On the lower bound for $B_i(K)B_i(K^*)$

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Abstract

In this paper, we establish the greatest lower bound for the product $B_i(K)B_i(K^*)$ of the width-integrals of index i of convex body K. Further, the greatest lower bound for the product of the mixed width-integrals $A_{p,i}(K)A_{p',i}(K^*)$ for the mixed width-integrals is given.

1 Introduction

Throughout the paper all convex bodies are assumed to contain the origin in their interior. Polar dual convex bodies are useful in geometry of numbers [16], Minkowski geometry [9, 10] and differential equations [11]. Chakerian [5] uses polar duals to discuss self-circumference of unit circles in a Minkowski plane. The upper bound for the product of volumes of a convex body and its polar dual is the well-known *The Blaschke-Santaló inequality* as follows.

If K is a convex body, then

$$V(K)V(K^*) \leq \omega_n^2, \tag{1.1}$$

with equality if and only if K is an ellipsoid, where K^* is polar dual of K and ω_n is the volume of the unit ball.

The Blaschke-Santaló inequality is due to Blaschke [2] for $n = 2, 3$ and Santaló [22] for $n \geq 2$ (See also the comments of Schneider [23]). For a good discussion of the Blaschke-Santaló inequality and a further list of references, see Lutwak [17].

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On the lower bound, Steinhardt [24] showed that for planar convex bodies,

$$W_1(K)W_1(K^*) \geq \omega_2^2 \quad \text{or} \quad S(K)S(K^*) \geq 4\omega_2^2,$$

where, K is a convex body in $\mathbb{R}^2$ and $S(K)$ is the surface area of K. Chai and Lee [4] also found a lower bound of $W_1(K)W_1(K^*)$ for all convex bodies K. On the other hand, Lutwak [18] (also see Ghandehari [14]) found a lower bound of $W_{n-1}(K)W_{n-1}(K^*)$ for all convex bodies K as follows

$$W_{n-1}(K)W_{n-1}(K^*) \geq \omega_n^2, \tag{1.2}$$

with equality if and only if K is a ball (centered at the origin).

This was obtained by Firey [8] for dimensions 2 and 3.

However, the problem of finding the lower bound of the product $W_i(K)W_i(K^*)$ for all convex bodies, for each i, is not solved completely yet. This is a open problem in Lutwak [18] and Ghandehari [14]. Also see Bambah [1], Dvoretzky and Rogers [6], Firey [7], Guggenheimer [12, 13], Heil [15], and Steinhardt [24] for partial results.

Lutwak [19] defined the mixed width-integral

$$A(K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} b(K_1, u) \cdots b(K_n, u) dS(u),$$

where $b(K, \cdot)$ is half the width of convex body K in the direction u.

Just as the cross-sectional measures $W_i(K)$ are defined to be the special mixed volumes $V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i})$, the width-integrals of index i, $B_i(K)$ (see Sec. 2) can be defined as the special mixed width-integrals $A(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i})$.

In the paper, for the width-integrals of index i, we discuss a problem similar to above open question: we first establish the greatest lower bound of the product $B_i(K)B_i(K^*)$ as follows.

For a convex body K and its polar dual K^*, and $0 \leq i \leq n$

$$B_i(K)B_i(K^*) \geq \omega_n^2, \tag{1.3}$$

with equality if and only if K is a n-ball.

For a real number, Lutwak [19] also defined the mixed width-integral of order p ($p \neq 0$) by

$$A_p(K_1, \ldots, K_n) = \omega_n \left[\frac{1}{n\omega_n}\int_{S^{n-1}} b(K_1, u)^p \cdots b(K_n, u)^p dS(u)\right]^{1/p}.$$

For p equal to $-\infty$, 0 or ∞ the mixed width-integral of order p was defined by

$$A_p(K_1, \ldots, K_n) = \lim_{s \to p} A_s(K_1, \ldots, K_n).$$

The width-integral of order $A_{p,i}(K)$ is defined as the special mixed width-integral $A_p(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i})$, and called the i-th width-integral of order p.

Another aim of the paper is to establish the greatest lower bound of the product $A_{p,i}(K)A_{p',i}(K^*)$.

For a convex body K and its polar dual K^*, if $0 \le i \le n$, and the conjugate exponent $p' = \dfrac{p}{p-1}$ and $p > 1$, then

$$A_{p,i}(K)A_{p',i}(K^*) \ge \omega_n^2, \tag{1.4}$$

with equality if and only if K is a n-ball.

2 Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n \ge 2)$. Let $\mathcal{K}^n$ denote set of all convex bodies (compact, convex subsets and contain the origin in their interior) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}.

