Computations in rational sectional category

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Abstract

We give simple upper bounds for rational sectional category and use them to compute invariants of the type of Farber's topological complexity of rational spaces. In particular we show that the sectional category of formal morphisms reaches its cohomological lower bound and give a method to compute higher topological complexity of formal spaces in terms of their cohomology.

Introduction

This paper concerns the rational sectional category of a continuous map $f\colon X \to Y$ and, in particular, the rational topological complexity of a space X. All the spaces considered will be supposed simply connected CW-complexes with finite Betti numbers.

Recall, [15], that the sectional category of f, $\operatorname{secat}(f)$, is the least integer m for which there is an open cover $\{U_0, \dots, U_m\}$ of Y and maps $s_i\colon U_i \to X$ such that $f \circ s_i$ is homotopic to the inclusion of U_i in Y. When X is contractible, the sectional category of f is the usual LS category of Y, see [1].

We give special attention to the particular case, introduced by Y.B. Rudyak in [14], of *higher topological complexity* of a space X, $\mathrm{TC}_n(X)$, defined as the sectional category of the n-diagonal map $\Delta_n\colon X \to X^n$. The case $n = 2$ yields M. Farber's

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well known *topological complexity*, $\mathrm{TC}(X)$, introduced in [3]. Explicit computations for topological complexity of rational spaces can be found in [11] and [12], for instance.

Denote by $H^*(X;R)$ the cohomology ring of X with coefficients in a ring R. It is well known, [1], that

$$\mathrm{nil}\ \ker(H^*(f;R)) \leq \mathrm{secat}(f),$$

where nil denotes the *nilpotency* of an ideal, $\mathrm{nil}\, I := \min\left\{k\colon\ I^{k+1} = 0\right\}$.

We will denote X_0 the rationalisation of X and $f_0\colon X_0 \to Y_0$ the rationalisation of f. Following the scheme of Jessup-Murillo-Parent in [11], the following approximation to rational sectional category can easily be deduced:

Let $\varphi\colon A \to B$ be a surjective **cdga** morphism. Define $\mathbf{sc}(\varphi)$ as the smallest integer m such that the quotient map

$$\rho_m\colon (A,d) \to \left(\frac{A}{K^{m+1}}, \overline{d}\right)$$

admits a homotopy retraction, where K denotes the kernel of φ. If f is a continuous map, define $\mathbf{sc}(f)$ as the least $\mathbf{sc}(\varphi)$ with φ a surjective model for f.

Observe that, for rational LS category, the main theorem of [5] asserts that $\mathrm{cat}(X_0) = \mathbf{sc}(* \hookrightarrow X)$. Inspired on this, the obvious questions to ask whether $\mathrm{secat}(f_0) = \mathbf{sc}(f)$. Although one of these inequalities is not known in general, the other one holds:

Proposition 1. *For any continuous map f,* $\mathrm{secat}(f_0) \leq \mathbf{sc}(f)$*.*

This proposition is used to establish results regarding higher topological complexity including generalizations of [12, Theorem 1.2] and [11, Theorem 1.4]. Namely, we prove that if X is a formal space then

$$\mathrm{TC}_n(X_0) = \mathrm{nil}\ \ker\left(\mu_n\colon H^*(X,\mathbb{Q})^{\otimes n} \to H^*(X,\mathbb{Q})\right),$$

being μ_n the multiplication. We also prove that if X is a space such that $\pi_*(X)\otimes\mathbb{Q}$ is finite dimensional and concentrated in odd degrees, then

$$\mathrm{TC}_n(X_0) = (n-1)\mathrm{cat}(X_0).$$

Finally, we introduce the concept of homology nilpotency, Hnil, as an improvement of the upper bound nil ker φ. We also study the case that $H^*(f,\mathbb{Q})$ is surjective, giving a dimensional upper bound for $\mathrm{secat}(f_0)$ and establishing

$$\mathrm{nil}\, H(K) \leq \mathrm{secat}(f_0) \leq \mathrm{Hnil}\, K.$$

1 Rational sectional category

Throughout this paper we will often use rational homotopy theory techniques for which much more than needed can be found in [6]. We now present some basic facts. To every simply connected CW-complex of finite type X one can associate a *rationalisation* map $\rho\colon X \to X_0$ where $\pi_*(X_0)$ is a rational vector space and $\pi_*(\rho) \otimes \mathbb{Q}$ is an isomorphism. The space X_0 is called the *rationalisation* of X. This construction is functorial in the sense that a map $f\colon X \to Y$ can also be *rationalized* to a map $f_0\colon X_0 \to Y_0$ commuting with rationalisations.

On the other hand, every such X has a (Sullivan) minimal model $(\Lambda V, d)$. This is a commutative differential graded algebra over $\mathbb{Q}$ (**cdga** for short), where ΛV denotes the free graded commutative algebra on a graded vector space V and where $d(V) \subset \Lambda^{\geq 2}V$, see [6, Chapter 12]. This correspondence yields an equivalence between the homotopy categories of rational 1-connected CW-complexes of finite type and 1-connected **cdga**'s of finite type. Moreover, every **cdga** morphism $\varphi\colon (A,d) \to (B,d)$ admits a minimal relative model

$$\begin{tikzcd}
(A,d) \arrow[r, "\varphi"] \arrow[dr, "i"'] & (B,d) \\
 & (A \otimes \Lambda V, d) \arrow[u, "\psi"', "\simeq"]
\end{tikzcd}$$

where i is the canonical injection, $\psi \circ i = \varphi$,

$$d(V) \subset \left(A^+ \otimes \Lambda V\right) + \left(A \otimes \Lambda^{\geq 2} V\right),$$

and ψ is a quasi-isomorphism.

