Operator-valued measurable functions

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Abstract

Let Ω be a measurable space and $\mathcal{M}$ be a σ-finite von Neumann algebra which is also a second dual space. On the set of functions from Ω into $\mathcal{M}$, it is supposed to give a criterion to illustrate τ-measurability where τ runs over some well-known locally convex topologies on $\mathcal{M}$ which is stronger than weak operator topology and weaker than the Arens-Mackey topology.

introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathbf{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. In the literature, there are some well-known locally convex topologies on $\mathbf{B}(\mathcal{H})$ which are given in the following diagram. For definitions and details see ([7, 8, 1, 6]). We just recall the Arens-Mackey topology which is less classical than the six other ones. This locally convex topology is given as the uniform convergence topology on $\sigma(\mathbf{B}(\mathcal{H}), \mathbf{B}(\mathcal{H})_*)$-compact convex subsets of $\mathbf{B}(\mathcal{H})_*$.

$$\begin{array}{ccccccc} \text{Arens-Mackey} & \supset & \sigma\text{-strong}^* & \supset & \sigma\text{-strong} & \supset & \sigma\text{-weak} \\ & & \cup & & \cup & & \cup \\ & & \text{strong}^* & \supset & \text{strong} & \supset & \text{weak} \end{array} \tag{0.1}$$

where $\supset$ means that the right-hand side is coarser than the left-hand side. In the sequel, the notation τ runs just over these seven topologies.

Let Ω be a measurable space. An operator valued function $f : \Omega \to \mathbf{B}(\mathcal{H})$ is called τ-measurable if $f^{-1}(O)$ is measurable for any arbitrary τ-open set O in $\mathbf{B}(\mathcal{H})$. Two main points are proved in [4], when $\mathcal{H}$ is a separable Hilbert space.

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At first $f : \Omega \to \mathbf{B}(\mathcal{H})$ is weak operator measurable if and only if it is strong operator measurable. Also the product of finitely many weak operator measurable functions is weakly measurable as well.

A von Neumann algebra $\mathcal{M}$ is σ-finite if it admits at most countably many orthogonal projections. It is also a second dual space if there is a Banach space with $\mathcal{M} = \mathcal{X}^{**}$. We extend results in [4] for functions $f : \Omega \to \mathcal{M}$ when $\mathcal{M}$ is a σ-finite second dual space. We show τ-measurability for all topologies in the above diagram are the same. Besides, we prove that the set of all τ-measurable functions $f : \Omega \to \mathcal{M}$ forms an involutive complex algebra and is closed under the point wise limit. Finally, we will observe that norm-measurability is strictly stronger than τ-measurability.

1 Main results

It is proved in [2] that a σ-finite von Neumann algebra $\mathcal{M}$ is a second dual space if and only if $\mathcal{M}$ is atomic (i.e, every projection majorizes a minimal projection) if and only if $\mathcal{M}$ is a direct sum of the $\mathbf{B}(\mathcal{H})$'s ($\mathcal{H}$ is a separable Hilbert space). Based on this fact, we continue with $\mathcal{M} = \mathbf{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, because our results will be valid in general case as well.

We commence by an interesting point which plays the key point in the main result. But first, we have to recall something concerning the Lindelöf spaces. A Lindelöf space is a topological space in which every open cover has a countable subcover. A topological space $\mathcal{X}$ is Lindelöf if and only if $\mathcal{X}$ has a cover $\{\mathcal{X}_n\}_1^\infty$ such that subspaces $\mathcal{X}_n$, under the relative topology, are all Lindelöf.

Proposition 1.1. *The topological space* $(\mathbf{B}(\mathcal{H}), \tau)$ *is Lindelöf.*

Proof. Closed bounded balls $\mathbf{B}(\mathcal{H})_{\|\cdot\|\leq n}$ are all compact under weak operator topology (res. σ-w topology). Therefore $(\mathbf{B}(\mathcal{H}), \tau)$ is clearly Lindelöf where τ is either WOT topology or σ-w topology. As for the rest of topologies, combination of the following items imply any closed ball $\mathbf{B}(\mathcal{H})_{\|\cdot\|\leq n}$ forms a second countable metrizable space which leads us to conclude $(\mathbf{B}(\mathcal{H}), \tau)$ is again Lindelöf.

- The unit ball of $\mathbf{B}(\mathcal{H})$ is metrizable under both σ-strong and σ-strong* topology (see [6] proposition 5.3).
- On the bounded parts of $\mathbf{B}(\mathcal{H})$, the Arens-Mackey topology and the σ-strong* topology coincide ([1] theorem II.7).
- It is well-known that on the bounded parts of $\mathbf{B}(\mathcal{H})$ the topologies given in each column of the diagram (0.1) are the same.
- Since the C*-algebra of compact operators $\mathbf{K}(\mathcal{H})$ is separable then the Kaplansky's density theorem implies the unit ball of $\mathbf{B}(\mathcal{H})$ has a countable dense subset under both strong and strong* operator topology. ∎

We fix $\{e_n\}$ an orthonormal basis for $\mathcal{H}$. For given function $f : \Omega \to \mathbf{B}(\mathcal{H})$, we consider ij^{th}-entry of f as a scalar valued function f_{ij} given by $x \to \langle f(x)e_j, e_i\rangle$.

Theorem 1.2. *For given function $f : \Omega \to \mathbf{B}(\mathcal{H})$ the following are equivalent:*

1. *The scalar valued functions f_{ij} are all measurable ($i, j \in \mathbb{N}$).*

2. *f is τ-measurable.*

Proof. We just need to show (1) implies that f is Arens-Mackey measurable. Assume the scalar-valued functions f_{ij} are all measurable.

