Dual cyclic Brunn-Minkowski inequalities

Chang-Jian Zhao*

Abstract

New cyclic Brunn-Minkowski inequalities for dual quermassintegrals of star bodies are established. These inequalities are obtained with respect to radial Minkowski addition as well as radial Blaschke and harmonic Blaschke addition.

1 Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. We use $V(K)$ for the n-dimensional volume of a body K. Let $|\cdot|_\infty$ denote the maximum-norm on the space of continuous functions $C(S^{n-1})$ on S^{n-1}.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot): S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, i.e., for $K, L \in \mathcal{S}^n$, $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$ (see e.g. [9] or [24]). In the following we recall some basic notions and results from the dual Brunn-Minkowski inequality (see, e.g., [12], [14], [15], [18], [22], [25] and the references therein).

*Research is supported by National Natural Science Foundation of China (11371334).

Received by the editors in December 2013 - In revised form in December 2014.

Communicated by J. Thas.

2010 *Mathematics Subject Classification* : 52A40, 52A30.

Key words and phrases : Radial Minkowski addition, radial Blaschke addition, harmonic Blaschke addition, dual Brunn-Minkowski inequality.

1.1 Radial Minkowski addition and dual mixed volumes

For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, was defined by Lutwak by (see [19])

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}.$$

It has the following important property, for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$,

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot) \qquad (1.1).$$

For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \ldots \tilde{+} \lambda_r K_r$ is a homogeneous polynomial of degree n in the λ_i,

$$V(\lambda_1 K_1 \tilde{+} \ldots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \qquad (1.2)$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (1.2) to be symmetric in their argument, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ and is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$, the dual mixed volumes $\tilde{V}(K_1, \ldots, K_n)$ is written as $\tilde{W}_i(K)$ (see e.g. [8]).

For $K_i \in \mathcal{S}^n$, the dual mixed volumes satisfy (see [20])

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \qquad (1.3)$$

For $K, L \in \mathcal{S}^n$ and $i \in \mathbb{R}$, the ith dual mixed volume of K and L, $\tilde{V}_i(K, L)$, is defined by,

$$\tilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i dS(u). \qquad (1.4)$$

From (1.4) and in view of $\rho(B, u) = 1$, we have

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u), \ i \in \mathbb{R}. \qquad (1.5)$$

1.2 Radial Blaschke addition

If $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$, then $\lambda \cdot K \check{+} \mu \cdot L$ is the star body whose radial function is given by

$$\rho(\lambda \cdot K \check{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \qquad (1.6)$$

This addition is called radial Blaschke addition (see e.g. [19]).

Lutwak established the following dual Brunn-Minkowski inequality for radial Blaschke addition.

If $K, L \in \mathcal{S}^n$, then

$$V(K \check{+} L)^{(n-1)/n} \leq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \qquad (1.7)$$

with equality if and only if K and L are dilates.

This inequality is dual to the Kneser-Süss inequality (see [19]).

1.3 Harmonic Blaschke addition

Another addition called harmonic Blaschke addition was defined by Lutwak [21]. Suppose $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$ (not both zero). In order to define the harmonic Blaschke linear combination, $\lambda K \hat{+} \mu L$, first let $\xi > 0$ be defined by

$$\xi^{1/(n+1)} = \frac{1}{n}\int_{S^{n-1}}[\lambda V(K)^{-1}\rho(K,u)^{n+1} + \mu V(L)^{-1}\rho(L,u)^{n+1}]^{n/(n+1)}dS(u).$$

Then $\lambda K \hat{+} \mu L \in \mathcal{S}^n$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1}\rho(K,\cdot)^{n+1} + \mu V(L)^{-1}\rho(L,\cdot)^{n+1}.$$

It follows immediately that $\xi = V(\lambda K \hat{+} \mu l)$, and hence

$$\frac{\rho(\lambda K \hat{+} \mu L, \cdot)^{n+1}}{V(\lambda K \hat{+} \mu L)} = \lambda\frac{\rho(K,\cdot)^{n+1}}{V(K)} + \mu\frac{\rho(L,\cdot)^{n+1}}{V(L)}. \tag{1.8}$$

Lutwak established the following dual Brunn-Minkowski inequality for harmonic Blaschke addition (see [21]). If $K, L \in \mathcal{S}^n$ and $\lambda, \mu > 0$, then

$$V(\lambda K \hat{+} \mu L)^{1/n} \geq \lambda V(K)^{1/n} + \mu V(L)^{1/n}, \tag{1.9}$$

with equality if and only if K and L are dilates.

