On different barrelledness notions in locally convex algebras

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Dedicated to the memory of Anastasios Mallios

Abstract

This is a synthetic presentation of several barrelledness notions, in locally convex algebras. These are characterized, as in locally convex spaces, via (algebra) seminorms. This approach reveals a new notion of barrelledness. The latter shows to be what is needed to have meaningful statements in locally uniformly convex algebras.

1 Introduction

Besides the very classical barrelledness in locally convex algebras, as locally convex spaces, several notions of a specific kind of barrelledness, pertaining to the algebra structure, have been introduced, according to the context someone is working in. The aim of this paper is to give an idea of all of them. We, in particular, provide characterizations and proceed to a complete comparison. The characterizations are, naturally, given in terms of (algebra) seminorms, which are the respective ones of vector space seminorms in locally convex spaces. All vector spaces and algebras, considered here, are over the field $\mathbb{C}$ of complexes, while the algebras are unital.

In Section 3, definitions are given by using a notation, which is helpful in comparisons. In particular, we encounter the very classical barrelledness, the

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m-barrelledness (i-barrelledness of S. Warner), the infrabarrelledness of A.K. Chilana and S. Sharma and a different one but under the same name according to A. Mallios.

Section 4 deals with characterizations of these kinds of barrelledness. As in locally convex spaces, the notions are naturally characterized via (algebra) seminorms (Propositions 4.1, 4.2 and 4.3). This leads to structural results. Thus, a unital locally uniformly convex algebra being moreover, $(\mathcal{B}, m\mathbb{B})$-barrelled is in fact normable (Proposition 6.10).

Various examples are given in Section 5 to distinguish the different notions of barrelledness. They are of different nature.

The $(\mathcal{B}, m\mathbb{B})$-barrelledness seems to be new. We have never met it. It turns out that it is the adequate tool in locally uniformly convex algebras (Proposition 6.9). In the Mackey complete case, $(\mathcal{B}, m\mathbb{B})$-barrelledness is sufficient (cf. Proposition 4.5).

2 Preliminaries

Let (E, τ) be a locally convex space. The bounded structure (bornology) of (E, τ), denoted by $\mathbb{B}\tau$, is the collection of all subsets B of E which are bounded in the sense of Kolmogorov-von Neumann, namely B is absorbed by every neighborhood of zero. We say that a locally convex space (E, τ) is Mackey complete (M-complete) if its bounded structure $\mathbb{B}\tau$ admits a fundamental system $\mathcal{B}$ of Banach discs ("completent" discs) that is, for every B in $\mathcal{B}$, the vector space generated by B is a Banach space when endowed with the gauge $\|.\|_B$ of B (see [7, p. 95] and [2, p. 402]).

For a reference to the bornological notions, we refer to [11]. We remind that a locally convex space E is called *barrelled*, if every barrel in E (viz. absorbing, balanced, convex and closed subset of E) is a neighborhood of zero (see e.g. [12, p. 212, Definition 1] and [15, p. 9]). An *m-barrel* $\mathcal{U}$ of an algebra is a multiplicative (viz. $\mathcal{U}\mathcal{U} \subseteq \mathcal{U}$) barrel. The term *idempotent barrel* is also used instead of m-barrel.

A topological algebra is an algebra E endowed with a topological vector space topology τ for which multiplication is separately continuous. Let (E, τ) be a locally convex algebra with separately continuous multiplication, whose topology τ is given by a family $(p_\lambda)_{\lambda \in \Lambda}$ of seminorms; we use the notation $(E, (p_\lambda)_{\lambda \in \Lambda})$. The algebra (E, τ) is said to be locally A-convex (see [6, p. 18, Definition 2.5]; see also [4], [5]) if, for every x and every λ, there is $M(x, \lambda) > 0$ such that

$$\max\left[p_\lambda(xy), p_\lambda(yx)\right] \leq M(x, \lambda) p_\lambda(y); \text{for all } y \in E.$$

If $M(x, \lambda) = M(x)$ depends only on x, we say that (E, τ) is a locally uniformly A-convex algebra [5, p. 477, Definition 3.1]. If

$$p_\lambda(xy) \leq p_\lambda(x) p_\lambda(y); \text{for all } x, y \in E, \text{ and all } \lambda \in \Lambda,$$

then (E, τ) is called a locally m-convex algebra ([16], see also [15]). Recall that a locally convex algebra has a continuous multiplication if, for every λ, there is a

