Special Open Sets in Manifold Calculus

Daniel Pryor

Abstract

Embedding Calculus, as described by Weiss, is a calculus of functors, suitable for studying contravariant functors from the poset of open subsets of a smooth manifold M, denoted $\mathcal{O}(M)$, to a category of topological spaces (of which the functor $\mathrm{Emb}(-, N)$ for some fixed manifold N is a prime example). Polynomial functors of degree k can be characterized by their restriction to $\mathcal{O}_k(M)$, the full subposet of $\mathcal{O}(M)$ consisting of open sets which are a disjoint union of at most k components, each diffeomorphic to the open unit ball. In this work, we replace $\mathcal{O}_k(M)$ by more general subposets and see that we still recover the same notion of polynomial cofunctor.

1 Introduction

In [8], Weiss develops manifold calculus, a variation on Goodwillie's calculus of homotopy functors in [4]. Manifold calculus studies contravariant topological space-valued functors on the poset of open subsets of a manifold M. Manifold calculus is especially good for studying spaces of smooth embeddings of one manifold into another by looking at the functor $\mathrm{Emb}(-, N)$ for a fixed manifold N, which is the apparent motivation behind [8]. The main goal of this work is to generalize Weiss' characterization of polynomial cofunctors.

Being a calculus of functors, manifold calculus has a notion of polynomial cofunctor. These are the cofunctors which satisfy an appropriate higher-order excision property, similar to the case of [4]. Weiss is able to characterize degree k polynomial cofunctors as follows. Let $\mathcal{O}$ be the poset of open sets of a d-dimensional manifold M, and let $\mathcal{O}_k$ be the full subposet of $\mathcal{O}$ whose objects are

disjoint unions of at most k components, each diffeomorphic to $\mathbb{R}^d$. Weiss calls objects of $\mathcal{O}_k$ special open sets. Then a degree k polynomial cofunctor $F : \mathcal{O} \to \textbf{Top}$ is determined (up to equivalence) by its restriction to $\mathcal{O}_k$. The main result in this paper (Theorem 6.12) is a statement that generalizes this characterization of a k-polynomial cofunctor by its restriction to a subposet $\mathcal{B}_k$ of $\mathcal{O}_k$. The objects of $\mathcal{B}_k$ are simply disjoint unions of the objects of $\mathcal{B}_1$. As long as the objects of $\mathcal{B}_1$, form a basis for the topology of M, then no homotopy theoretic information is lost when forming the polynomial approximation to a cofunctor using $\mathcal{B}_k$ instead of $\mathcal{O}_k$ (Corollary 6.3 and Corollary 6.13).

We now give a brief outline of this work. In Section 2, we quickly go over some of the conventions and basic notions from homotopy theory that we will need. Then, in Section 3 and Section 4, we briefly introduce Manifold Calculus, summarizing some of the main results of Weiss. We define $\mathcal{O}_k(M)$, the special open sets of Weiss mentioned earlier. We recall Weiss' construction of polynomial cofunctors as suitable extensions of cofunctors defined only on $\mathcal{O}_k(M)$.

In Section 5, we recall and prove some fairly general results about functors from any category $\mathcal{C}$ to $\textbf{Top}$. Then, in Section 6, we generalize the notion of special open set as mentioned earlier. Namely, we let $\mathcal{B}_k(M)$ be full subposets of $\mathcal{O}_k(M)$ which still contain enough open sets so that we lose no information when we develop the analogous theory. The primary example of interest here occurs when M is a smooth codimension zero submanifold of $\mathbb{R}^d$, and $\mathcal{B}_k(M)$ contains all open sets which are disjoint unions of at most k open balls (in the euclidean metric sense).

We prove the analogs of several results of Weiss in this more general setting, adding in a few details as well. Most proofs go through with very little change, but one notable exception is Theorem 4.1, where having all of $\mathcal{O}_k$ as special open sets is crucial to his proof. Then we prove the main result (Theorem 6.12) that any valid choice (as described above) of special open sets yields equivalent notions of polynomial cofunctors and polynomial approximation by polynomial cofunctors.

2   Conventions

In this section, we introduce some conventions and recall some basic notions from homotopy theory. We work in the category of weak Hausdorff compactly generated topological spaces, which we denote $\textbf{Top}$. For topological spaces X and Y, we let $\text{Map}(X, Y)$ denote the space of maps (continuous functions) from X to Y, with the compact-open topology.

If $p : E \to B$ is a map of (unbased) topological spaces, its mapping path space, $\text{mps}(p)$, is defined to be the subspace

$$\text{mps}(p) = \{(e, \varphi) \in E \times \text{Map}(I, B) \mid \varphi(0) = p(e)\}$$

of $E \times \mathrm{Map}(I, B)$. Any such map can be factored as

$$E \to \mathrm{mps}(p) \to B.$$

The lefthand map sends a point $e \in E$ to the point $(e, \mathrm{const}_{p(e)})$, where $\mathrm{const}_{p(e)}$ denotes the constant path at $p(e)$; this map is a homotopy equivalence. The right-hand map sends a point $(e, \varphi) \in \mathrm{mps}(p)$ to the point $\varphi(1) \in B$; this map is a fibration.