For more information and characterizations of additions and related binary operations on convex and star bodies we refer to [6], [10], [11] and [13].

2 Statements of main results

The dual Brunn-Minkowski inequality for radial Minkowski addition was established in [20]. If $K, L \in \mathcal{S}^n$, then

$$V(K \tilde{+} L)^{\frac{1}{n}} \leq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{2.1}$$

with equality if and only if K and L are dilates.

A general version of the dual Brunn-Minkowski inequality is the following (see [27]):

Theorem A *If $K, L \in \mathcal{S}^n$ and $i < n-1$, then*

$$\tilde{W}_i(K \tilde{+} L)^{\frac{1}{n-i}} \leq \tilde{W}_i(K)^{\frac{1}{n-i}} + \tilde{W}_i(L)^{\frac{1}{n-i}}, \tag{2.2}$$

with equality if and only if K and L are dilates.

The following cyclic inequality for dual mixed volumes was established in [20].

Theorem B *If $K, L \in \mathcal{S}^n$ and $j < i < k$, then*

$$\tilde{W}_i(K)^{k-j} \leq \tilde{W}_j(K)^{k-i}\tilde{W}_k(K)^{i-j} \tag{2.3}$$

with equality if and only if K is an n-ball centered at the origin.

The first aim of the present paper is to establish the following new dual cyclic Brunn-Minkowski inequality for radial Minkowski addition, which has inequalities (2.1), (2.2) and (2.3), as special cases!

Theorem 2.1 *Let $K, L \in \mathcal{S}^n$ and $i, j, k \in \mathbb{R}$. If $j \leq i \leq k$ and $i < n-1$, or $k \leq i \leq j$ and $i < n-1$, then*

$$\tilde{W}_i(K\tilde{+}L)^{\frac{1}{n-i}} \leq \tilde{W}_j(K)^{\frac{k-i}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{i-j}{(n-i)(k-j)}} + \tilde{W}_j(L)^{\frac{k-i}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{i-j}{(n-i)(k-j)}} \quad (2.4)$$

with equality for $j \neq k$ if and only if K and L are n-balls centered at the origin, and for $i = j$ or $i = k$ if and only if K and L are dilates.

The inequality is reversed if $k \geq j \geq i$ and $n-1 < i < n$, or $k \leq j \leq i$ and $n-1 < i < n$.

Remark 2.1 Taking $i = j = 0$ or $k = i = 0$ in (2.4), (2.4) reduces to (2.1).

Taking $i = j$ or $k = i$ in (2.4), (2.4) changes to the dual Brunn-Minkowski inequality (2.2). If K or L is a single point in (2.4), (2.4) becomes the cyclic inequality inequality (2.3). This shows that inequality (2.4) is not only a generalization of inequalities (2.2) and (2.3), but also a perfect fusion of inequalities (2.2) and (2.3).

Another aim of the present paper is to establish the following new cyclic Brunn-Minkowski inequalities for radial Blaschke addition and harmonic Blaschke addition, respectively.

Theorem 2.2 *Let $K, L \in \mathcal{S}^n$ and $i, j, k \in \mathbb{R}$. If $j \leq i \leq k$ and $i < 1$, or $k \leq i \leq j$ and $i < 1$, then*

$$\tilde{W}_i(K\check{+}L)^{\frac{n-1}{n-i}} \leq \tilde{W}_j(K)^{\frac{(k-i)(n-1)}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{(i-j)(n-1)}{(n-i)(k-j)}} + \tilde{W}_j(L)^{\frac{(k-i)(n-1)}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{(i-j)(n-1)}{(n-i)(k-j)}} \quad (2.5)$$

with equality for $j \neq k$ if and only if K and L are n-balls centered at the origin, and for $i = j$ or $i = k$ if and only if K and L are dilates.