λ'such that

$$p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y); \text{for all } x, y \in E.$$

If $(E, (p_\lambda)_{\lambda\in\Lambda})$ is a unital locally-A-convex algebra, then it can be endowed with a stronger m-convex topology $M(\tau)$, where τ is the topology of E. The latter topology is determined by the family $(q_\lambda)_{\lambda\in\Lambda}$ of seminorms given by

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}. \tag{2.1}$$

If $(E, (p_\lambda)_{\lambda\in\Lambda})$ is not unital, consider its topological unitization E_1 and then take the restriction of $M(\tau_1)$ to E (see Section 3 in [10]). A locally convex algebra is said to be *pseudo-complete* if every closed bounded and idempotent (alias multiplicative) disc is *completent*.

An absolutely convex subset of a locally convex algebra (E, τ) is said to be bornivorous (resp. m-bornivorous) if it absorbs every bounded (resp. m-bounded) subset of (E, τ). A locally convex algebra (E, τ) is said to be *barrelled* (resp. *m-barrelled*) if every barrel (resp. m-barrel) is a neighborhood of zero (the latter class of locally convex algebras was introduced in [14]). Furthermore, a locally convex algebra is called *m-infrabarrelled* in the sense of A.K. Chilana and S. Sharma if every bornivorous m-barrel is a neighborhood of zero [3, p. 140]. In this paper, we also use the term of an *m-infrabarrelled algebra* in the sense of A. Mallios, namely a locally convex algebra (E, τ) is said to be *m-infrabarrelled* if every m-bornivorous m-barrel in E is a neighborhood of zero (see [15, p. 307, Definition 9.4]). The unified terminology applied in Definition 3.1 below, enables us to distinguish the last two aspects of m-infrabarrelledness.

3 Definitions and first comparisons

In this section, we provide relations between the six classes of locally convex algebras that are characterized through barreldness and bornivority (see Definition 3.1). Throughout the paper, the letters $\mathcal{B}$, m and $\mathbb{B}$ will be used for a barrel, idempotence (multiplicativity) and bornivorousness, respectively. Moreover, all spaces (resp. algebras) are locally convex ones. So, we use $(\mathcal{B}, 0)$ for barrels without any property of bornivorousness, $(\mathcal{B}, \mathbb{B})$ for barrels which are bornivorous, $(m\mathcal{B}, 0)$ for m-barrels without any property of bornivorousness, $(m\mathcal{B}, \mathbb{B})$ for bornivorous m-barrels, $(\mathcal{B}, m\mathbb{B})$, to indicate m-bornivorous barrels, and $(m\mathcal{B}, m\mathbb{B})$ for m-bornivorous m-barrels. We also make use of the following unified terminology.

Definition 3.1. We say that a locally convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$ is

(i) A $(\mathcal{B}, 0)$-*barrelled algebra* if it is a barrelled space (namely, every barrel is a neighborhood of zero).

(ii) A $(\mathcal{B}, \mathbb{B})$-*barrelled algebra* if every bornivorous barrel is a neighborhood of zero.

(iii) An $(m\mathcal{B}, 0)$-*barrelled algebra*, if every m-barrel is a neighborhood of zero. This is i-barrelledness, in the sense of S. Warner [21, p. 192].

(iv) A $(\mathcal{B}, m\mathbb{B})$-*barrelled algebra* if every m-bornivorous barrel is a neighborhood of zero.

(v) An $(m\mathcal{B}, \mathbb{B})$-*barrelled algebra* if every bornivorous m-barrel is a neighborhood of zero. This is m-infrabarrelledness in the sense of Chilana and Sharma.

(vi) An $(m\mathcal{B}, m\mathbb{B})$-*barrelled algebra* if every m-bornivorous m-barrel is a neighborhood of zero. This is m-infrabarrelledness in the sense of A. Mallios.

A study of m-infrabarrelledness, in the context of locally convex algebras has been given in [10]. As a first step, we easily get the next.