Let $\mathcal{C}$ be a small category, and suppose $F : \mathcal{C} \to \mathbf{Top}$ is a functor. Along the lines of [1], we write $\mathrm{srep}(F)$ to denote the simplicial replacement of F, a simplicial space whose geometric realization gives the homotopy colimit of F. Dually, we let $\mathrm{crep}(F)$ be the cosimplicial replacement of F, a cosimplicial space whose totalization gives the homotopy limit of F. In this work, geometric realization and totalization will mean homotopy invariant geometric realization and totalization.

3 Manifold Calculus Preliminaries

Let M be a d-dimensional smooth manifold without boundary. Define $\mathcal{O} = \mathcal{O}(M)$ to be the poset of open subsets of M, considered as a category. If V is an open subset of M, let $\mathcal{O}(V)$ be the full subposet of $\mathcal{O}$ whose objects are contained in V. Equivalently, we can think of $\mathcal{O}(V)$ as the comma category $(\mathcal{O}(M) \downarrow V)$.

Definition 3.1. A cofunctor F from a subcategory of $\mathcal{O}$ to **Top** will be called an isotopy cofunctor if F takes all inclusions which are isotopy equivalences to homotopy equivalences. A cofunctor $F : \mathcal{O} \to \mathbf{Top}$ will be called good if it satisfies the following conditions:

(a) F is an isotopy cofunctor.

(b) If $V_0 \to V_1 \to \cdots$ is a string of inclusions in $\mathcal{O}$, then the natural map

$$F\left(\bigcup_{i=0}^{\infty} V_i\right) \to \underset{i}{\mathrm{holim}}\, F(V_i)$$

is a homotopy equivalence.

Example 3.2. For a fixed manifold N, Proposition 1.4 in [8] shows that the cofunctors $\mathrm{Emb}(-, N)$ and $\mathrm{Imm}(-, N)$ are good, where Emb and Imm denote the spaces of smooth embeddings and immersions, respectively.

Let V be an open subset of M, and let $C_0, \ldots, C_k$ be pairwise disjoint closed subsets of V. For $S \subseteq \{0, \ldots, k\}$, let

$$V_S = V \setminus \bigcup_{i \in S} C_i = \bigcap_{i \in S} (V \setminus C_i).$$

We thus have a $(k+1)$-cube of spaces $S \mapsto F(V_S)$. We say that this cube is homotopy cartesian if the natural map

$$F(V) \to \underset{S \neq \varnothing}{\mathrm{holim}}\, F(V_S)$$

is a homotopy equivalence. For example, if $k = 1$ this becomes the requirement that

$$
\begin{array}{ccc}
F(V = V_\varnothing) & \longrightarrow & F(V_{\{0\}}) \\
\downarrow & & \downarrow \\
F(V_{\{1\}}) & \longrightarrow & F(V_{\{0,1\}})
\end{array}
$$

is a homotopy pullback square. See [3] for more background on cubical diagrams of spaces.

Definition 3.3. A good cofunctor F is **polynomial** of degree $\leq k$ if for any choices of V and $C_0, \ldots, C_k$, the $(k+1)$-cube $S \mapsto F(V_S)$ is homotopy cartesian.

Example 3.4. Example 2.3 in [8] shows that $\mathrm{Imm}(-, N)$ is linear (polynomial of degree ≤ 1) if $\dim M < \dim N$, or if $\dim M = \dim N$ and M has no compact components.

4   Polynomial Cofunctors and the Taylor Tower

Let $\mathcal{O}_k$ be the full subposet of $\mathcal{O}$ consisting of those open sets which are diffeomorphic to a disjoint union of at most k copies of $\mathbb{R}^d$. Weiss calls these special open sets and shows [8] that k-polynomial cofunctors are determined by their restriction to $\mathcal{O}_k$. More precisely, we have the following.

Theorem 4.1 ([8], 5.1). *If $F, G : \mathcal{O} \to$ **Top** are polynomial cofunctors of degree $\leq k$, and $\gamma : F \to G$ is a natural transformation such that $\gamma_V : F(V) \to G(V)$ is a homotopy equivalence for all $V \in \mathcal{O}_k$, then γ_V is a homotopy equivalence for all $V \in \mathcal{O}$.*

Moreover, Weiss shows [8] that any (isotopy) cofunctor $\mathcal{O}_k \to$ **Top** has a canonical extension to a k-polynomial cofunctor $\mathcal{O} \to$ **Top**. We introduce some notation and then give the result.
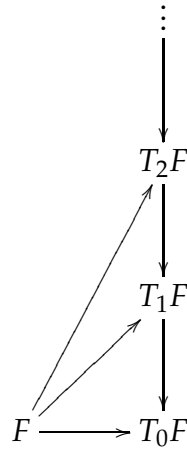
Notation 4.2. Let $\mathcal{N}$ be a subposet of $\mathcal{O}$. If F is a cofunctor from $\mathcal{N} \to$ **Top**, then define $F^! : \mathcal{O} \to$ **Top** to be the homotopy right Kan extension of F along the inclusion functor $\mathcal{N} \to \mathcal{O}$. An explicit formula given on objects is:

$$
F^!(V) = \operatorname*{holim}_{U \in \mathcal{N}(V)} F(U).
$$

Theorem 4.3 ([8], 3.8 and 4.1). *If $F : \mathcal{O}_k \to$ **Top** is an isotopy cofunctor, then $F^! : \mathcal{O} \to$ **Top** is (good and) polynomial of degree $\leq k$.*

Combining these two results leads to the notion of the polynomial approximation of a cofunctor. Specifically, for $F : \mathcal{O}(M) \to$ **Top** a good cofunctor, Weiss defines the degree k polynomial approximation to F, written $T_k F$, by the formula $T_k F = (F|_{\mathcal{O}_k})^!$. That is, $T_k F$ is obtained by first restricting F to $\mathcal{O}_k(M)$, then by extending it back to all of $\mathcal{O}(M)$. Thus, T_k is an endofunctor on the category of good cofunctors $\mathcal{O}(M) \to$ **Top**.