Step one: At first we prove (2) for the strong* topology (s*-topology). Since $(\mathbf{B}(\mathcal{H}), s^*)$ is Lindelöf then every s^*-open set is a countable union of basic s^*-open sets. Therefore it is enough to show $f^{-1}(O)$ is measurable for all s^*-subbasic open sets O in $\mathbf{B}(\mathcal{H})$. To do this, we have to prove for given $a \in \mathbf{B}(\mathcal{H})$ and vector $h \in \mathcal{H}$ the following map is measurable

$$x \to \| (f(x) - a)h \|^2 + \| (f(x) - a)^* h \|^2$$

To see this

$$\begin{aligned} \| (f(x) - a)h \|^2 &= \sum_{i=1}^{\infty} |\langle f(x)h - ah, e_i \rangle|^2 \\ &= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \underbrace{\langle h, e_j \rangle \langle (f(x) - a)e_j, e_i \rangle}_{\text{scalar-valued measurable function}} \right|^2 \end{aligned}$$

which shows $x \to \| (f(x) - a)h \|^2$ is a point-wise limit of measurable functions and so it is itself measurable. Similarly one may show $x \to \| (f(x) - a)^* h \|^2$ is measurable.

Step two: In this part we show the σ-algebra generated by the Arens-Mackey topology and the σ-algebra generated by s^*-topology coincide on $\mathbf{B}(\mathcal{H})$. Since the Arens-Mackey topology is finer than the s^*-topology then it is enough to show any open set in the Arens-Mackey topology is contained in the σ-algebra generated by the s^*-open sets. Let $\widetilde{O} \subseteq \mathbf{B}(\mathcal{H})$ be an open set in the Arens Mackey topology. On the bounded subsets of $\mathbf{B}(\mathcal{H})$, the Arens Mackey topology and s^*-topology are the same. There is a sequence of s^*-open sets O_n in $\mathbf{B}(\mathcal{H})$ with

$$\widetilde{O} = \bigcup_{1}^{\infty} \{O_n \cap \mathbf{B}(\mathcal{H})_{\|\cdot\| \leq n} : n \in \mathbb{N}\}$$

Since bounded closed balls in $\mathbf{B}(\mathcal{H})$ are all s^*-closed the proof will be complete. ∎

Based on the previous theorem one may speak of "measurable function" instead τ-measurable function. The involution and absolute value of $f : \Omega \to \mathcal{M}$ are naturally given by

$$f^*(x) := f(x)^* \quad , \quad |f|(x) := |f(x)|$$

We also say $f : \Omega \to \mathcal{M}$ is the point-wise limit of a sequence $f_n : \Omega \to \mathcal{M}$ if for all $x \in \Omega$

$$f(x) = \text{wot-}\lim f_n(x)$$

Theorem 1.3. *The set of all $\mathcal{M}$-valued measurable functions on Ω forms an involutive complex algebra under the natural operator multiplication. The absolute value of a measurable function is also measurable. The point-wise limit of a sequence $\phi^n : \Omega \to \mathcal{M}$ of measurable functions is measurable as well.*

Proof. Let f, g be two measurable functions from Ω into $\mathcal{M}$. We have then

$$\begin{aligned}(fg)_{ij}(x) &= \sum_{k=1}^{\infty} f_{ik}(x) g_{kj}(x) \\ f_{ij}(x)^* &= \overline{\langle f(x)e_i, e_j \rangle} \\ |f|(x) &= \lim_{n\to\infty} p_n(f(x)^* f(x)) \\ \phi_{ij}(x) &= \text{wot-}\lim \phi^n_{ij}(x)\end{aligned}$$

where p_n is a sequence of polynomials uniformly convergent to the square root function. All assertions follow from item(1) in theorem (1.2). ∎

We end by an example to show the norm measurability is strictly stronger than measurability.

Example 1.4. We give a countability argument on the closed unit ball of radius 2 in $\mathbf{B}(\mathcal{H})$ to show the σ-algebra generated by the norm topology has higher cardinality than the σ-algebra generated by the weak operator topology: The orthonormal basis $\{e_i\}_{i=1}^{\infty}$ has uncountably many subsets namely $\{E_i\}_{i\in I}$. We denote q_i by the orthogonal projection onto the closed subspace generated by E_i. We have then $\| q_i - q_j \| = 1$ $(i \neq j)$. Let's note,
(i) One may construct uncountably many closed balls B_i (centered at q_i) with $\text{dist}(B_i, B_j) \geq \frac{1}{2}$. Therefore any arbitrary union of these balls is norm closed. Hence $\{\cup_{i\in J} B_i : J \subseteq I\}$ forms a collection of closed subsets in $\mathbf{B}(\mathcal{H})_{\|\cdot\|\leq 2}$ with the cardinal number 2^c. This point forces the cardinal number of the σ-algebra generated by the norm topology to be at least 2^c.
(ii) The topological space $(\mathbf{B}(\mathcal{H})_{\|\cdot\|\leq 2}, \text{wot})$ is second countable then the collection of weakly-measurable sets has the cardinality of the continuum.
Combination of **(i)** , **(ii)** shows the inclusion map $\iota : (\mathbf{B}(\mathcal{H})_{\|\cdot\|\leq 2}, \text{wot}) \to \mathbf{B}(\mathcal{H})$ is norm measurable if and only if $\mathcal{H}$ is finite dimensional.

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