r,f) = T(r,\frac{P_k}{h_k}) + o(T(r,f)) = T(r,h_k) + o(T(r,f)),\ k = 1, 2.$$

Consequently, putting $g = g_2 - g_1$, g belongs to $\mathcal{M}_f(\mathbb{K})$ (resp. to $\mathcal{M}_f(d(0,R^-))$) and satisfies

$$\frac{P_1}{h_1} = \frac{P_2}{h_2} + g.$$

Therefore $P_1h_2 - P_2h_1 = gh_1h_2$ and hence

$$T(r, P_1h_2 - P_2h_1) = T(r, gh_1h_2). \tag{2}$$

Now, by Lemma J we have

$$T(r, P_1h_2 - P_2h_1) \le \max(T(r, P_1h_2), T(r, P_2h_1)) + O(1) \le \max(T(r, h_1), T(r, h_2)) + o(T(r, f))$$

and hence by (1), we obtain

$$T(r, P_1h_2 - P_2h_1) \le T(r, f) + o(T(r, f)). \tag{3}$$

On the other hand, by Lemma L, we have $T(r, gh_1h_2) = T(r, h_1h_2) + T(r, g) = 2T(r, f) + o(T(r, f))$, a contradiction to (3).

Suppose now that f belongs to $\mathcal{A}(\mathbb{K})$ and that there exists a function $w \in \mathcal{M}_f(\mathbb{K})$ such that $f - w$ has finitely many zeros. Set $w = \frac{l}{t}$ where l and t belong to $\mathcal{A}_f(\mathbb{K})$ and have no common zeros. Thus, $f - w = \frac{tf - l}{t}$ and each zero of $tf - l$ cannot be a zero of t, hence it is zero of $f - w$. Consequently, since $f - w$ has finitely many zeros, $tf - l$ also has finitely many zeros and hence is a polynomial. But since l belongs to $\mathcal{A}_f(\mathbb{K})$, when r is big enough we have $|f|(r) > |l|(r)$ and hence $|tf|(r) > |l|(r)$ since $t \in \mathcal{A}_f(\mathbb{K})$, therefore $|tf - l|(r) = |tf|(r)$. And since f is transcendental, by Lemma M for every fixed $q \in \mathbb{N}$, $|f|(r) > r^q$ when r is big enough. Similarly, $|tf - l|(r) > r^q$ when r is big enough. Consequently, by Lemma M, $tf - l$ is not a polynomial, which proves that w does not exist.

Suppose finally that f belongs to $\mathcal{A}_u(d(0,R^-))$ and that there exists a function $w \in \mathcal{M}_f(d(0,R^-))$ such that $f - w$ has finitely many zeros. Without loss of generality, we can assume that the field $\mathbb{K}$ is spherically complete because both f and w have continuation to an algebraically closed spherically complete extension of $\mathbb{K}$ where their zeros are the same as in $\mathbb{K}$. Consequently, by results of [14], we can write $w = \frac{l}{t}$ where l and t have no common zeros. Now, the zeros of $tf - l$ are those of $f - w$, hence $tf - l$ has finitely many zeros and hence, is bounded in $d(0,R^-)$. But since w belongs to $\mathcal{M}_f(d(0,R^-))$, so does l and hence $|tf|(r) > |l|(r)$ when r tends to R. Consequently, $|tf - l|(r) = |tf|(r)$ and hence $tf - l$ is not bounded in $d(0,R^-)$, a contradiction proving again that w does not exist.

Proof of Theorem 5: Let $\lambda = \frac{\rho(f) - \rho(g)}{2}$. There exists a sequence of intervals $[r'_n, r''_n]$ such that

$$\frac{\log(\log(|f|(r)))}{\log r} > \frac{\log(\log(|g|(r)))}{\log r} + \lambda \ \forall r \in [r'_n, r''_n],\ \forall n \in \mathbb{N}$$

and $\lim_{n\to+\infty} r'_n = +\infty$. Therefore

$$\log(\log(|f|(r))) > \log(\log(|g|(r))) + \lambda \log(r)\ \forall r \in [r'_n, r''_n],\ \forall n \in \mathbb{N},$$

hence

$$\log(|f|(r)) > \log(|g|(r))(r^{\lambda})\ \forall r \in [r'_n, r''_n],\ \forall n \in \mathbb{N}.$$

Consequently, putting $S_n = \sup_{r\in[r'_n,r''_n]} \dfrac{\log(|g|(r))}{\log(|f|(r))}$ for every $n \in \mathbb{N}$, we have $\lim_{n\to+\infty} S_n = 0$ and hence $\liminf_{r\to+\infty} \dfrac{T(r,g)}{T(r,f)} = 0$, which ends the proof.