The polynomial approximations to F fit into a tower:

$$
\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
T_2F & & \\
\nearrow \; \downarrow & & \\
T_1F & & \\
\nearrow \; \downarrow & & \\
F \longrightarrow T_0F & &
\end{array}
$$

called the Taylor tower of F. The vertical maps $T_kF \to T_{k-1}F$ come from applying T_k to the natural map $F \to T_{k-1}F$ and then observing that T_kT_{k-1} is naturally equivalent to T_{k-1}.

5   Categorical Lemmas

Definition 5.1. A map $p : E \to B$ is a quasifibration if for all $e \in E$, the canonical inclusion of the fiber of p over $p(e)$ into the homotopy fiber of p over $p(e)$ is a weak homotopy equivalence.

Notation 5.2. If $\mathcal{C}$ is a category, then we write $|\mathcal{C}|$ for the classifying space of $\mathcal{C}$, that is, for the geometric realization of the nerve of $\mathcal{C}$.

Theorem 5.3 (Quillen-Dwyer Theorem). *Let $\mathcal{C}$ be a small category, and let $F : \mathcal{C} \to$* **Top** *be a functor which sends all morphisms to homotopy equivalences. Then the projection $p : \operatorname{hocolim} F \to |\mathcal{C}|$ is a quasifibration. Moreover, if $p' : \operatorname{mps}(p) \to |\mathcal{C}|$ is the associated fibration, then $\operatorname{holim} F$ is equivalent to the space of sections of p'.*

The quasifibration statement of this result is due to Quillen in [5], and the identification of the homotopy limit of F with the space of sections of the associated fibration is due to Dwyer in [2].

Lemma 5.4. *Let $\mathcal{D}$ be a small category, and let $\mathcal{C}$ be a subcategory of $\mathcal{D}$. Let $F : \mathcal{D} \to$* **Top** *be a functor taking all morphisms to homotopy equivalences. If the inclusion $|\mathcal{C}| \hookrightarrow |\mathcal{D}|$ is a homotopy equivalence, then so is the map*

$$\operatorname*{holim}_{\mathcal{D}} F \to \operatorname*{holim}_{\mathcal{C}} F|_{\mathcal{C}}.$$

Proof. Let $p : \operatorname{hocolim}_{\mathcal{C}} F|_{\mathcal{C}} \to |\mathcal{C}|$ and $q : \operatorname{hocolim}_{\mathcal{D}} F \to |\mathcal{D}|$ be the projections.

Consider the diagram

$$\begin{array}{ccccc}
\underset{\mathcal{C}}{\mathrm{hocolim}}\, F|_{\mathcal{C}} & \overset{\simeq}{\longrightarrow} & \mathrm{mps}(p) & \overset{p'}{\longrightarrow} & |\mathcal{C}| \\
\downarrow & & \downarrow & & \downarrow \\
\underset{\mathcal{D}}{\mathrm{hocolim}}\, F & \overset{\simeq}{\longrightarrow} & \mathrm{mps}(q) & \overset{q'}{\longrightarrow} & |\mathcal{D}| \,.
\end{array}$$

By Theorem 5.3, p and q are quasifibrations, and in this diagram, we have (functorially) factored these maps through their mapping path spaces as homotopy equivalences followed by fibrations. The map of homotopy limits will be a homotopy equivalence if the associated map of section spaces, $\Gamma_{q'} \to \Gamma_{p'}$ is also. This will be the case if both of the right two vertical maps are homotopy equivalences. The righthand one is by hypothesis, and the middle one will be if the lefthand one is.

Choose a basepoint in $|\mathcal{C}|$, a 0-simplex corresponding to some object c. Then the fiber of the map

$$p : \underset{\mathcal{C}}{\mathrm{hocolim}}\, F|_{\mathcal{C}} \to |\mathcal{C}| \simeq \underset{\mathcal{C}}{\mathrm{hocolim}} *$$

can be taken to be just $F(c)$. See this by looking at the induced map of simplicial replacements, then observing that we get the constant simplicial space $F(c)$ as the fiber. So now we have a map of quasifibration sequences

$$\begin{array}{ccccc}
F(c) & \longrightarrow & \underset{\mathcal{C}}{\mathrm{hocolim}}\, F|_{\mathcal{C}} & \overset{p}{\longrightarrow} & |\mathcal{C}| \\
\downarrow & & \downarrow & & \downarrow \\
F(c) & \longrightarrow & \underset{\mathcal{D}}{\mathrm{hocolim}}\, F & \overset{q}{\longrightarrow} & |\mathcal{D}|
\end{array}$$

Again, the righthand map is a homotopy equivalence, and the lefthand map can be assumed to be the identity (again looking at simplicial replacements). The choice of basepoint was arbitrary (arbitrary enough, as we have at least one for each path component of $|\mathcal{C}|$), so the middle map is also a homotopy equivalence, and the lemma is proved. ∎

Lemma 5.5. *Let $\mathcal{C}_0 \hookrightarrow \mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \cdots$ be an increasing inclusion of small categories, and call their union $\mathcal{C}$. Let $F : \mathcal{C} \to$ **Top** be a functor and denote its restriction to $\mathcal{C}_i$ by F_i. Then the natural map*

$$\underset{\mathcal{C}}{\mathrm{holim}}\, F \to \underset{i}{\mathrm{holim}}\, \underset{\mathcal{C}_i}{\mathrm{holim}}\, F_i,$$

is a homotopy equivalence.