The inequality is reversed if $k \geq j \geq i$ and $1 < i < n$, or $k \leq j \leq i$ and $1 < i < n$.

Remark 2.2 Taking for $i = j = 0$ or $i = k = 0$ in Theorem 2.2, (2.5) reduces to (1.7).

Taking $k = i$ or $i = j$ in (2.5), (2.5) changes the following result: If $K, L \in \mathcal{S}^n$ and $i < 1$, then

$$\tilde{W}_i(K\check{+}L)^{\frac{n-1}{n-i}} \leq \tilde{W}_i(K)^{\frac{n-1}{n-i}} + \tilde{W}_i(L)^{\frac{n-1}{n-i}},$$

with equality if and only if K and L are dilates. This inequality is reversed if $1 < i < n$.

This Brunn-Minkowski type inequality for radial Blaschke addition was previously established in [28].

Moreover, if K or L is a single point, then (2.5) reduces to (2.3).

Theorem 2.3 *Let $K, L \in \mathcal{S}^n$, $\lambda, \mu \geq 0$, and $i, j, k \in \mathbb{R}$. If $j \leq i \leq k$ and $i < -1$, or $k \leq i \leq j$ and $i < -1$, then*

$$\frac{\tilde{W}_i(\lambda K\hat{+}\mu L)^{\frac{n+1}{n-i}}}{V(\lambda K\hat{+}\mu L)} \leq \lambda\frac{\tilde{W}_j(K)^{\frac{(k-i)(n+1)}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{(i-j)(n+1)}{(n-i)(k-j)}}}{V(K)} + \mu\frac{\tilde{W}_j(L)^{\frac{(k-i)(n+1)}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{(i-j)(n+1)}{(n-i)(k-j)}}}{V(L)} \quad (2.6)$$

with equality for $j \neq k$ if and only if K and L are n-balls centered at the origin, and for $i = j$ or $i = k$ if and only if K and L are dilates.

The inequality is reversed if $k \geq j \geq i$ and $-1 < i < n$, or $k \leq j \leq i$ and $-1 < i < n$.

Remark 2.3 Taking $i = j = 0$ or $i = k = 0$ in Theorem 2.3, (2.6) reduces to (1.9).

Taking $i = j$ or $k = i$ in Theorem 2.3, (2.6) reduces to the following result. If $K, L \in \mathcal{S}^n$ and $i < -1$, then

$$\frac{\tilde{W}_i(\lambda K \hat{+} \mu L)^{\frac{n+1}{n-i}}}{V(\lambda K \hat{+} \mu L)} \leq \lambda \frac{\tilde{W}_i(K)^{\frac{n+1}{n-i}}}{V(K)} + \mu \frac{\tilde{W}_j(L)^{\frac{n+1}{n-i}}}{V(L)},$$

with equality if and only if K and L are dilates. This inequality is reversed if $-1 < i < n$.

This is an inequality previously established in [28].

Taking $\lambda = 1$ and $\mu = 0$, or $\mu = 1$ and $\lambda = 0$ in Theorem 2.3 reduces to the cyclic inequality (2.3).

For more information on Brunn-Minkowski type inequalities for different geometric functionals we refer to the recent articles [1], [2], [3], [5], [7], [23] and [26].

3 Proofs of the main results

In order to prove our main results, we first derive a new norm inequality. In the following let $f, g \in C((S^{n-1})$ and denote by $\|\cdot\|_q$ the usual L_q-norm for functions on S^{n-1}.

Lemma 3.1 *Let $f, g \geq 0$ and $r, s, t \in \mathbb{R}$. If $s \geq r \geq t$ and $r > 1$, or $t \geq r \geq s$ and $r > 1$, then*

$$\|f(x) + g(x)\|_r \leq \|f(x)\|_s^{\frac{s(r-t)}{r(s-t)}} \|f(x)\|_t^{\frac{t(s-r)}{r(s-t)}} + \|g(x)\|_s^{\frac{s(r-t)}{r(s-t)}} \|g(x)\|_t^{\frac{t(s-r)}{r(s-t)}}, \tag{3.1}$$

with equality for $s \neq t$ if and only if f and g are constants, and for $r = s$ or $r = t$ if and only if f and g are proportional.