Let $i\colon X \rightarrowtail Y$ be a cofibration, the m-fat-wedge of i is the subspace of Y^{m+1} defined as

$$T^m(i) := \left\{(x_0, \dots, x_m) \in Y^{m+1}\colon x_j \in i(X) \text{ for some } j = 0, \dots, m\right\},$$

with inclusion $W_m(i)\colon T^m(i) \to Y^{m+1}$.

Theorem 2. *([4]) Let f be a map and $i\colon X \to Y$ a cofibration replacement for f with Y a paracompact space. Then* $\operatorname{secat}(f)$ *is the smallest m such that there exists a map r making the following diagram homotopy commutative:*

$$\begin{tikzcd}
 & T^m(i) \arrow[d, "W_m(i)"] \\
Y \arrow[ur, dashed, "r"] \arrow[r, "\Delta_{m+1}"'] & Y^{m+1}.
\end{tikzcd}$$

Remark 3. *In order to take care of unnecessary technical conditions, we will consider the statement of previous theorem as the definition for sectional category.*

Now let L be a simplicial complex on m vertices, then the *polyhedral product* of the pair (Y,X) and L is defined as

$$\underline{(Y,X)}^L = \bigcup_{\sigma \in L} \left(\prod_{j=1}^{m} A_\sigma^j \right) \subset X^m,$$

where

$$A_\sigma^j := \begin{cases} Y & \text{if } j \in \sigma \\ X & \text{if } j \notin \sigma. \end{cases}$$

The m-fat-wedge $T^m(i)$ can be written in terms of the polyhedral product $\underline{(Y,X)}^{\partial \Delta^m}$. Therefore, if $A \twoheadrightarrow B$ is a surjective **cdga** model for i with kernel K, then [8, Thm. 1] tells us that a model for the inclusion $W_m(i)\colon T^m(i) \rightarrowtail Y^{m+1}$ is the projection

$$q_m\colon A^{\otimes m+1} \longrightarrow \frac{A^{\otimes m+1}}{K^{\otimes m+1}}.$$

Recall also that if A is a **cdga** model for Y then the diagonal map $\Delta_{m+1}\colon Y \to Y^{m+1}$ is modelled by the multiplication morphism $\mu_{m+1}\colon A^{\otimes m+1} \to A$. These remarks lead us to

Definition 4. *The* sectional category *of a surjective* **cdga** *morphism* $\varphi\colon A \twoheadrightarrow B$, secat$(\varphi)$*, is the smallest m for which there exists a* **cdga** *morphism* τ *such that* $\tau \circ i_m = \mu_{m+1}$,

$$\begin{tikzcd}
A^{\otimes m+1} \arrow[rr, "q_m"] \arrow[drr, tail, "i_m"] \arrow[dd, "\mu_{m+1}"'] & & \frac{A^{\otimes m+1}}{K^{\otimes m+1}} \\
 & & (A^{\otimes m+1} \otimes \Lambda W_m, D) \arrow[u, "\simeq"'] \arrow[dll, dashed, "\tau"] \\
A, & &
\end{tikzcd}$$

where $K = \ker\ \varphi$ *and* i_m *is a relative Sullivan model for* q_m*. The* sectional category *of any morphism is defined as the sectional category of any of its surjective replacements.*

Taking the pushout

$$\begin{tikzcd}
A^{\otimes m+1} \arrow[r, tail, "i_m"] \arrow[d, "\mu_{m+1}"'] & (A^{\otimes m+1} \otimes \Lambda W_m, D) \arrow[d] \\
A \arrow[r, tail, "j_m"'] & (A \otimes \Lambda W_m, \overline{D})
\end{tikzcd}$$

one can easily check, thanks to pushout's universal property, that secat$(\varphi) \leq m$ if and only if j_m admits a retraction. In fact, if φ models a map f then j_m is a model for the m-th Ganea map, $G_m(f)$.

Definition 5. *Let φ be a surjective* **cdga** *morphism and consider previous diagram.*

(i) The module sectional category *of φ,* $\mathrm{msecat}(\varphi)$*, is the smallest m such that j_m admits an A-module retraction,*

(ii) the homology sectional category *of φ,* $\mathrm{Hsecat}(\varphi)$*, as the smallest m such that $H(j_m)$ is injective.*

(iii) If a continuous map f is modelled by φ, define $\mathrm{msecat}(f) := \mathrm{msecat}(\varphi)$ *and* $\mathrm{Hsecat}(f) := \mathrm{Hsecat}(\varphi)$*.*

The module sectional category of this paper coincides with the one introduced in [9]. The expected cohomological lower bound follows:

Proposition 6. *Let $\varphi\colon A \to B$ be a surjective* **cdga** *morphism. Then*

$$\mathrm{nil}\ \ker\ H(\varphi) \leq \mathrm{Hsecat}(\varphi).$$

Proof. Suppose $\mathrm{Hsecat}(\varphi) = m$, then $H(j_m)$ is injective, where j_m is as in previous diagram. Let $[x_0], \dots, [x_m] \in \ker\ H(\varphi)$, since φ is surjective, there are $a_0, \dots, a_m \in A$ such that $x_i - da_i \in K$, for $i = 0, \dots, m$. We have constructed a cycle $z := (x_0 - da_0) \otimes \cdots \otimes (x_m - da_m) \in K^{\otimes m+1}$ therefore there exists $\xi \in A^{\otimes m+1} \otimes \Lambda W_m$ such that $D\xi = z$. Thus $H(j_m)([x_0] \cdots [x_m]) = [0]$ and since $H(j_m)$ is injective, $[x_0] \cdots [x_m] = [0]$, proving that $\mathrm{nil}\ \ker\ H(\varphi) \leq m$. ■

The following chain of inequalities is now clear for a surjective morphism φ,

$$\mathrm{nil}\ \ker\ H(\varphi) \leq \mathrm{Hsecat}(\varphi) \leq \mathrm{msecat}(\varphi) \leq \mathrm{secat}(\varphi).$$

We now prove:

Theorem 7. *Let φ be a surjective* **cdga** *model for f, then*

$$\mathrm{secat}(f_0) = \mathrm{secat}(\varphi).$$

Proof. Transform f_0 into a cofibration $i\colon X \to Y$. Denote by $\alpha\colon (\Lambda V, d) \xrightarrow{\simeq} A$ be a minimal model for A. By construction $(\Lambda V, d)$ is a minimal model of Y, and as explained at the beginning of this section, if $K = \ker\ \varphi$, then the projection

$$q_m\colon A^{\otimes m+1} \to \frac{A^{\otimes m+1}}{K^{\otimes m+1}}$$

is a model for the map $W_m(i)\colon T^m(i) \to Y^{m+1}$. Denote by $((\Lambda V)^{\otimes m+1} \otimes \Lambda W_m, D)$ a relative model for q_m. Then we have a commutative diagram

$$\begin{tikzcd}
A^{\otimes m+1} \arrow[r, "q_m"] & \frac{A^{\otimes m+1}}{K^{\otimes m+1}} \\
(\Lambda V)^{\otimes m+1} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] \arrow[r, tail] & ((\Lambda V)^{\otimes m+1} \otimes \Lambda W_m, D) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
A_{PL}(Y^{m+1}) \arrow[r, "A_{PL}(W_m(i))"'] & A_{PL}(T^m(i))
\end{tikzcd}$$

Suppose $\operatorname{secat}(f_0) = m$, then, by Theorem 2, there is $\theta\colon Y \to T^m(i)$ such that the following diagram homotopy commutes

$$\begin{tikzcd}
T^m(i) \arrow[rr, "W_m(i)"] & & Y^{m+1} \\
& Y. \arrow[ul, "\theta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Then the relative lifting lemma gives a morphism γ making commutative the upper triangle and homotopy commutative the lower triangle in

$$\begin{tikzcd}
(\Lambda V)^{\otimes m+1} \arrow[rrr, "\mu_{m+1}"] \arrow[d, tail] & & & \Lambda V \arrow[d, "\simeq"'] \\
(\Lambda V)^{\otimes m+1} \otimes \Lambda W_m \arrow[rrru, dashed, "\alpha"] \arrow[r, "\simeq"'] & A_{PL}(T^m(i)) \arrow[rr, "A_{PL}(\theta)"'] & & A_{PL}(Y).
\end{tikzcd}$$

The desired τ is given by pushout's universal property:

$$\begin{tikzcd}
(\Lambda V)^{\otimes m+1} \arrow[r, tail] \arrow[d, "\simeq"'] \arrow[dr, phantom, "po"] & (\Lambda V)^{\otimes m+1} \otimes \Lambda W_m \arrow[d, "\simeq"] \arrow[ddr, bend left, "\alpha\circ\gamma"] & \\
A^{\otimes m+1} \arrow[r, tail] \arrow[drr, bend right, "\mu_{m+1}"'] & A^{\otimes m+1} \otimes \Lambda W_m \arrow[dr, dashed, "\tau"] & \\
& & A.
\end{tikzcd}$$

This proves that $\operatorname{secat}(\varphi) \leq \operatorname{secat}(f_0)$. For the second inequality, just apply spatial realization functor, [6, Chapt. 17]. ■

Previous theorem combined with [2, Thm. 23] gives

Corollary 8. *Given f a continuous map, then*

$$\operatorname{secat}(f_0) \leq \operatorname{secat}(f).$$

Then we have for a map f that

$$\operatorname{nil}\ \ker H^*(f, \mathbb{Q}) \leq \operatorname{Hsecat}(f) \leq \operatorname{msecat}(f) \leq \operatorname{secat}(f_0) \leq \operatorname{secat}(f).$$

2 The invariant sc(f)

Recall that a **cdga** morphism $\psi\colon A \to B$ admits a homotopy retraction if there exists a map $r\colon (A \otimes \Lambda V, D) \to A$ such that $r \circ i = \mathrm{Id}_A$, where $i\colon A \rightarrowtail (A \otimes \Lambda V, D)$ is a relative Sullivan model for ψ. We now introduce the following upper bound to rational sectional category:

Definition 9. *Let $\varphi\colon A \to B$ be a surjective* **cdga** *morphism with kernel K and consider the projection*

$$\rho_m\colon (A, d) \to \left(\frac{A}{K^{m+1}}, \overline{d}\right).$$

Define:

(i) $\mathbf{sc}(\varphi)$ *as the smallest integer* m *such that* ρ_m *admits a homotopy retraction,*

(ii) $\mathrm{m}\mathbf{sc}(\varphi)$ *the smallest* m *such that* ρ_m *admits a homotopy retraction as* A*-modules,*

(iii) $\mathrm{H}\mathbf{sc}(\varphi)$ *the smallest* m *such that* $H(\rho_m)$ *is injective.*

If X is a space modelled by $(\Lambda V, d)$ and $\epsilon\colon (\Lambda V, d) \to \mathbb{Q}$ is the augmentation then $\mathrm{m}\mathbf{sc}(\epsilon)$ is the classical module category of X_0 and $\mathrm{H}\mathbf{sc}(\epsilon)$ is the rational Toomer invariant of X. Also, if $\mu\colon (\Lambda V, d) \otimes (\Lambda V, d) \to (\Lambda V, d)$ is the multiplication, then $\mathbf{sc}(\mu) = \mathbf{tc}(X)$ and $\mathrm{m}\mathbf{sc}(\mu) = \mathbf{mtc}(X)$, as defined in [11].

Observe that $\mathbf{sc}(\varphi)$, $\mathrm{m}\mathbf{sc}(\varphi)$ and $\mathrm{H}\mathbf{sc}(\varphi)$ are not invariants of the weak homotopy type of φ. This can be seen explicitly in

Example 10. *Consider* $A := (\Lambda(a,b)/(a^2), d)$ *the* **cdga** *defined as* $|a| = 4$, $|b| = 3$, $db = a$ *and* $\varphi\colon A \to \mathbb{Q}$ *the augmentation. Since* $B := (\Lambda v_7, 0)$ *is a minimal model for* A*, the augmentation* $\psi\colon B \to \mathbb{Q}$ *is weakly equivalent to* φ*. It is easy to see that* $\mathrm{secat}(\psi) = 1$ *while* $\mathrm{H}\mathbf{sc}(\varphi) \geq 2$.

This definition extends to continuous maps:

Definition 11. *Let* f *be a continuous map. Define:*

(i) $\mathbf{sc}(f)$ *as the least* $\mathbf{sc}(\varphi)$ *with* φ *a surjective model for* f*,*

(ii) $\mathrm{m}\mathbf{sc}(f)$ *as the smallest* $\mathrm{m}\mathbf{sc}(\varphi)$ *with* φ *a surjective model for* f*,*

(iii) $\mathrm{H}\mathbf{sc}(f)$ *as the smallest* $\mathrm{H}\mathbf{sc}(\varphi)$ *with* φ *a surjective model for* f*.*

Proposition 12. *For any surjective* **cdga** *morphism* φ*, we have*

(i) $\mathrm{secat}(\varphi) \leq \mathbf{sc}(\varphi)$,

(ii) $\mathrm{msecat}(\varphi) \leq \mathrm{m}\mathbf{sc}(\varphi)$,

(iii) $\mathrm{Hsecat}(\varphi) \leq \mathrm{H}\mathbf{sc}(\varphi)$.

Proof. Denote by $(A \otimes \Lambda Z_m, D) \xrightarrow{\simeq} \frac{A}{K^{m+1}}$ a relative Sullivan model for ρ_m. Since multiplication induces a map

$$\overline{\mu}\colon \frac{A^{\otimes m+1}}{K^{\otimes m+1}} \longrightarrow \frac{A}{K^{m+1}},$$

the relative lifting lemma gives a morphism α making commutative the diagram

$$\begin{tikzcd}
A^{\otimes m+1} \arrow[r, "\mu_{m+1}"] \arrow[d, tail] & A \arrow[r, tail] & A \otimes \Lambda Z_m \arrow[d, "\simeq"] \\
A^{\otimes m+1} \otimes \Lambda W_m \arrow[r, "\simeq"'] \arrow[urr, dashed, "\alpha"] & \frac{A^{\otimes m+1}}{K^{\otimes m+1}} \arrow[r, "\overline{\mu}"'] & A/K^{m+1},
\end{tikzcd}$$

If $r\colon (A \otimes \Lambda Z_m, D) \to A$ is a homotopy retraction for ρ_m then the desired map τ is given by $r \circ \alpha$. ∎

The following corollary includes Proposition 1.

Corollary 13. *If f is a continuous map, then*

(i) $\mathrm{secat}(f_0) \leq \mathbf{sc}(f)$,

(ii) $\mathrm{msecat}(f) \leq \mathrm{m}\mathbf{sc}(f)$,

(iii) $\mathrm{Hsecat}(f) \leq \mathrm{H}\mathbf{sc}(f)$.

We now prove that for computing $\mathbf{sc}(f)$ we can restrict to models for f between Sullivan algebras and the answer does not depend on the choice of the model. The following lemma is straightforward.