Proof. We can view this as a map of totalizations of cosimplicial spaces:

$$\text{Tot}(\text{crep } F) \rightarrow \underset{i}{\text{holim}}\, \text{Tot}(\text{crep } F_i).$$

But totalization commutes with homotopy limits (since for example, $\text{Tot } X \simeq \text{holim}_\Delta X$ by [1]). So this becomes the map

$$\text{Tot}(\text{crep } F) \rightarrow \text{Tot}(\underset{i}{\text{holim}}\, \text{crep } F_i).$$

On the righthand side, $\text{crep } F_i$ is the diagram

$$
\begin{array}{ccc}
\text{crep}(F_0)^0 \rightrightarrows \text{crep}(F_0)^1 \Rrightarrow \cdots \\
\uparrow \qquad\qquad \uparrow \\
\text{crep}(F_1)^0 \rightrightarrows \text{crep}(F_1)^1 \Rrightarrow \cdots \\
\uparrow \qquad\qquad \uparrow \\
\cdots \qquad\qquad \cdots
\end{array}
$$

So, after interchanging the totalization with the homotopy limit, we are now taking homotopy limits in the vertical direction first, then the totalization of the resulting cosimplicial space second. But the vertical maps are all fibrations (projections onto subproducts in fact), so we can just take ordinary limits. But note that in codegree q, this limit is just $\text{crep}(F)^q$. $\blacksquare$

Let $\mathcal{C}$ be a small category, and let $F : \mathcal{C} \rightarrow \mathbf{Cat}$ be a functor. The Grothendieck construction is a new category $\mathcal{C} \int F$ (alternatively $F \rtimes \mathcal{C}$) whose objects are pairs (c, x) where c is an object of $\mathcal{C}$, and x is an object of $F(c)$. A morphism $(c, x) \rightarrow (c', x')$ is a pair (f, g), where $f : c \rightarrow c'$ is a morphism in $\mathcal{C}$, and $g : F(f)(x) \rightarrow x'$ is a morphism in $F(c')$. If $(f, g) : (c, x) \rightarrow (c', x')$ and $(f', g') : (c', x') \rightarrow (c'', x'')$ are morphisms, the composite $(f', g') \circ (f, g)$ is defined to be

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ F(f')(g)) : (c, x) \rightarrow (c'', x'').$$

Example 5.6. If $\mathcal{C}$ is a group G (as a category with one object), and F takes the object of $\mathcal{C}$ to another group H in $\mathbf{Cat}$, then we can consider F as a group homomorphism $\varphi : G \rightarrow \text{Aut}(H)$. In this case, the Grothendieck construction recovers the usual semidirect product $H \rtimes_\varphi G$.

Thomason [7] proves the following.

Theorem 5.7 (Thomason's homotopy colimit theorem)**.** *For $\mathcal{C}$ and F as above, there is a natural homotopy equivalence*

$$\left| \mathcal{C} \int F \right| \simeq \underset{c \in \mathcal{C}}{\text{hocolim}}\, |F(c)|$$

6 Special Open Sets

We are now ready to tackle our main goal. Namely, we would like to characterize k-polynomial cofunctors by their restriction to some smaller class of special open sets than all of $\mathcal{O}_k$. We can do this provided that we leave enough special open sets in place. More precisely, for each $k \geq 0$, let $\mathcal{B}_k$ be a full subposet of $\mathcal{O}_k$ satisfying each of the following conditions:

(a) The objects of $\mathcal{B}_1$ form a basis for the topology of M.

(b) The objects of $\mathcal{B}_k$ are exactly those which are a union of at most k pairwise disjoint objects of $\mathcal{B}_1$.

Note that the second condition implies that each $\mathcal{B}_k$ contains the empty set as one of its objects. Also note that once $\mathcal{B}_1$ is chosen, the rest of the $\mathcal{B}_k$ are determined automatically. Now let $\mathcal{A}_k$ be the wide subposet (same objects, but possibly fewer morphisms) of $\mathcal{B}_k$ with the weaker order where $U \leq V$ if the inclusion of U into V is an isotopy equivalence.

Example 6.1. One possible choice for $\mathcal{B}_k$ is $\mathcal{O}_k$ itself. This is the case that Weiss considers in [8], where he uses the notation $\mathcal{O}k$ instead. In this example, our $\mathcal{A}_k$ is exactly Weiss' $\mathcal{I}k$.

Example 6.2. If M is given as a smooth codimension zero submanifold of $\mathbb{R}^d$, then we can take $\mathcal{B}_k$ to be the subsets of M which are unions of at most k pairwise disjoint open balls (with respect to the euclidean metric), or cubes, simplices, or convex d-bodies more generally.

First, we can very easily strengthen Weiss' characterization of polynomial cofunctors by considering their restriction to $\mathcal{B}_k$ as follows.