A set A is said to be centered if $-x \in A$ whenever $x \in A$, and centrally symmetric if there is a vector c such that the translate $A - c$ of A by $-c$ is centered. For each direction $u \in S^{n-1}$, we define the support function $h(K, u)$ on S^{n-1} of the convex body K by

$$h(K, u) = \max\{u \cdot x | x \in K\},$$

and the radial function $\rho(K, u)$ on S^{n-1} of the convex body K is

$$\rho(K, u) = \max\{\lambda > 0 | \lambda \mu \in K\}.$$

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

The polar dual of a convex body K that contains the origin in its interior, denoted by K^*, is another convex body defined by

$$K^* = \{y | x \cdot y \le 1, \text{ for all } x \in K\}.$$

The polar dual has the following well known property:

$$h(K^*, u) = \frac{1}{\rho(K, u)} \quad \text{and} \quad \rho(K^*, u) = \frac{1}{h(K, u)}. \tag{2.1}$$

The outer parallel set of K at the distance $\lambda > 0$, K_λ, is given by

$$K_\lambda = K + \lambda B.$$

Then the volume $V(K_\lambda)$ is a polynomial in λ whose coefficients $W_i(K)$ are geometric invariants of K:

$$V(K + \lambda B) = \sum_{i=1}^{n} \tbinom{n}{i} W_i(K) \lambda^i.$$

The functionals $W_i(K)(i = 0, \ldots, n)$ are called the $i - th$ quermassintegrals of K. The following is true:

$$W_0(K) = V(K);\ \ nW_1(K) = S(K);\ \ W_n(K) = \omega_n, \tag{2.2}$$

where $V(K)$ and $S(K)$ are the volume and surface area of K, respectively and ω_n is the volume of the unit ball B in $\mathbb{R}^n$.

For $u \in S^{n-1}$,

$$b(K, u) := \frac{1}{2}(h(K, u) + h(K, -u)) \tag{2.3}$$

is defined to be half the width of K in the direction u. Two convex bodies K and L are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$. Width-integrals were first considered by Blaschke (see [3]). The width-integral of index i is defined by Lutwak [20]. For $K \in \mathcal{K}^n, i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u). \tag{2.4}$$

The width-integral of index i is a map $B_i : \mathcal{K}^n \to \mathbb{R}$. It is positive, continuous, homogeneous of degree $(n - i)$ and invariant under motion.

3 The lower bound for $B_i(K)B_i(K^*)$

Theorem 3.1 *If $K_1, \ldots, K_n \in \mathcal{K}^n$, then,*

$$A(K_1, \ldots, K_n)A(K_1^*, \ldots, K_n^*) \geq \omega_n^2, \tag{3.1}$$

with equality if and only if $K_i (i = 1, \ldots, n)$ are n-balls.

Proof From (2.3) and in view of the definition of the mixed width-integrals, we have

$$A(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} \frac{1}{2}(h(K_i, u) + h(K_i, -u))dS(u).$$

For any convex body K, in view of the following fact

$$\frac{h(K, u) + h(K, -u)}{2} \geq \frac{\rho(K, u) + \rho(K, -u)}{2}. \tag{3.2}$$

Notes that for any convex body K

$$h(K, u) \geq \rho(K, u),$$

with equality for all u if and only if K is a ball centered at the origin. This follows that the equality in (3.2) holds if and only if K is n-ball (centered at the origin).

On the other hand, by using the Arithmetic-Harmonic means inequality (see [21, p.27]), we have

$$\frac{\rho(K, u) + \rho(K, -u)}{2} \geq \frac{2}{\rho(K, u)^{-1} + \rho(K, -u)^{-1}}, \tag{3.3}$$

with equality if and only if $\rho(K,u)=\rho(K,-u)$, it follows if and only if K is n-ball (centered at the origin).

Hence

$$A(K_1,\ldots,K_n)\geq\frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\frac{2}{\rho(K_i,u)^{-1}+\rho(K_i,-u)^{-1}}dS(u). \tag{3.4}$$

On the other hand, from (2.1) and (2.3), we have

$$\begin{aligned}A(K_1^*,\ldots,K_n^*)&=\frac{1}{n}\int_{S^{n-1}}b(K_1^*,u)\cdots b(K_n^*,u)dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\frac{1}{2}(h(K_i^*,u)+h(K_i^*,-u))dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\frac{\rho(K_i,u)^{-1}+\rho(K_i,-u)^{-1}}{2}dS(u)\end{aligned} \qquad (3.5).$$

Therefore, from (3.4) and (3.5), we obtain

$$\begin{aligned}A(K_1,\ldots,K_n)A(K_1^*,\ldots,K_n^*)\geq\frac{1}{n^2}&\int_{S^{n-1}}\prod_{i=1}^{n}\frac{2}{\rho(K_i,u)^{-1}+\rho(K_i,-u)^{-1}}dS(u)\\&\times\int_{S^{n-1}}\prod_{i=1}^{n}\frac{\rho(K_i,u)^{-1}+\rho(K_i,-u)^{-1}}{2}dS(u).\end{aligned}$$

In view of the well-known inequality: If $f\in C(S^{n-1})$ and $f(u)>0$, then

$$\int_{S^{n-1}}f(u)du\int_{S^{n-1}}f(u)^{-1}du\geq n^2\omega_n^2, \tag{3.6}$$

with equality if and only if $f(u)$ is constant.