Lemma 14. *Consider the commutative* **cdga** *diagram where ω is a quasi-isomorphism,*

$$\begin{tikzcd} A \arrow[r, "\omega", "\simeq"'] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"] \\ C \arrow[r] & D. \end{tikzcd}$$

If ψ admits a homotopy retraction, then so does φ.

We can now prove

Lemma 15. *Let $\varphi\colon A \to B$ be a surjective* **cdga** *morphism and $\psi\colon (\Lambda T, d) \xrightarrow{\simeq} A$ a surjective Sullivan model for A. Then $\mathbf{sc}(\varphi \circ \psi) \leq \mathbf{sc}(\varphi)$, $\mathrm{m}\mathbf{sc}(\varphi \circ \psi) \leq \mathrm{m}\mathbf{sc}(\varphi)$ and $\mathrm{H}\mathbf{sc}(\varphi \circ \psi) \leq \mathrm{H}\mathbf{sc}(\varphi)$.*

Proof. The morphism ψ induces a diagram

$$\begin{tikzcd} \Lambda T \arrow[r, "\psi", "\simeq"'] \arrow[d] & A \arrow[d] \\ \frac{\Lambda T}{L^m} \arrow[r] & \frac{A}{K^m}, \end{tikzcd}$$

where L denotes the kernel of $\varphi \circ \psi$. The result follows by previous lemma. ■

Lemma 16. *Let $\varphi\colon (\Lambda V, d) \to B$ be a surjective* **cdga** *morphism and consider ϕ an extension of φ,*

$$\begin{tikzcd} (\Lambda V, d) \arrow[rr, two heads, "\varphi"] \arrow[dr, hook, "\simeq"'] & & B \\ & (\Lambda V, d) \otimes (\Lambda W, d). \arrow[ur, two heads, "\phi"'] & \end{tikzcd}$$

Then $\mathbf{sc}(\varphi) = \mathbf{sc}(\phi)$, $\mathrm{m}\mathbf{sc}(\varphi) = \mathrm{m}\mathbf{sc}(\phi)$ and $\mathrm{H}\mathbf{sc}(\varphi) = \mathrm{H}\mathbf{sc}(\phi)$.

Proof. Remark that W admits a basis of the form $\{v_i, w_i\}$ with $dv_i = w_i$ and that, by a change of variable, one can suppose that $\phi(W) = 0$. Now define on ΛW a derivation s of degree -1 by $s(w_i) = v_i$ and $s(v_i) = 0$. For each $l \geq 1$,

$$s \circ d + d \circ s \colon \Lambda^l W \to \Lambda^l W$$

is multiplication by l and thus $H(\Lambda^l W, d) = 0$. Now, denoting $K = \ker \varphi$ and $L = \ker \phi$, we have that $L = K \oplus \Lambda V \otimes \Lambda^+ W$ and $L^m = K^m \oplus I$ with

$$I = \sum_{l=0}^{m-1} K^l \otimes \Lambda^{\geq m-l} W,$$

with $K^0 := \Lambda V$. Since, as a vector space,

$$I = K^{m-1} \otimes \Lambda^+ W \oplus \frac{K^{m-2}}{K^{m-1}} \otimes \Lambda^{\geq 2} W \oplus \cdots \oplus \frac{K}{K^2} \otimes \Lambda^{\geq m-1} W \oplus \frac{\Lambda V}{K} \otimes \Lambda^{\geq m} W,$$

an inductive argument shows that $H(I) = 0$. As the five lemma gives a diagram

$$\begin{tikzcd} \Lambda V \arrow[r, tail, "\simeq"] \arrow[d] & \Lambda V \otimes \Lambda W \arrow[d] \\ \frac{\Lambda V}{K^m} \arrow[r, "\simeq"] & \frac{\Lambda V \otimes \Lambda W}{L^m}, \end{tikzcd}$$

the lemma follows. ■

Corollary 17. *Let f be a continuous map and φ be a surjective model for f between Sullivan algebras. Then $\mathbf{sc}(f) = \mathbf{sc}(\varphi)$, $\mathrm{m}\mathbf{sc}(f) = \mathrm{m}\mathbf{sc}(\varphi)$, $\mathrm{H}\mathbf{sc}(f) = \mathrm{H}\mathbf{sc}(\varphi)$.*

3 The case $H(\varphi)$ surjective

Suppose $\varphi\colon A \to B$ is a surjective morphism with $H(\varphi)$ also surjective and write $K = \ker\ \varphi$. Then the short exact sequence

$$\begin{tikzcd} 0 \arrow[r, hook] & K \arrow[r, hook] & A \arrow[r, two heads] & B \arrow[r, two heads] & 0 \end{tikzcd}$$

yields the short exact sequence

$$\begin{tikzcd} 0 \arrow[r, hook] & H(K) \arrow[r, hook] & H(A) \arrow[r, two heads] & H(B) \arrow[r, two heads] & 0, \end{tikzcd}$$

which tells us that $\operatorname{nil}\ \ker H(\varphi) = \operatorname{nil} H(\ker \varphi)$. Moreover, the homology of the projection

$$q_m\colon A^{\otimes m+1} \longrightarrow \frac{A^{\otimes m+1}}{K^{\otimes m+1}}$$

is given by

$$H(q_m)\colon H(A)^{\otimes m+1} \longrightarrow \frac{H(A)^{\otimes m+1}}{H(K)^{\otimes m+1}}.$$