Corollary 6.3. *Let F_1 and F_2 be good cofunctors from $\mathcal{O} \to$ **Top**, both polynomial of degree $\leq k$. If $\gamma : F_1 \to F_2$ is a natural map such that $\gamma_V : F_1(V) \to F_2(V)$ is a homotopy equivalence for all $V \in \mathcal{B}_k$, then it is a homotopy equivalence for all $V \in \mathcal{O}$.*

Proof. Let $V \in \mathcal{O}_k$. Then there is a $U \in \mathcal{B}_k$ such that the inclusion of U into V is an isotopy equivalence. We get a commutative square,

$$
\begin{array}{ccc}
F_1(V) & \longrightarrow & F_1(U) \\
\downarrow{\scriptstyle \gamma_V} & & \downarrow{\scriptstyle \gamma_U} \\
F_2(V) & \longrightarrow & F_2(U).
\end{array}
$$

The top and bottom maps are induced by isotopy equivalences, so are themselves homotopy equivalences since F_1 and F_2 are good. And γ_U is a homotopy equivalence by assumption, so therefore γ_V is a homotopy equivalence as well. Then Theorem 4.1 implies that γ_V is in fact a homotopy equivalence for all $V \in \mathcal{O}$. ∎

Remark 6.4. The rest of the development of this section mimics that of Weiss, but in slightly more generality. Still, this development is logically independent from [8]. However, the previous characterization of polynomial cofunctors appealed directly to Weiss' result. His proof (Theorem 5.1 in [8]) relied on using all of $\mathcal{O}_k$ in a critical way; it does not seem that the argument can be made valid when using some arbitrary choice for $\mathcal{B}_k$.

Notation 6.5. Let X be a topological space. Let $C(X, k)$ denote the (ordered) configuration space of k points in X. That is,

$$C(X, k) = \left\{ (x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ for } i \neq j \right\},$$

with the subspace topology. Let $\binom{X}{k}$ denote the unordered configuration space of k points in X. That is, the quotient of $C(X, k)$ by the (free) action of the symmetric group Σ_k which permutes coordinates.

The following result is the analog of Lemma 3.5 in [8]. This proof is based on that of Weiss, but treats the case for general j all at once.

Proposition 6.6.

$$|\mathcal{A}_k(M)| \simeq \coprod_{j=0}^{k} \binom{M}{j}.$$

Proof. First, note that $\mathcal{A}_k$ is a disjoint union

$$\mathcal{A}_k = \coprod_{j=0}^{k} \mathcal{A}^{(j)},$$

where $\mathcal{A}^{(j)}$ is the full subposet of $\mathcal{A}_j$ consisting of the objects with exactly j components. Thus it suffices to show that $\left| \mathcal{A}^{(k)} \right| \simeq \binom{M}{k}$. For $k = 0$, this is trivial; now let k be a positive integer.

Define the space W to be the subspace of $\left| \mathcal{A}^{(k)} \right| \times \binom{M}{k}$ consisting of all points (x, y) where x is in the interior of a nondegenerate r-simplex corresponding to the string $V_0 \to \cdots \to V_r$, and each component of V_r contains exactly one point of y. We claim W is an open set. To see this, fix a point (x, y) as above, and let A be the subset of $\left| \mathcal{A}^{(k)} \right|$ consisting of all interiors of simplices which correspond to nondegenerate strings $U_0 \to \cdots \to U_s$ which contain $V_0 \to \cdots \to V_r$ as a substring. Note that A is an open set since it is a union of open cells, and whenever an s-cell is in A, then every $(s+1)$-cell having it as a face is also in A. And now $(x, y) \in A \times V_r \subseteq W$, so W is open.

Since W is open, it follows that the projections from W to each factor are almost locally trivial in the sense of [6]. By a theorem of Segal [6], if each fiber of an almost locally trivial map is contractible, then the map is a homotopy equivalence. So if we can show each fiber of each projection is contractible, then we have proved the lemma.

First, the fiber over any point of $\left|\mathcal{A}^{(k)}\right|$ will be diffeomorphic to $\mathbb{R}^{kd}$. Second, fix a $y \in \binom{M}{k}$, and let W_y be the fiber of the projection at y, considered as a subspace of $\left|\mathcal{A}^{(k)}\right|$. Thus W_y is the union of the interiors of simplices corresponding to nondegenerate strings $V_0 \to \cdots \to V_r$, where each component of V_r contains exactly one point from y. Consider the subspace W_y' of W_y to be the union of the interiors of only those simplices for which each component of V_0 contains exactly one point from y.

Note that W_y' is homeomorphic to the classifying space of the full subposet of $\mathcal{A}^{(k)}$ consisting of all objects which contain exactly one point of y in each component. Since the objects of $\mathcal{A}_1$ form a basis for the topology of M, this subposet is codirected, and hence its classifying space is contractible by [5]. Thus $W_y' \simeq *$, and we finish the proof of the lemma by showing that W_y' is a deformation retract of W_y.

Consider a nondegenerate string $V_0 \to \cdots \to V_r$ with each component of V_r containing one point of y. Let q be the smallest index such that each component of V_q contains one point of y. Let Δ be the r-simplex in $\left|\mathcal{A}^{(k)}\right|$ corresponding to $V_0 \to \cdots \to V_r$. Note that a point $x \in \Delta$ with barycentric coordinates $(x_0, \ldots, x_r)$ is in W_y iff $x_q, \ldots, x_r$ are not all 0. For each such $x \in \Delta \cap W_y$, let $\overline{x}$ be the point in $\Delta \cap W_y'$ with coordinates

$$\frac{(0, \ldots, 0, x_q, \ldots, x_r)}{x_q + \cdots + x_r}.$$

We now define a homotopy $H : W_y \times I \to W_y$ piecewise on (the appropriate part of) each simplex Δ by the formula $H(x, t) = (1 - t)x + t\overline{x}$.