The inequality is reversed if $t \leq s \leq r$ and $0 < r < 1$, or $t \geq s \geq r$ and $0 < r < 1$.

Proof We first suppose that $s > r > t$ or $t > r > s$. Notice that

$$s > r > t \ \text{ or } \ t > r > s \text{ implies } \frac{s-t}{r-t} > 1.$$

Using the Minkowski inequality ([4, p.21]) for $r > 1$ and followed by the Hölder inequality ([4, p.22]) with exponents $\frac{s-t}{r-t} > 1$ and $\frac{s-t}{s-r}$, we obtain

$$\begin{aligned} \|f(x) + g(x)\|_r &\leq \|f(x)\|_r + \|g(x)\|_r \\ &= \|f(x)^{\frac{s(r-t)}{r(s-t)}} f(x)^{\frac{t(s-r)}{r(s-t)}}\|_r + \|g(x)^{\frac{s(r-t)}{r(s-t)}} g(x)^{\frac{t(s-r)}{r(s-t)}}\|_r \end{aligned}$$

$$\leq \|f(x)\|_s^{\frac{s(r-t)}{r(s-t)}} \|f(x)\|_t^{\frac{t(s-r)}{r(s-t)}} + \|g(x)\|_s^{\frac{s(r-t)}{r(s-t)}} \|g(x)\|_t^{\frac{t(s-r)}{r(s-t)}}. \tag{3.2}$$

In the following, we discuss the equality conditions. In view of the equality conditions of the Minkowski and the Hölder inequalities, it follows that equality in (3.1) holds if and only if f and g are proportional, f^λ and f^μ are proportional and g^λ and g^μ are proportional, where $\lambda = \frac{r-t}{s-t}$ and $\mu = \frac{s-r}{s-t}$. Since $s \neq t$, it follows that f and g have to be constant.

Next, consider the case $r = t$ or $r = s$. If $r = t$ or $r = s$, then using only the Minkowski integral inequality for $r > 1$, we have

$$\|f(x) + g(x)\|_r \leq \|f(x)\|_r + \|g(x)\|_r$$

$$= \|f(x)\|_s^{\frac{s(r-t)}{r(s-t)}} \|f(x)\|_t^{\frac{t(s-r)}{r(s-t)}} + \|g(x)\|_s^{\frac{s(r-t)}{r(s-t)}} \|g(x)\|_t^{\frac{t(s-r)}{r(s-t)}} \tag{3.3}$$

with equality if and only if f and g are proportional. This proves inequality (3.1) in the case $r = t$ or $r = s$ and with equality if and only if f and g are proportional.

Theorem 3.1 *Let $K, L \in \mathcal{S}^n$ and $i, j, k \in \mathbb{R}$. If $j \leq i \leq k$ and $i < n-1$, or $k \leq i \leq j$ and $i < n-1$, then*

$$\tilde{W}_i(K \tilde{+} L)^{\frac{1}{n-i}} \leq \tilde{W}_j(K)^{\frac{k-i}{(n-i)(k-j)}} \tilde{W}_k(K)^{\frac{i-j}{(n-i)(k-j)}} + \tilde{W}_j(L)^{\frac{k-i}{(n-i)(k-j)}} \tilde{W}_k(L)^{\frac{i-j}{(n-i)(k-j)}} \tag{3.4}$$

with equality for $j \neq k$ if and only if K and L are n-balls centered at the origin, and for $i = j$ or $i = k$ if and only if K and L are dilates.

The inequality is reversed if $k \geq j \geq i$ and *$n-1 < i < n$, or $k \leq j \leq i$ and $n-1 < i < n$.*

Proof We first consider the case that: $j \leq i \leq k$ and $i < n-1$, *or* $k \leq i \leq j$ and $i < n-1$.