Example 18. *Consider the surjective morphism*

$$\varphi : (\Lambda(a_3, b_3, x_5);\ dx = ab) \longrightarrow (\Lambda(a,b)/(ab), 0)$$

whose kernel is $K := (ab, x)$. *We have that* $\mathbf{sc}(\varphi) = 2$ *since the projection* $\rho_1 \colon \Lambda(a, b, x) \to \frac{\Lambda(a,b,x)}{(abx)}$ *is not injective in homology and* $\operatorname{nil} K = 2$. *On the other hand, because of previous remarks and the fact that* $H(K) = H^{\geq 8}(K)$, *we have a commutative diagram*

$$\begin{tikzcd}
(\Lambda(a,b,x))^{\otimes 2} \arrow[r] \arrow[dr, hook] \arrow[dd, "\mu_2"'] & \frac{(\Lambda(a,b,x))^{\otimes 2}}{K^{\otimes 2}} \\
 & (\Lambda(a,b,x;d)^{\otimes 2} \otimes \Lambda W_1, D) \arrow[u, "\simeq"'] \arrow[dl, "\tau"] \\
\Lambda(a,b,x;d) &
\end{tikzcd}$$

with $W_1 = W_1^{\geq 15}$ *and* $\tau(W_1) = 0$. *This shows that* $\operatorname{secat}(\varphi) = 1 < 2 = \mathbf{sc}(\varphi)$.

The idea for computing $\operatorname{secat}(\varphi)$ in the previous example can be generalized:

Proposition 19. *Let* $\varphi \colon A \to B$ *be a surjective* **cdga** *morphism such that* $H(\varphi)$ *is also surjective,* $A = A^{<l}$ *and* $H(K) = H^{\geq k}(K)$. *Then*

$$\operatorname{secat}(\varphi) \leq \frac{l+1}{k}.$$

Proof. In this case, since $W_m = W_m^{\geq (m+1)k-1}$, a morphism r making the diagram

$$\begin{tikzcd}
A^{\otimes m+1} \arrow[r] \arrow[dr, hook, "i"] \arrow[dd, "\mu_{m+1}"'] & \frac{A^{\otimes m+1}}{K^{\otimes m+1}} \\
 & (A^{\otimes m+1} \otimes \Lambda W_m, D) \arrow[u, "\simeq"'] \arrow[dl, "\tau"] \\
A, &
\end{tikzcd}$$

commute can be defined as $r(a) := \mu_{m+1}(a)$, for $a \in A^{\otimes m+1}$ and $r(W_m) := 0$. ■

4 Homology nilpotency of an ideal

Consider a surjective **cdga** morphism $\varphi \colon A \twoheadrightarrow B$ with $K = \ker \varphi$, then, by Proposition 12, $\operatorname{secat}(\varphi) \leq \operatorname{nil} K$ but when A is a Sullivan algebra then it is very *likely* that $\operatorname{nil} K = \infty$.

Definition 20. *Let* I *be an ideal of a* **cdga** A, *the* homology nilpotency *of* I *is*

$$\operatorname{Hnil} I := \min \left\{ k \colon I^{k+1} \subset J,\ J \textit{ acyclic ideal of } A \right\}.$$

Remark that if K^{m+1} is included in an acyclic ideal J of A then we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, two heads, "\rho_m"] \arrow[dr, "\simeq"'] & \frac{A}{K^{m+1}} \arrow[d] \\
 & \frac{A}{J}
\end{tikzcd}$$

which can be used to deduce a homotopy retraction of ρ_m. As a consequence we have

Proposition 21. *Let φ be a surjective* **cdga** *morphism with $K := \ker \varphi$. Then,*

$$\operatorname{nil}\ker H(\varphi) \leq \operatorname{secat}(\varphi) \leq \mathbf{sc}(\varphi) \leq \operatorname{Hnil} K \leq \operatorname{nil} K,$$

and, if $H(\varphi)$ is surjective,

$$\operatorname{nil} H(K) \leq \operatorname{secat}(\varphi) \leq \operatorname{Hnil}(K).$$

Example 22. *Consider $A = (\Lambda a_2, x; dx = a^2)$ and ϵ the augmentation on A. Then, since $K^2 \subset (a^2, x)$, $\operatorname{cat}(A) = \operatorname{secat}(\epsilon) = \mathbf{sc}(\epsilon) = \operatorname{Hnil}(K) = 1$, and $\operatorname{nil}(K) = \infty$.*

5 Sectional category of formal morphisms

As the LS category of formal spaces, the sectional category of formal maps is very easy to compute.

Definition 23. *A* **cdga** *morphism φ is said to be* formal *if it is weakly equivalent to $H(\varphi)$. A continuous map is said to be* formal *if it admits a formal* **cdga** *model.*

For more on formal morphisms the reader is referred to [7] and [16]. It is obvious from this definition that if $\varphi\colon A \to B$ is formal then both A and B are formal as well.