This is tentatively a deformation retraction of W_y onto W_y', but we must still verify that H is well-defined; the given formula must agree on the intersection of simplices in W_y. So, suppose $x \in \Delta$ has coordinates $(x_0, \ldots, x_r)$ and corresponds to the string $V_0 \to \cdots \to V_r$, and $x' \in \Delta'$ has coordinates $(x_0', \ldots, x_{r'}')$ and corresponds to the string $V_0' \to \cdots \to V_{r'}'$. Suppose further that x and x' represent the same point in W_y. That is, their corresponding strings share a (necessarily nonempty) maximal substring

$$V_{i_0} \to \cdots \to V_{i_s} = V_{i_0'}' \to \cdots \to V_{i_s'}'.$$

Furthermore, the only nonzero entries of $(x_0, \ldots, x_r)$ are $x_{i_0}, \ldots, x_{i_s}$, the only nonzero entries of $(x_0', \ldots, x_{r'}')$ are $x_{i_0'}', \ldots, x_{i_s'}'$, and $x_{i_j} = x_{i_j'}'$ for all $0 \leq j \leq s$. Note that when defining the formula for $\overline{x}$, we could have equivalently chosen q to be the smallest index for which $y \in V_q$ *and* $x_q \neq 0$. With this in mind, we see that $\overline{x}$ and $\overline{x}'$ represent the same point, and so we are done.   ∎

Notation 6.7. For $p \geq 0$, let $\mathcal{A}_k(\mathcal{B}_k)_p = \mathcal{A}_k(\mathcal{B}_k)_p(M)$ be the category whose objects are strings of p composable morphisms in $\mathcal{B}_k$, $V_0 \to \cdots \to V_p$, and whose morphisms are natural transformations of such diagrams whose component maps all lie in $\mathcal{A}_k$.

The next result is the analog of 3.6 and 3.7 in [8].

Lemma 6.8. *The homotopy fiber (over some 0-simplex W) of the map*

$$\left|\mathcal{A}_k(\mathcal{B}_k)_p(M)\right| \to \left|\mathcal{A}_k(M)\right|$$

induced by the functor sending $(V_0 \to \cdots \to V_p) \mapsto V_p$ *is* $\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right|$. *Furthermore, the functor* $\left|\mathcal{A}_k(\mathcal{B}_k)_p(-)\right| : \mathcal{O}(M) \to$ **Top** *takes isotopy equivalences to homotopy equivalences.*

Proof. The proof is by induction on p. First consider the case $p = 0$. Note that $|\mathcal{A}_k(\mathcal{B}_k)_0(U)|$ is just $|\mathcal{A}_k(U)|$, which by Proposition 6.6 is homotopy equivalent to $\coprod_0^k \binom{U}{j}$. Thus if $U \to U'$ is an isotopy equivalence in $\mathcal{O}(M)$, then the induced map $\binom{U}{j} \to \binom{U'}{j}$ is a homotopy equivalence for each j.

Now for $p > 0$, the Grothendieck construction gives us an isomorphism of categories

$$\mathcal{A}_k(\mathcal{B}_k)_p(M) \cong \mathcal{A}_k(M) \int \mathcal{A}_k(\mathcal{B}_k)_{p-1}(-)$$
$$V_0 \to \cdots \to V_p \leftrightarrow (V_p, V_0 \to \cdots \to V_{p-1}).$$

Combining this with Thomason's homotopy colimit theorem (Theorem 5.7), we get a homotopy equivalence

$$\operatorname*{hocolim}_{V \in \mathcal{A}_k(M)} \left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(V)\right| \simeq \left|\mathcal{A}_k(\mathcal{B}_k)_p(M)\right|.$$

The map in question then corresponds to the usual projection of the homotopy colimit (which we take to be the usual representation as the realization of the appropriate simplicial replacement) to the nerve of the indexing category. By the induction hypothesis and Theorem 5.3, this map is a quasifibration, so the homotopy fiber we are interested in has the same homotopy type as the actual fiber of this map. By the proof of Lemma 5.4, this fiber can be taken to be $\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right|$, as we needed to show.

For the second part of the lemma, let $V \to V'$ be an isotopy equivalence in $\mathcal{O}(M)$. We have a map of quasifibration sequences

$$
\begin{array}{ccccc}
\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right| & \longrightarrow & \left|\mathcal{A}_k(\mathcal{B}_k)_p(V)\right| & \longmapsto & \left|\mathcal{A}_k(V)\right| \\
\downarrow & & \downarrow & & \downarrow \\
\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right| & \longrightarrow & \left|\mathcal{A}_k(\mathcal{B}_k)_p(V')\right| & \longrightarrow & \left|\mathcal{A}_k(V')\right|
\end{array}
$$

The lefthand vertical map can be taken to be the identity, and the righthand vertical map is a homotopy equivalence as mentioned above. Therefore, so is the middle vertical map. ∎

Let $F : \mathcal{B}_k \to$ **Top** be an isotopy cofunctor. Define $F_p : \mathcal{A}_k(\mathcal{B}_k)_p \to$ **Top** by $F_p(U_0 \to \cdots \to U_p) = F_p(U_0)$. Define $F_p^! : \mathcal{O} \to$ **Top** as:

$$F_p^!(V) = \operatorname*{holim}_{\mathcal{A}_k(\mathcal{B}_k)_p(V)} F_p.$$

Lemma 6.9. *$F_p^!$ is a good cofunctor.*

Proof. For part (a) of goodness, let $V \to V'$ be an isotopy equivalence in $\mathcal{O}$. Since F_p takes all morphisms to homotopy equivalences, then by Lemma 5.4 it suffices to show that the inclusion $\left|\mathcal{A}_k(\mathcal{B}_k)_p(V)\right| \hookrightarrow \left|\mathcal{A}_k(\mathcal{B}_k)_p(V')\right|$ is a homotopy equivalence, and it is by Lemma 6.8.