From (1.1), (1.5), and Lemma 3.1, it follows that $s \geq r \geq t$ and $r > 1$, or $t \geq r \geq s$ and $r > 1$, then we have with $r = n - i$,

$$\tilde{W}_i(K \tilde{+} L)^{\frac{1}{r}} = \left(\frac{1}{n}\int_{S^{n-1}} \rho(K \tilde{+} L, u)^r dS(u)\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} (\rho(K, u) + \rho(L, u))^r dS(u)\right)^{\frac{1}{r}}$$

$$\leq \left(\frac{1}{n}\int_{S^{n-1}} \rho(K, u)^s dS(u)\right)^{\frac{r-t}{r(s-t)}} \left(\frac{1}{n}\int_{S^{n-1}} \rho(K, u)^t dS(u)\right)^{\frac{s-r}{r(s-t)}}$$

$$+ \left(\frac{1}{n}\int_{S^{n-1}} \rho(L, u)^s dS(u)\right)^{\frac{r-t}{r(s-t)}} \left(\frac{1}{n}\int_{S^{n-1}} \rho(L, u)^t dS(u)\right)^{\frac{s-r}{r(s-t)}}.$$

By (1.5), we have for $K \in \mathcal{S}^n$ and $s, t \in \mathbb{R}$

$$\tilde{W}_{n-s}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^s dS(u), \quad \tilde{W}_{n-t}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^t dS(u).$$

Hence,

$$\left(\tilde{W}_i(K \tilde{+} L)\right)^{\frac{1}{r}} \leq \tilde{W}_{n-s}(K)^{\frac{r-t}{r(s-t)}} \tilde{W}_{n-t}(K)^{\frac{s-r}{r(s-t)}} + \tilde{W}_{n-s}(L)^{\frac{r-t}{r(s-t)}} \tilde{W}_{n-t}(L)^{\frac{s-r}{r(s-t)}}. \tag{3.5}$$

If we take $r = n - i$, $s = n - j$ and $t = n - k$ in (3.5), then since $r > 1$ we have $i < n - 1$ and

$$s \geq r \geq t \text{ or } t \geq r \geq s \text{ implies } j \leq i \leq k \text{ or } k \leq i \leq j.$$

Thus, we obtain

$$\left(\tilde{W}_i(K\check{+}L)\right)^{\frac{1}{n-i}} \leq \tilde{W}_j(K)^{\frac{k-i}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{i-j}{(n-i)(k-j)}} + \tilde{W}_j(L)^{\frac{k-i}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{i-j}{(n-i)(k-j)}}.$$

From that equality conditions of Lemma 3.1, it follows the equality in (3.4) holds for $j \neq k$ if and only if $\rho(K,u)$ and $\rho(L,u)$ are constants, it follows that equality in (3.4) holds for $j \neq k$ if and only if K and L are n-balls centered at the origin. Moreover, if $i = j$ or $i = k$, equality in (3.4) holds if and only if K and L are dilates.

Similar to the above proof, we can also establish the reverse inequality for $k \geq j \geq i$ and $n - 1 < i < n$, *or* $k \leq j \leq i$ and $n - 1 < i < n$. Here, we omit the details.

Theorem 3.2 *Let $K, L \in \mathcal{S}^n$ and $i, j, k \in \mathbb{R}$. If $j \leq i \leq k$ and $i < 1$, or $k \leq i \leq j$ and $i < 1$, then*

$$\tilde{W}_i(K\check{+}L)^{\frac{n-1}{n-i}} \leq \tilde{W}_j(K)^{\frac{(k-i)(n-1)}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{(i-j)(n-1)}{(n-i)(k-j)}} + \tilde{W}_j(L)^{\frac{(k-i)(n-1)}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{(i-j)(n-1)}{(n-i)(k-j)}}, \quad (3.6)$$

with equality for $j \neq k$ if and only if K and L are n-balls centered at the origin, and for $i = j$ or $i = k$ if and only if K and L are dilates.

The inequality is reversed if $k \geq j \geq i$ and $1 < i < n$, or $k \leq j \leq i$ and $1 < i < n$.