Theorem 24. *Let $\varphi\colon A \to B$ be a formal morphism with $H(\varphi)$ surjective. Then*

$$\operatorname{secat}(\varphi) = \operatorname{nil}(\ker H(\varphi)).$$

Proof. By formality, $\operatorname{secat}(\varphi) = \operatorname{secat}(H(\varphi))$. Write $m := \operatorname{nil}(\ker H(\varphi))$, we must prove that $\operatorname{secat}(H(\varphi)) \leq m$ but this is direct consequence of Proposition 12 and the fact that $(\ker H(\varphi))^{m+1} = \{0\}$. ■

6 Applications

As a direct consequence of Theorem 7 we have a rational model for topological complexity:

Proposition 25. *Let A be a* **cdga** *model for a space X and $\mu_n\colon A^{\otimes n} \to A$ the n-multiplication. Then*

$$\mathrm{TC}_n(X_0) = \operatorname{secat}(\mu_n).$$

Proof. Since rationalisation commutes with limits, we have

$$\mathrm{TC}_n(X_0) = \operatorname{secat}(\Delta^n_{X_0}) = \operatorname{secat}((\Delta^n_X)_0) = \operatorname{secat}(\mu_n). \quad ■$$

We extend this proposition to

Definition 26. *Let X be a space, then define* *(i)* $\mathrm{mTC}_n(X) := \mathrm{msecat}(\Delta_n)$,

(ii) $\mathrm{HTC}_n(X) := \mathrm{Hsecat}(\Delta_n)$.

It was proven by L. Lechuga and A. Murillo in [12] that the topological complexity of formal spaces equals the nilpotency of the kernel of the multiplication $H^*(X,\mathbb{Q}) \otimes H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{Q})$. We extend this result:

Theorem 27. *If X is a formal space, then*

$$\mathrm{TC}_n(X_0) = \mathrm{nil}\ \ker\left(\mu_n \colon H^*(X,\mathbb{Q})^{\otimes n} \to H^*(X,\mathbb{Q})\right),$$

and thus $\mathrm{TC}_n(X_0) = \mathrm{mTC}_n(X_0) = \mathrm{HTC}_n(X_0)$.

Proof. If X is a formal space, Δ_X^n is a formal map modelled by μ_n. The result follows from Theorem 24. ■

Observe now that if G is a subset of a **cdga** A and I the ideal of A generated by G then nil I is the largest m for which there exist $x_1, \dots, x_m \in G$ such that $x_1 \cdots x_m \neq 0$. Also, if A^+ is generated by $\{x_i\}_{i \in I}$, then K_n, the kernel of the n-multiplication morphism $\mu_n \colon A^{\otimes n} \to A$, is generated by $\{x_{i,j} - x_{i,1} : i \in I, 2 \leq j \leq n\}$ where

$$x_{i,j} = 1 \otimes \cdots \otimes 1 \otimes x_i \otimes 1 \otimes \cdots \otimes 1 \in A^{\otimes j-1} \otimes A^+ \otimes A^{\otimes n-j}.$$

This is consequence of the fact that an element $x_1 \otimes \cdots \otimes x_n$ can be written in the form

$$\begin{aligned}
&(x_{1,1} \cdots x_{n-1,n-1}) \cdot (x_{n,n} - x_{n,n-1}) + \\
&(x_{1,1} \cdots x_{n-2,n-2}) \cdot (x_{n-1,n-1}x_{n,n-1} - x_{n-1,n-2}x_{n,n-2}) + \\
&\qquad \vdots \\
&(x_{1,1}) \cdot (x_{2,2} \cdots x_{n,2} - x_{2,1} \cdots x_{n,1}) + \\
&(x_{1,1} \cdots x_{n,1}).
\end{aligned}$$

Proposition 28. *Let A be a* **cdga** *and K_n the kernel of the n-multiplication morphism $A^{\otimes n} \to A$. Then for $n \geq 3$,*

$$\mathrm{nil}\, K_n \geq \mathrm{nil}\, K_{n-1} + \mathrm{nil}\, A^+.$$

Proof. Write $r = \mathrm{nil}\, K_{n-1}$ and $s = \mathrm{nil}\, A^+$. Consider $\omega \neq 0$ a product of r factors in K_{n-1} and $\alpha = a_1 \cdots a_s \neq 0$ with $a_i \in A^+$. Then the element $(\omega \otimes 1)$ $(a_{1,n} - a_{1,1}) \cdots (a_{s,n} - a_{s,1}) = (\omega \otimes \alpha) + \xi$ with $\xi \in A^{\otimes n-1} \otimes A^{<|\alpha|}$ must be nonzero. This proves that nil $K_n \geq r + s$. ■

As pointed out by Bárbara Gutiérrez, [10], the inverse inequality does not hold:

Example 29. *Consider the* **cdga** $A = \Lambda(a,b,c,d,e)/I$ *generated by elements of odd degree where* I *is the ideal generated by* ad, ae, bcd, bce. *The only non-zero products of length* 3 *of* A *are* abc, bde *and* cde *and there are no non-zero products of length* 4, *then we have that* $\mathrm{nil}\ A = 3$ $\mathrm{nil}\ K_2 = 5$. *But* $\mathrm{nil}\ K_3 = 9$, *since*

$$\omega := \prod_{x=a}^{e}(x\otimes 1\otimes 1 - 1\otimes x\otimes 1)\prod_{x=b}^{e}(x\otimes 1\otimes 1 - 1\otimes 1\otimes x),$$

is a non-zero element of K_3^9 *because the non-zero summand* $bde \otimes abc \otimes cde$ *appears only once when we develop* ω.