Proof of Theorem 6: Without loss of generality, we can suppose that $\sigma(f) > \sigma(g)$. Put $\rho(f) = t$. There exist $\lambda > 0$ and a sequence $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lim_{n\to+\infty} r_n = +\infty$ and

$$\frac{\log(|f|(r_n))}{(r_n)^t} \geq \frac{\log(|g|(r_n))}{(r_n)^t} + \lambda\ \forall n \in \mathbb{N}$$

hence

$$\log(|f|(r_n)) \geq \lambda (r_n)^t + \log(|g|(r_n))\ \forall n \in \mathbb{N}$$

consequently,

$$T(r_n, f) \geq \lambda (r_n)^t + T(r_n, g)\ \forall n \in \mathbb{N}. \tag{1}$$

As in Theorem 3, we can write f and g in the form $f = h\widetilde{f}$, $g = h\widetilde{g}$ were $h \in \mathcal{A}(\mathbb{K})$ and $\widetilde{f}$, $\widetilde{g}$ have no common zero. Then $T(r,f) = T(r,h) + T(r,\widetilde{f})$, $T(r,g) = T(r,h) + T(r,\widetilde{g})$, hence by (1),

$$T(r_n, \widetilde{f}) \geq \lambda (r_n)^t + T(r_n, \widetilde{g})\ \forall n \in \mathbb{N}.$$

Consequently,

$$T(r_n, \phi) = T(r_n, \widetilde{f}) \tag{2}$$

when n is big enough. Suppose now that ϕ admits 4 perfectly branched values b_j, $j = 1, 2, 3, 4$ and let q be the total number of zeros of order 1 of the $\phi - b_j$, $j = 1, 2, 3, 4$. Applying Theorem N2 to ϕ, we have

$$3T(r_n, \phi) \leq \sum_{j=1}^{4} \overline{Z}(r_n, \phi - b_j) + q\log(r_n) + \overline{N}(r_n, \phi) - \log(r_n) + O(1)$$

$$\leq \frac{4T(r_n, \widetilde{f})}{2} + (q-1)\log(r_n) + T(r_n, \widetilde{g}) + O(1) \tag{3}$$

hence by (2),

$$3T(r_n, \widetilde{f}) \leq 2T(r_n, \widetilde{f}) + T(r_n, \widetilde{f}) + (q-1)\log(r_n) - \lambda(r_n)^t + O(1).$$

Clearly $\lim_{n\to+\infty}((q-1)\log(r_n)-\lambda(r_n)^t)=-\infty$ and hence that inequality is absurd when n is big enough, which ends the proof of the first claim.

Suppose now that $2\sigma(g)<\sigma(f)$ and set $\beta=\dfrac{\sigma(f)}{2}-\sigma(g)$. So, there exists a sequence $(r_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} r_n=+\infty$ and

$$\frac{2T(r_n,g)}{(r_n)^t}+2\beta\leq\frac{T(r_n,f)}{(r_n)^t}$$

hence

$$T(r_n,\widetilde{g})+T(r_n,h)\leq\frac{T(r_n,\widetilde{f})+T(r_n,h)}{2}-\beta(r_n)^t\ \forall n\in\mathbb{N}$$

therefore

$$T(r_n,\widetilde{g})\leq\frac{T(r_n,\widetilde{f})}{2}-\beta(r_n)^t\ \forall n\in\mathbb{N}. \tag{4}$$

Suppose now that ϕ has three perfectly branched values b_j, $j=1,\,2,\,3$. As in the proof of Theorem 2, we have $o(T(r,\phi))=o(T(r,\widetilde{f}))$. Similarly to (3), thanks to (4) now we can get

$$2T(r_n,\phi)=2T(r_n,\widetilde{f})\leq\sum_{j=1}^{3}\overline{Z}(r_n,\phi-b_j)+(q-1)\log(r_n)+\overline{Z}(r_n,\widetilde{g})+O(1)$$

$$\leq\frac{3T(r_n,\widetilde{f})}{2}+\frac{T(r_n,\widetilde{f})}{2}+(q-1)\log(r_n)-\beta(r_n)^t+O(1)).$$

Clearly, $\lim_{n\to+\infty}(q-1)\log(r_n)-\beta(r_n)^t=-\infty$, a contradiction which finishes the proof.

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