Proof From (1.5), (1.6), and Lemma 3.1, it follows that if $s \geq r \geq t$ and $r > 1$, or $t \geq r \geq s$ and $r > 1$, we have with $r = \frac{n-i}{n-1}$

$$\begin{aligned}
\tilde{W}_i(K\check{+}L)^{\frac{n-1}{n-i}} &= \left(\frac{1}{n}\int_{S^{n-1}} \rho(K\check{+}L,u)^{n-i} dS(u)\right)^{\frac{n-1}{n-i}} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} (\rho(K,u)^{n-1} + \rho(L,u)^{n-1})^r dS(u)\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{s(n-1)} dS(u)\right)^{\frac{r-t}{r(s-t)}} \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{t(n-1)} dS(u)\right)^{\frac{s-r}{r(s-t)}} \\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{s(n-1)} dS(u)\right)^{\frac{r-t}{r(s-t)}} \left(\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{t(n-1)} dS(u)\right)^{\frac{s-r}{r(s-t)}}.
\end{aligned}$$

By (1.5), we have

$$\tilde{W}_{n-s(n-1)}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{s(n-1)} dS(u), \quad \tilde{W}_{n-t(n-1)}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{t(n-1)} dS(u).$$

Hence,

$$\tilde{W}_i(K\check{+}L)^{\frac{1}{r}} \leq \tilde{W}_{n-s(n-1)}(K)^{\frac{r-t}{r(s-t)}}\tilde{W}_{n-t(n-1)}(K)^{\frac{s-r}{r(s-t)}} + \tilde{W}_{n-s(n-1)}(L)^{\frac{r-t}{r(s-t)}}\tilde{W}_{n-t(n-1)}(L)^{\frac{s-r}{r(s-t)}}. \quad (3.7)$$

Taking $r=\frac{n-i}{n-1}$, $s=\frac{n-j}{n-1}$ and $t=\frac{n-k}{n-1}$ in (3.7), then since $r>1$ we have $i<1$ and

$$s\geq r\geq t \text{ or } t\geq r\geq s \text{ implies } j\leq i\leq k \text{ or } k\leq i\leq j,$$

Thus, we obtain

$$\left(\tilde{W}_i(K\check{+}L)\right)^{\frac{n-1}{n-i}}\leq \tilde{W}_j(K)^{\frac{(k-i)(n-1)}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{(i-j)(n-1)}{(n-i)(k-j)}}+\tilde{W}_j(L)^{\frac{(k-i)((n-1))}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{(i-j)(n-1)}{(n-i)(k-j)}}.$$

From that equality conditions of Lemma 3.1, it follows the equality in (3.6) holds for $j\neq k$ if and only if $\rho(K,u)$ and $\rho(L,u)$ are constants, it follows that equality in (3.6) holds for $j\neq k$ if and only if K and L are n-balls centered at the origin. Moreover, if $i=j$ or $i=k$, equality in (3.6) holds if and only if K and L are dilates.

Similar to the above proof, we can also establish the reverse inequality for $k\geq j\geq i$ and $1<i<n$, or $k\leq j\leq i$ and $1<i<n$. Here, we omit the details.

Theorem 3.3 *Let $K,L\in\mathcal{S}^n$, and $\lambda,\mu\geq 0$ and $i,j,k\in\mathbb{R}$. If $j\leq i\leq k$ and $i<-1$, or $k\leq i\leq j$ and $i<-1$, then*

$$\frac{\tilde{W}_i(\lambda K\hat{+}\mu L)^{\frac{n+1}{n-i}}}{V(\lambda K\hat{+}\mu L)}\leq \lambda\frac{\tilde{W}_j(K)^{\frac{(k-i)(n+1)}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{(i-j)(n+1)}{(n-i)(k-j)}}}{V(K)}+\mu\frac{\tilde{W}_j(L)^{\frac{(k-i)(n+1)}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{(k-j)(n+1)}{(n-i)(k-j)}}}{V(L)}, \tag{3.8}$$

with equality for $j\neq k$ if and only if K and L are n-balls centered at the origin, and for $i=j$ or $i=k$ if and only if K and L are dilates.

The inequality is reversed if $k\geq j\geq i$ and $-1<i<n$, or $k\leq j\leq i$ and $-1<i<n$.