Proposition 3.2. *The following hold true.*

(i) The class of $(\mathcal{B}, 0)$*-barreled algebras is contained in each one of the others.*

(ii) $(\mathcal{B}, \mathbb{B})$*-barrelled algebras are* $(m\mathcal{B}, \mathbb{B})$*-barrelled.*

(iii) $(m\mathcal{B}, 0)$*-barrelled algebras are* $(m\mathcal{B}, \mathbb{B})$*-barrelled and* $(m\mathcal{B}, m\mathbb{B})$*-ones.*

(iv) $(\mathcal{B}, m\mathbb{B})$*-barrelled algebras are* $(m\mathcal{B}, m\mathbb{B})$*,* $(m\mathcal{B}, \mathbb{B})$ *and* $(\mathcal{B}, \mathbb{B})$*-barrelled.*

(v) Any $(m\mathcal{B}, m\mathbb{B})$*-barrelled algebra is* $(m\mathcal{B}, \mathbb{B})$*-barrelled.*

Statement (v), in the previous proposition, amends a comment in [10, p. 474], concerning the relation between $(m\mathcal{B}, m\mathbb{B})$-barrelled algebras and $(m\mathcal{B}, \mathbb{B})$-barrelled ones.

In Section 5, we give some examples of the topological algebras defined in Proposition 3.2, as well as counter-examples that ensure, whose of them are not identical to each other. We also give some examples of locally convex algebras that do not belong to some of the classes, given in Definition 3.1. From (v) in the previous proposition, *the notion of an m-infrabarrelled algebra in Chilana and Sharma sense is a generalization of that in the sense of Mallios*. Examples 5.7 and 5.12 below, ensure the distinction between these two notions.

The following diagram displays the relations between those classes of locally convex algebras which are characterized via barreldness and bornivority, as in Definition 3.1.

$$\begin{tikzcd}
 & (\mathcal{B}, \mathbb{B}) \arrow[r] & (m\mathcal{B}, \mathbb{B}) & \\
(\mathcal{B}, 0) \arrow[ur] \arrow[urr] \arrow[rrr] \arrow[dr] \arrow[drr] & & & (m\mathcal{B}, 0) \arrow[ul] \arrow[dl] \\
 & (\mathcal{B}, m\mathbb{B}) \arrow[uu] \arrow[uur] \arrow[r] & (m\mathcal{B}, m\mathbb{B}) \arrow[uu] &
\end{tikzcd}$$

4 Characterizations

We know that in the context of locally convex spaces E, "*barrelledness of E*" is equivalent to "*every lower semi-continuous seminorm on E is continuous*" (see e.g. [12, p. 219]). We can get analogous results in the algebra setting. Namely, in the next three propositions we characterize barrelledness in the sense of Mallios, Warner, and Chilana and Sharma, respectively. Recall that a seminorm is said to be *m-bounded* if it is bounded on the m-bounded subsets.

Proposition 4.1. *Let (E, τ) be a Hausdorff locally convex algebra. The following are equivalent:*

(1) *(E, τ) is an $(m\mathcal{B}, m\mathbb{B})$-barrelled algebra.*

(2) *Every algebra seminorm p which is lower semi-continuous and m-bounded is automatically continuous.*

Proof. (1) $\Rightarrow$ (2): Put $V = \{x \in E : p(x) \leq 1\}$. Since p is lower semi-continuous, V is closed (see e.g. [12, p. 219]), and finally, an m-barrel. It is also m-bornivorous, since p is m-bounded. So, by assumption, V is a neighborhood of zero, and hence p is continuous.

(2) $\Rightarrow$ (1): Let U be an m-bornivorous m-barrel. The respective gauge function p_U is an algebra seminorm. Moreover, $V = \{x \in E : p_U(x) \leq 1\}$ is closed. Therefore, p_U is lower semi-continuous. Besides, p_U is m-bounded, since U is m-bornivorous. Hence, by hypothesis, p_U is continuous, and therefore U is a neighborhood of zero. Namely, E is $(m\mathcal{B}, m\mathbb{B})$-barrelled. ∎

By (iii) and (iv) of Proposition 3.2, any $(m\mathcal{B}, 0)$-barrelled algebra is an $(m\mathcal{B}, \mathbb{B})$-barrelled algebra, and also an $(m\mathcal{B}, m\mathbb{B})$-barrelled algebra. So an obvious modification of the proof of Proposition 4.1 leads to the next two propositions.