Corollary 30. *Suppose* A *is a* **cdga** *satisfying* $A = A^{\text{even}}$. *Then*

$$\mathrm{nil}\ K_n = n(\mathrm{nil}\ A^+).$$

Proof. For $n = 2$, choose elements $x_1,\ldots,x_r \in A^+$ with $r = \mathrm{nil}\ A^+$ such that the product $x_1\cdots x_r$ is non-zero. Then $\prod_{i=1}^{r}(x_i\otimes 1 - 1\otimes x_i)^2$ is a non-zero element in K_2^{2r}. Since $\mathrm{nil}\ K_2 \leq \mathrm{nil}\ (A\otimes A)^+ = 2(\mathrm{nil}\ A^+)$, we get $\mathrm{nil}\ K_2 = 2(\mathrm{nil}\ A^+)$. The result now follows by induction and Proposition 28. ■

Since for formal spaces X, $cat(X_0) = \mathrm{nil}\ H^+(X,\mathbb{Q})$ (apply Theorem 27 to $* \hookrightarrow X$), Proposition 28 combined with Theorem 27 directly implies

Theorem 31. *Let* X *be a formal space, then for* $n \geq 2$,

$$\mathrm{TC}_n(X_0) \geq \mathrm{TC}(X_0) + (n-2)\mathrm{cat}(X_0).$$

Recall that the wedge of two formal spaces remains a formal space.

Proposition 32. *Let* X *and* Y *be formal spaces, then*

$$\mathrm{TC}(X\vee Y) \geq \mathrm{cat}(X) + \mathrm{cat}(Y).$$

Proof. Let w_1 and w_2 be monomials in $H^*(X)$ and $H^*(Y)$ of maximal length n and m, $w_1 = a_1\cdots a_n$ and $w_2 = b_1\cdots b_m$. Then, using the notation $a^- = 1\otimes a - a\otimes 1$, the identity

$$a_1^-\cdots a_n^- b_1^-\cdots b_m^- = w_1\otimes w_2 \pm w_2\otimes w_1$$

shows that $\mathrm{TC}(X\vee Y) \geq n+m$. ■

Example 33. *Let* X *be the wedge* $(S^3\times S^3)\vee(S^3\times S^3)$. *Then* $\mathrm{TC}(S^3\times S^3) = 2$ *but* $\mathrm{TC}(X) = 4$.

The cohomology of X is $\Lambda(x,y,z,t)/(xz,xt,yz,yt)$, with x,y,z,t in degree 3. Then writing $x^- = 1\otimes x - x\otimes 1$ and so on, we see that $x^-y^-z^-t^- = xy\otimes zt + zt\otimes xy$. This shows that $\mathrm{nil}\ \ker\mu_2 = 4$ and $\mathrm{TC}(X) \geq 4$. On the other hand, $\mathrm{TC}(X) \leq \mathrm{cat}(X\times X) = 4$.

We now generalize Theorem 1.4 in [11].

Theorem 34. *If* $\pi_*(X)\otimes\mathbb{Q}$ *is finite dimensional and concentrated in odd degrees. Then* $\mathrm{TC}_n(X_0) = (n-1)\mathrm{cat}(X_0)$.

Proof. Let $(A,d)=(\Lambda\,(x_1,\dots,x_r)\,,d)$ be a model of X with x_i in odd degree. Then

$$(A,d)^{\otimes n}=(\Lambda\,(x_{1,1},\dots,x_{r,1},x_{1,2},\dots,x_{2,r},\dots,x_{1,n},\dots,x_{r,n})\,,d)$$

and K the kernel of the n multiplication is generated by the elements

$$\left\{x_{i,j}-x_{i,1}:\ 1\le i\le r,\ 2\le j\le n\right\}.$$

Since the square of these elements is zero, we have that $K^{r(n-1)+1}=0$ and so $\mathrm{TC}_n(X)\geq r(n-1)$.

Now consider the pullback diagram

$$\begin{tikzcd}
PX \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "pb"] & X^{[0,1]} \arrow[d, "p"] \\
X^{n-1}\times * \arrow[r] & X^n
\end{tikzcd}$$

where $p(\omega)=\Big(\omega(0),\omega(\frac{1}{n-1}),\omega(\frac{2}{n-1}),\dots,\omega(1)\Big)$. Since $\mathrm{TC}_n(X)=\mathrm{secat}(p)$ (Proposition 12), $\mathrm{cat}(X^{n-1})=\mathrm{secat}(q)$ and $\mathrm{secat}(q)\leq\mathrm{secat}(p)$, we have

$$\mathrm{TC}_n(X_0)\leq r(n-1)=\mathrm{nil}(A^{\otimes n-1})=\mathrm{cat}(X_0^{n-1})\leq\mathrm{TC}_n(X_0). \qquad \blacksquare$$

A similar result can be found in [13] for integral H-spaces.

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