Proof From (1.5), (1.8), and Lemma 3.1, it follows that if $s\geq r\geq t$ and $r>1$, or $t\geq r\geq s$ and $r>1$, we have with $r=\frac{n-i}{n+1}$

$$\tilde{W}_i(\lambda K\hat{+}\mu L)^{\frac{n+1}{n-i}}=\left(\frac{1}{n}\int_{S^{n-1}}\rho(\lambda K\hat{+}\mu L,u)^{n-i}dS(u)\right)^{\frac{n+1}{n-i}}$$

$$=\left(\frac{1}{n}\int_{S^{n-1}}\left(\xi\lambda V(K)^{-1}\rho(K,u)^{n+1}+\xi\mu V(L)^{-1}\rho(L,u)^{n+1}\right)^r dS(u)\right)^{\frac{1}{r}}$$

$$\leq\left(\frac{1}{n}\int_{S^{n-1}}\left(\xi\lambda V(K)^{-1}\right)^s\rho(K,u)^{s(n+1)}dS(u)\right)^{\frac{r-t}{r(s-t)}}$$

$$\left(\frac{1}{n}\int_{S^{n-1}}\left(\xi\lambda V(K)^{-1}\right)^t\rho(K,u)^{t(n+1)}dS(u)\right)^{\frac{s-r}{r(s-t)}}$$

$$+\left(\frac{1}{n}\int_{S^{n-1}}\left(\xi\mu V(L)^{-1}\right)^s\rho(L,u)^{s(n+1)}dS(u)\right)^{\frac{r-t}{r(s-t)}}$$

$$\left(\frac{1}{n}\int_{S^{n-1}}\left(\xi\mu V(L)^{-1}\right)^t\rho(L,u)^{t(n+1)}dS(u)\right)^{\frac{s-r}{r(s-t)}}.$$

By (1.5), we have

$$\tilde{W}_{n-s(n+1)}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{s(n+1)} dS(u), \quad \tilde{W}_{n-t(n+1)}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{t(n+1)} dS(u).$$

Hence,

$$\tilde{W}_i(K\tilde{+}L)^{\frac{1}{r}} \leq \left[(\xi\lambda V(K)^{-1})^s \tilde{W}_{n-s(n+1)}(K)\right]^{\frac{r-t}{r(s-t)}} \left[(\xi\lambda V(K)^{-1})^t \tilde{W}_{n-t(n+1)}(K)\right]^{\frac{s-r}{r(s-t)}} + \left[(\xi\mu V(L)^{-1})^s \tilde{W}_{n-s(n+1)}(L)\right]^{\frac{r-t}{r(s-t)}} \left[(\xi\mu V(L)^{-1})^t \tilde{W}_{n-t(n+1)}(L)\right]^{\frac{s-r}{r(s-t)}}. \tag{3.9}$$

Taking $r = \frac{n-i}{n+1}$, $s = \frac{n-j}{n+1}$ and $t = \frac{n-k}{n+1}$ in (3.9), then since $r > 1$ we have $i < -1$ and

$$s \geq r \geq t \text{ or } t \geq r \geq s \text{ implies } j \leq i \leq k \text{ or } j \geq i \geq k.$$

Thus, we obtain

$$\tilde{W}_i(\lambda K\tilde{+}\mu L)^{\frac{n+1}{n-i}} \leq \xi\lambda V(K)^{-1}\tilde{W}_j(K)^{\frac{(k-i)(n+1)}{(n-i)(k-j)}}\tilde{W}_k(K)^{\frac{(i-j)(n+1)}{(n-i)(k-j)}} + \xi\mu V(L)^{-1}\tilde{W}_j(L)^{\frac{(k-i)(n+1)}{(n-i)(k-j)}}\tilde{W}_k(L)^{\frac{(i-j)(n+1)}{(n-i)(k-j)}},$$

where $\xi = V(\lambda K\tilde{+}\mu L)$.

From that equality conditions of Lemma 3.1, it follows the equality in (3.8) holds for $j \neq k$ if and only if $\rho(K,u)$ and $\rho(L,u)$ are constants, it follows that equality in (3.8) holds for $j \neq k$ if and only if K and L are n-balls centered at the origin. Moreover, if $i = j$ or $i = k$, equality in (3.8) holds if and only if K and L are dilates.

Similar to the above proof, we we can also establish the reverse inequality for $k \geq j \geq i$ and $-1 < i < n$, or $k \leq j \leq i$ and $-1 < i < n$. Here, we omit the details.