Proposition 4.2. *Let (E, τ) be a Hausdorff locally convex algebra. The following are equivalent:*

(1) *(E, τ) is an $(m\mathcal{B}, 0)$-barrelled algebra.*

(2) *Every algebra seminorm p which is lower semi-continuous is automatically continuous.*

Proposition 4.3. *Let (E, τ) be a Hausdorff locally convex algebra. The following are equivalent:*

(1) *(E, τ) is an $(m\mathcal{B}, \mathbb{B})$-barrelled algebra.*

(2) *Every algebra seminorm p which is lower semi-continuous and bounded is automatically continuous.*

Note 4.4. Concerning the last results, we note that all other notions can be characterized in analogous ways.

By Proposition 3.2, (iv), every $(\mathcal{B}, m\mathbb{B})$-barrelled algebra is a $(\mathcal{B}, \mathbb{B})$-barrelled one. In the next proposition we give a context in which the converse is also true.

Proposition 4.5. *Let (E, τ) be a unital Mackey-complete locally uniformly A-convex algebra. Then E is a $(\mathcal{B}, \mathbb{B})$-barrelled algebra if and only if it is a $(\mathcal{B}, m\mathbb{B})$-barrelled algebra.*

Proof. We only have to prove that every m-bornivorous barrel is a neighborhood of zero, and since every bornivorous barrel is, by hypothesis, a neighborhood of zero, we restrict ourselves to the bounded structure $\mathbb{B}\tau$ of the topological algebra. Since E is locally uniformly A-convex, there is an algebra norm $\|\cdot\|_0$ on E, that yields a topology stronger than $M(\tau)$. If $(p_\lambda)_{\lambda\in\Lambda}$ is the family of seminorms defining τ, the stronger topology is given by

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda \in \Lambda\} \tag{4.1}$$

(viz. $p_\lambda(x) \leq \|x\|_0$, for all $\lambda \in \Lambda$, and for all $x \in E$; see [5, p. 477, (**) and Lemma 3.2]). For the definition of the q_λ's, see (2.1). In this case, $\mathbb{B}\tau = \mathbb{B}_{\|\cdot\|_0}$. Thus, all of them are m-bounded. ■

By Proposition 3.2, (v) every $(mB, m\mathbb{B})$-barrelled algebra is an $(mB, \mathbb{B})$-barrelled one. So, arguing as in Proposition 4.5, we get that *for unital Mackey-complete locally uniformly A-convex algebras, the notions "$(mB, m\mathbb{B})$-barrelled algebra" and "$(mB, \mathbb{B})$-barrelled algebra" coincide.*

5 Examples and counter-examples

Example 5.1. Any seminormed algebra (E, p) (in particular, a normed one) is an example of each of (ii) to (vi) barrelledness algebra notions (see Definition 3.1), described through barrels and bornivorousness. Indeed, the unit ball $B_p = \{x : p(x) \leq 1\}$ is included in every bornivorous subset of E.

For the next example see Remark 3.4 in [10].

Example 5.2. *A complete locally m-convex algebra that is $(\mathcal{B}, 0)$-barrelled*. Let X be a non compact, locally compact and metrizable space such that $X = \cup K_n$ where (K_n) is an exhaustive sequence of compact subsets of X. Take the complex algebra $\mathcal{K}(X)$ of continuous functions with compact support, and the subalgebras $E_n = \mathcal{K}(X, K_n)$ of functions with support in K_n. It is known that $\mathcal{K}(X) \equiv E$ is algebraically the inductive limit of the E_n's (see for instance, [15, p. 128, (4.6); see also p. 127, 4.(1)]). Consider the strict inductive limit topology τ of the Banach algebras $(E_n, \|.\|_n)$, where $\|.\|_n$ is the supremum norm on E_n. Then (E, τ) is a *complete locally m-convex algebra and $(\mathcal{B}, 0)$-barrelled* (namely, a barrelled one) and it is not a normed algebra.