For part (b) of goodness, let $V_0 \to V_1 \to \cdots$ be a string in $\mathcal{O}$. We need to show that the map

$$F_p^! \left(\bigcup_i V_i \right) \to \operatorname*{holim}_i F_p^!(V_i)$$

is a homotopy equivalence. Rewrite and factor this as

$$\operatorname*{holim}_{\mathcal{A}_k(\mathcal{B}_k)_p(\bigcup V_i)} F_p \to \operatorname*{holim}_{\bigcup \mathcal{A}_k(\mathcal{B}_k)_p(V_i)} F_p \to \operatorname*{holim}_i \operatorname*{holim}_{\mathcal{A}_k(\mathcal{B}_k)_p(V_i)} F_p.$$

The right map is a homotopy equivalence by Lemma 5.5. Furthermore, if we can show that the inclusion

$$\left| \bigcup_i \mathcal{A}_k(\mathcal{B}_k)_p(V_i) \right| \hookrightarrow \left| \mathcal{A}_k(\mathcal{B}_k)_p \left(\bigcup_i V_i \right) \right|$$

is a homotopy equivalence, then Lemma 5.4 would imply that the left map is a homotopy equivalence as well.

Note that the nerve of a union is the same as the union of nerves. First, if $p = 0$, we have the inclusion

$$\bigcup_i |\mathcal{A}_k(V_i)| \hookrightarrow \left| \mathcal{A}_k \left(\bigcup_i V_i \right) \right|,$$

which by Proposition 6.6 is the same as the map

$$\bigcup_i \coprod_{j=0}^{k} \binom{V_i}{j} \hookrightarrow \coprod_{j=0}^{k} \binom{\bigcup_i V_i}{j}.$$

This is actually a homeomorphism since the union and coproduct commute, and since the configuration spaces in question are selecting only a finite number of points. For the $p > 0$ case, consider the inclusion of homotopy fibration sequences over some base point W in some $\mathcal{A}_k(V_i)$.

$$
\begin{array}{ccccc}
\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right| & \longrightarrow & \bigcup_i \left|\mathcal{A}_k(\mathcal{B}_k)_p(V_i)\right| & \longrightarrow & \bigcup_i |\mathcal{A}_k(V_i)| \\
\Big\| & & \Big\downarrow & & \Big\downarrow \\
\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right| & \longrightarrow & \left|\mathcal{A}_k(\mathcal{B}_k)_p \left(\bigcup_i V_i \right)\right| & \longrightarrow & \left|\mathcal{A}_k \left(\bigcup_i V_i \right)\right|
\end{array}
$$

The righthand map is a homotopy equivalence by the $p = 0$ case above, so the middle map is as well. ∎

Lemma 6.10. *For any $V \in \mathcal{O}$, the projection*

$$\mathrm{Tot}(p \mapsto F_p^!(V)) \to F^!(V)$$

is a homotopy equivalence.

Proof. Let $(\mathcal{A}_k)_q \mathcal{B}_k = (\mathcal{A}_k)_q \mathcal{B}_k(M)$ be the category whose objects are strings of q composable morphisms in $\mathcal{A}_k$, and whose morphisms are natural transformations of such diagrams with component maps in $\mathcal{B}_k$. Now, the domain of the projection can be thought of as a totalization of a totalization (of the cosimplicial replacement of F_p). We can switch the order of the totalizations and rewrite the domain as $\mathrm{Tot}(q \mapsto \widehat{F}_q^!(V))$, where $\widehat{F}_q : (\mathcal{A}_k)_q \mathcal{B}_k(V) \to \mathbf{Top}$ is the functor sending $U_0 \to \cdots \to U_q \mapsto F(U_0)$. Thus, if the map $\widehat{F}_q^!(V) \to F^!(V)$ is a homotopy equivalence for each q, then so will the original map be.

We can write this map as the map

$$\operatorname*{holim}_{(\mathcal{A}_k)_q \mathcal{B}_k(V)} \widehat{F}_q \to \operatorname*{holim}_{\mathcal{B}_k(V)} F$$

which is induced by the inclusion functor $J : \mathcal{B}_k(V) \to (\mathcal{A}_k)_q \mathcal{B}_k(V)$ sending $U \mapsto U \to \cdots \to U$. We claim that J is homotopy terminal. That is, for any $U_0 \to \cdots \to U_q \in (\mathcal{A}_k)_q \mathcal{B}_k(V)$, the comma category $(U_0 \to \cdots \to U_q \downarrow J)$ is nonempty and contractible. This follows since the comma category has an intial object, namely $U_q \to \cdots \to U_q$. Therefore, the induced map of homotopy limits is a homotopy equivalence, and this proves the lemma. ∎