Acknowledgements The author expresses his grateful thanks to the referee for his excellent suggestions and comments.

References

[1] J. Abardia, A. Bernig, Projection bodies in complex vector spaces, *Adv. Math.*, **227** (2011), 830-846.

[2] J. Abardia, T. Wannerer, Aleksandrov-Fenchel inequalities for unitary valuations of degree 2 and 3, *Calc. Var. Partial Differential Equations*, in press.

[3] S. Alesker, A. Bernig, F. E. Schuster, Harmonic analysis of translation invariant valuations, *Geom. Funct. Anal.*, **21** (2011), 751-773.

[4] E. F. Beckenbach, R. Bellman, Inequalities, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1961.

[5] A. Berg, L. Parapatits, F. E. Schuster, M. Weberndorfer, Log-concavity properties of Minkowski valuations, arXiv:1411.7891.

[6] F. Besau, F. E. Schuster, Binary operations in spherical convex geometry, arXiv:1407.1153.

[7] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn-Minkowski-inequality, *Adv. Math.*, **231** (2012), 1974-1997.

[8] Y. D. Burago, V. A. Zalgaller, Geometric Inequalities, Springer-Verlag, Berlin, 1988.

[9] R. J. Gardner, Geometric Tomography, Cambridge Univ. Press, New York, 1996.

[10] R. J. Gardner, D. Hug, W. Weil, Operations between sets in geometry, *J. Eur. Math. Soc.* (JEMS), **15** (2013), 2297-2352.

[11] R. J. Gardner, D. Hug, W. Weil, The Orlicz-Brunn-Minkowski theory: A general framework, additions, and inequalities, *J. Differential Geom.*, **97** (2014), 427-476.

[12] R. J. Gardner, A. Koldobsky, T. Schlumprecht, An analytic solution to the Busemann- Petty problem on sections of convex bodies, *Ann. Math.,* **149** (1999) 691-703.

[13] R. J. Gardner, L. Parapatits, F. E. Schuster, A Characterization of Blaschke Addition, *Adv. Math.,* **254** (2014), 396-418.

[14] C. Haberl, L_p intersection bodies, *Adv. Math.,* **217** (2008), 2599-2624.

[15] C. Haberl, M. Ludwig, A characterization of L_p intersection bodies, *Int. Math. Res. Not.*, 2006, Art. ID 10548, 29 pp.

[16] H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer, Berlin, 1957.

[17] G. H. Hardy , J. E. Littlewood and G. Pólya,Inequalities, Cambridge Univ. Press. Cambridge, 1934.

[18] M. Ludwig, Intersection bodies and valuations, *Amer. J. Math.,* **128** (2006), 1409-1428.

[19] E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.*, **71**(1988), 232-261.

[20] E. Lutwak, Dual mixed volumes, *Pacific J. Math.*, **58**(1975), 531-538.

[21] E. Lutwak, Centroid bodies and dual mixed volumes, *Proc. London Math. Sci.*, **60**(1990), 365-391.

[22] E. Lutwak, The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas, *Adv. Math.,* **118** (1996), 244-294.

[23] L. Parapatits, F. E. Schuster, The Steiner formula for Minkowski valuations, *Adv. Math.,* **230** (2012), 978-994.

[24] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge Univ. Press, 1993.

[25] F. E. Schuster, Valuations and Busemann-Petty type problems, *Adv. Math.,* **219** (2008), 344-368.

[26] F. E. Schuster, Crofton Measures and Minkowski Valuations, *Duke Math. J.,* **154** (2010), 1-30.

[27] C. J. Zhao, G. S. Leng, Brunn-Minkowski inequality for mixed intersection bodies, *J. Math. Anal. Appl.,* **301**(2005), 115-123.

[28] C. J. Zhao, G. S. Leng, L. Debnath, Some new Brunn-Minkowski-type inequalities in convex bodies, *Inter. J. Math. Math. Sci.,* **2005**(6)(2005), 895-915.

Department of Mathematics, China Jiliang University,
Hangzhou 310018, China
Email: chjzhao@163.com chjzhao@aliyun.com