Example 5.3. [B.E. Johnson] *A $(\mathcal{B}, 0)$-barrelled (topological) algebra (jointly continuous multiplication)*. Let X be a Hausdorff topological space. Take the algebra $\mathcal{C}(X)$ of all continuous complex valued functions on X. Let $\mathcal{S}$ be the class of the subsets of X filtered by the relation "$\subset$", in the sense that if $S_1, S_2 \in \mathcal{S}$, then there is an $S \in \mathcal{S}$ such that $S_1 \cup S_2 \subset S$. Consider the subalgebra $\mathcal{C}_0(\mathcal{S})$ of those bounded elements $f \in \mathcal{C}(X)$ so that $f(x) = 0$ in the complement $X \backslash S$ of some $S \in \mathcal{S}$. It is easy to see that, for each $S \in \mathcal{S}$, the set

$$I_S = \{f : f \in \mathcal{C}_0(\mathcal{S}), f(x) = 0 \text{ if } x \in X \backslash S\}$$

is an ideal in $\mathcal{C}_0(\mathcal{S})$, which under the supremum (or uniform) norm becomes a Banach space. On $\mathcal{C}_0(\mathcal{S})$ we consider the inductive limit of these topologies. Then *$\mathcal{C}_0(\mathcal{S})$ is a $(\mathcal{B}, 0)$-barrelled (topological) algebra* (jointly continuous multiplication). We note that, the topology considered on $\mathcal{C}_0(\mathcal{S})$ is finer than the supremum norm topology. See [13, p. 603, Theorem 1].

Example 5.4. [S. Warner] *A locally m-convex algebra which is an $(m\mathcal{B}, 0)$-barrelled algebra but not a $(\mathcal{B}, 0)$-barrelled space*. Consider the algebra $\mathbb{K}[x]$ of all polynomials in a variable x over a field $\mathbb{K}$. Take the subalgebra E of $\mathbb{K}[x]$ of all polynomials without constant term, endowed with the strongest locally

m-convex topology (viz. the collection of all convex, balanced, multiplicative, absorbing sets is a fundamental system of neighborhoods of zero). According to Theorem 3.1 in [15, p. 18] these neighborhoods of zero can be considered closed). Then *E is an $(m\mathcal{B}, 0)$-barrelled algebra which is not a $(\mathcal{B}, 0)$-barrelled space* (see [21, p. 193, Example 5]). Moreover, *it is an $(m\mathcal{B}, \mathbb{B})$-barrelled algebra, but not a $(\mathcal{B}, 0)$-barrelled algebra*.

In the sequel, the topology induced by the norm $\|\cdot\|$ will be denoted by $\tau_{\|\cdot\|}$.

Example 5.5. *A complete locally convex algebra which belongs to none of the six classes of barrelled algebras*. (See Example 3.8 in [10]). Let $\mathcal{C}_b(\mathbb{R})$ be the algebra of all complex continuous bounded functions on the real field $\mathbb{R}$ with the usual pointwise operations. Denote by $\mathcal{C}_0^+(\mathbb{R})$ the set of all strictly positive real-valued continuous functions on $\mathbb{R}$ vanishing at infinity (as elements of $\mathcal{C}_b(\mathbb{R})$). Consider the family $\{p_\varphi : \varphi \in \mathcal{C}_0^+(\mathbb{R})\}$ of seminorms given by

$$p_\varphi(f) = \sup\{|f(x)\varphi(x)| : x \in \mathbb{R}\}; f \in \mathcal{C}_b(\mathbb{R}),$$

that determines a locally convex topology, say β, on $\mathcal{C}_b(\mathbb{R})$. The space $(C_b(\mathbb{R}), \beta)$ is actually, a complete (non-metrizable) locally convex algebra. Since

$$p_\varphi(fg) \leq \|f\|_\infty p_\varphi(g) \text{ for all } g$$

the previous algebra is a (unital) locally uniformly A-convex algebra ([19]; see also [18]). Notice that $\beta \prec \tau_{\|\cdot\|_\infty}$. But then, $(\mathcal{C}_b(\mathbb{R}), (p_\varphi))$ can be endowed with a locally m-convex topology $M(\beta)$ stronger than β. This topology is defined by the seminorms

$$q_\varphi(f) = \sup\{p_\varphi(fg) : p_\varphi(g) \leq 1\}.$$

Thus, we have $\beta \prec M(\beta) \prec \tau_{\|\cdot\|_0}$, where

$$\|f\|_0 = \sup\{q_\varphi(fg) : q_\varphi(g) \leq 1\}$$

(see also the proof of Proposition 4.5). By the previous argument, $\mathbb{B}\beta = \mathbb{B}\tau_{\|\cdot\|_0}$. Consider the unit ball $B_{\infty,1} = \{f : \|f\|_\infty \leq 1\}$. This is a β-barrel, which is not a neighborhood of zero. Otherwise, the considered algebra would be normable, which is not the case. So, $(C_b(\mathbb{R}), \beta)$ is not of any kind of barrelled algebras, as in Definition 3.1.