Theorem 6.11. *$F^!$ is a good cofunctor.*

Proof. Let $V \to V'$ be an isotopy equivalence in $\mathcal{O}$. By the previous lemma, the horizontal maps in the square

$$
\begin{array}{ccc}
\mathrm{Tot}(p \mapsto F_p^!(V)) & \xrightarrow{\simeq} & F^!(V) \\
\uparrow & & \uparrow \\
\mathrm{Tot}(p \mapsto F_p^!(V')) & \xrightarrow{\simeq} & F^!(V')
\end{array}
$$

are homotopy equivalences, so showing the righthand map is a homotopy equivalence for part (a) of goodness is equivalent to showing so for for the lefthand map. In codegree p, the lefthand map is just the map $F_p^!(V') \to F_p^!(V)$, which is a homotopy equivalence since $F_p^!$ is good. So the overall map of totalizations is a homotopy equivalence since the underlying map of cosimplicial spaces is a homotopy equivalence in each codegree.

For part (b) of goodness, let $V_0 \to V_1 \to \cdots$ be a string in $\mathcal{O}$. Similarly, to show that the map

$$F^!\left(\bigcup_i V_i\right) \to \operatorname*{holim}_i F^!(V_i)$$

is a homotopy equivalence, it suffices to show that the map

$$\text{Tot}\left[p \mapsto F_p^!\left(\bigcup_i V_i\right)\right] \to \operatorname*{holim}_i \text{Tot}\left[p \mapsto F_p^!(V_i)\right]$$

is one. But the homotopy limit and totalization commute, so this is really a map of totalizations of cosimplicial spaces which in codegree p is the map

$$F_p^!\left(\bigcup_i V_i\right) \to \operatorname*{holim}_i F_p^!(V_i).$$

And again, this is a homotopy equivalence since $F_p^!$ is good.   ∎

We come to our main result.

Theorem 6.12. *Suppose $\{\mathcal{B}_k\}$ and $\{\mathcal{B}_k'\}$ are two choices of special open sets with $\mathcal{B}_1' \subseteq \mathcal{B}_1$ (which implies that $\mathcal{B}_k' \subseteq \mathcal{B}_k$ as well). Let $F : \mathcal{B}_k \to$ **Top** be an isotopy cofunctor, and let G denote its restriction to $\mathcal{B}_k'$. Then the map $F^!(V) \to G^!(V)$ is a homotopy equivalence for all $V \in \mathcal{O}$.*

Proof. Let $V \in \mathcal{O}$. By a similar argument as before, it suffices to check that the map $F_p^!(V) \to G_p^!(V)$ is a homotopy equivalence for all p. And by Lemma 5.4, it is enough to check that the inclusion $\left|\mathcal{A}_k'(\mathcal{B}_k')_p(V)\right| \hookrightarrow \left|\mathcal{A}_k(\mathcal{B}_k)_p(V)\right|$ is a homotopy equivalence. Note that if $p = 0$, then this is really the inclusion $\left|\mathcal{A}_k'(V)\right| \hookrightarrow \left|\mathcal{A}_k(V)\right|$, and this is a homotopy equivalence by Proposition 6.6. If $p > 0$, choose a basepoint $W \in \left|\mathcal{A}_k'(V)\right|$ and use Lemma 6.8 to get a diagram of homotopy fibration sequences

$$
\begin{array}{ccccc}
\left|\mathcal{A}_k'(\mathcal{B}_k')_{p-1}(W)\right| & \longrightarrow & \left|\mathcal{A}_k'(\mathcal{B}_k')_p(V)\right| & \longrightarrow & \left|\mathcal{A}_k'(V)\right| \\
\downarrow & & \downarrow & & \downarrow \\
\left|\mathcal{A}_k(\mathcal{B}_k)_{p-1}(W)\right| & \longrightarrow & \left|\mathcal{A}_k(\mathcal{B}_k)_p(V)\right| & \longrightarrow & \left|\mathcal{A}_k(V)\right|
\end{array}
$$

The righthand vertical map is again the $p = 0$ case, so is a homotopy equivalence. The lefthand vertical map can be assumed to be a homotopy equivalence by induction on p. Therefore, the middle vertical map is a homotopy equivalence as well, as we needed to show.   ∎

As an immediate corollary, we get a strengthening of Weiss' construction of polynomial cofunctors (Theorem 4.3).

Corollary 6.13. *Let F be an isotopy cofunctor from $\mathcal{B}_k \to$ **Top**. Then $F^!$ is polynomial of degree $\leq k$.*

Proof. This follows from Theorem 4.3 and Theorem 6.12.   ∎

References

[1] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[2] W. G. Dwyer. The centralizer decomposition of *BG*. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 167–184. Birkhäuser, Basel, 1996.

[3] Thomas G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.

[4] Thomas G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.

[5] Daniel Quillen. Higher algebraic *K*-theory. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 171–176. Canad. Math. Congress, Montreal, Que., 1975.

[6] Graeme Segal. Classifying spaces related to foliations. *Topology*, 17(4):367–382, 1978.

[7] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.*, 85(1):91–109, 1979.

[8] Michael Weiss. Embeddings from the point of view of immersion theory. I. *Geom. Topol.*, 3:67–101 (electronic), 1999.

IRMP, Université Catholique de Louvain
2, chemin du Cyclotron
B-1348 Louvain-la-Neuve, Belgium
email:daniel.pryor@uclouvain.be