Hence

$$A(K_1,\ldots,K_n)A(K_1^*,\ldots,K_n^*)\geq\omega_n^2. \tag{3.7}$$

If equality holds in (3.7) then equality must hold in particular in (3.2) and hence K must be a ball centered at the origin.

Taking for $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=B$ in (3.1), (3.1) reduces to the following result stated in the introduction. If $K\in\mathcal{K}^n$ and $0\leq i\leq n$, then

$$B_i(K)B_i(K^*)\geq\omega_n^2,$$

with equality if and only if K is a n-ball.

Theorem 3.2 *If $K_1,\ldots,K_n\in\mathcal{K}^n$, then*

$$A_p(K_1,\ldots,K_n)A_{p'}(K_1^*,\ldots,K_n^*)\geq\omega_n^2, \tag{3.8}$$

with equality if and only if $K_i(i=1,\ldots,n)$ are n-balls.

Proof From (2.1), (2.3) and the definition of the mixed width-integrals of index p, and in view of the Arithmetic-Harmonic means inequality, we obtain

$$A_{p'}(K_1^*,\ldots,K_n^*)=\omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\prod_{i=1}^{n}\left(\frac{h(K_i^*,u)+h(K_i^*,-u)}{2}\right)^{p'}dS(u)\right)^{1/p'}$$

$$= \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \prod_{i=1}^{n} \left(\frac{\rho(K_i,u)^{-1} + \rho(K_i,-u)^{-1}}{2} \right)^{p'} dS(u) \right)^{1/p'}$$

$$\geq \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \prod_{i=1}^{n} \left(\frac{2}{\rho(K_i,u) + \rho(K_i,-u)} \right)^{p'} dS(u) \right)^{1/p'}, \tag{3.9}$$

with equality if and only if $\rho(K_i,u) = \rho(K_i,-u), i = 1,\ldots,n$.

On the other hand, from (2.3), we have.

$$A_p(K_1,\ldots,K_n) = \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} b^p(K_1,u)\cdots b^p(K_n,u) dS(u) \right)^{1/p}$$

$$= \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \prod_{i=1}^{n} \left(\frac{h(K_i,u) + h(K_i,-u)}{2} \right)^{p} dS(u) \right)^{1/p}. \tag{3.10}$$

From (3.9), (3.10) and in view of Hölder's inequality, we obtain

$$A_p(K_1,\ldots,K_n) A_{p'}(K_1^*,\ldots,K_n^*) \geq \frac{\omega_n}{n} \left(\int_{S^{n-1}} \prod_{i=1}^{n} \left(\frac{h(K_i,u) + h(K_i,-u)}{2} \right)^{p} dS(u) \right)^{1/p} \times \left(\int_{S^{n-1}} \prod_{i=1}^{n} \left(\frac{2}{\rho(K_i,u) + \rho(K_i,-u)} \right)^{p'} dS(u) \right)^{1/p'}$$

$$\geq \frac{\omega_n}{n} \int_{S^{n-1}} \prod_{i=1}^{n} \frac{h(K_i,u) + h(K_i,-u)}{\rho(K_i,u) + \rho(K_i,-u)} dS(u). \tag{3.11}$$

For any convex body K

$$h(K,u) \geq \rho(K,u), \tag{3.12}$$

with equality for all u if and only if K is a ball centered at the origin.

Hence

$$A_p(K_1,\ldots,K_n) A_{p'}(K_1^*,\ldots,K_n^*) \geq \omega_n^2. \tag{3.13}$$

From the equality conditions of (3.9), (3.12) and Hölder inequality, it follows that the single of equality of (3.13) holds if and only if $K_i (i = 1,\ldots,n)$ are n-balls.

Taking for $K_1 = \cdots = K_{n-i} = K, K_{n-i+1} = \cdots = K_n = B$ in (3.8), (3.8) changes to the following result stated in the introduction

$$A_{p,i}(K) A_{p',i}(K^*) \geq \omega_n^2,$$

with equality if and only if K is a n-ball.

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