Remark. The reasoning of Example 5.5 is actually *true for any unital Mackey complete locally uniformly A-convex algebra.* For a realization, see Example 5.10, below.

Example 5.6. [10, Example 4.5]. *A unital complete locally m-convex algebra which belongs to none of the six barrelledness classes of Definition* 3.1. Consider the algebra $(\mathcal{C}[0,1], |\cdot|_{K_d})$ of all continuous complex valued functions on the interval $[0,1]$, where K_d are compact subsets of $[0,1]$ and $|f|_{K_d} = \sup_{x\in K_d}|f(x)|$. Denote by τ the respective topology, under which the considered topological algebra is not metrizable. Moreover, we have

$$|fg|_{K_d} = \sup_{x\in K_d}|f(x)g(x)| \leq \|f\|_\infty \|g\|_{K_d}, \text{ for all } g.$$

For the bounded structures, we have $\mathbb{B}\tau = \mathbb{B}\tau_{\|\cdot\|_\infty}$. Now, since

$$B_\infty = \{f : \|f\|_\infty \leq 1\}$$

is a bornivorous (hence m-bornivorous) m-barrel which is not a τ-neighborhood of zero, $(\mathcal{C}[0,1], |\cdot|_{K_d})$ belongs to none of the six classes of barrelled algebras.

In what follows, $m\mathbb{B}\tau$ will denote the set of all multiplicative bounded sets with respect to the topology τ, of a locally convex space (E,τ).

Example 5.7. ([10, Remark 3.7]) *A pseudo-complete A-normed algebra which is* $(\mathcal{B},\mathbb{B})$*-barrelled and* $(m\mathcal{B},\mathbb{B})$*-one, and does not belong to the rest of the barrelledness classes.* Endow the algebra $\mathcal{C}[0,1]$ with the L_1-norm, $\|f\|_1 = \int_0^1 |f(t)|\,dt$. Then $(\mathcal{C}[0,1], \|\cdot\|_1)$ is a pseudo-complete A-normed algebra, which is not m-infrabarrelled.
Notice that

$$\|fg\|_1 = \int_0^1 |fg(t)|\,dt = \int_0^1 |f(t)g(t)|\,dt \leq \|f\|_\infty \|g\|_1, \text{ for all } g.$$

Moreover, the m-bounded subsets, with respect to $\|\cdot\|_1$, are exactly the m-bounded subsets for the uniform norm $\|\cdot\|_\infty$. Namely, $m\mathbb{B}\tau_{\|\cdot\|_1} = m\mathbb{B}\tau_{\|\cdot\|_\infty}$. Consider the unit ball

$$B_{0,1} = \{f : \|f\|_0 \leq 1\}$$

where $\|f\|_0 = \sup\{\|fg\|_1, \|g\|_1 \leq 1\}$. Then $B_{0,1}$ is a $\|\cdot\|_1$-m-barrel, that can not be a neighborhood of zero, since $\tau_{\|\cdot\|_1} \prec \tau_{\|\cdot\|_\infty}$. Moreover, $B_{0,1}$ is not bornivorous. Otherwise, it would exist some $\alpha > 0$ such that $B_{1,1} \subset \alpha B_{0,1}$, where $B_{1,1}$ is a neighborhood of zero. So, $B_{0,1}$ would be a neighborhood of zero, which is not. Take now B a bornivorous subset. Then there exists a $\lambda > 0$ with $B_{1,1} \subset \lambda B$, where $B_{1,1}$ is a neighborhood of zero for $\|\cdot\|_1$. Thus, $\frac{1}{\lambda}B_{1,1}$ is a neighborhood of zero contained in B. Namely, every bornivorous set is a neighborhood of zero. Thus, $(\mathcal{C}[0,1], \|\cdot\|_1)$ is, as asserted.

Example 5.8. *A* $(\mathcal{B},\mathbb{B})$*-barrelled algebra which is not* $(\mathcal{B},0)$*-barrelled.* Let (E,d) be a non-barrelled metric space. Under the trivial multiplication ($xy = 0$ for all $x,y \in E$), every barrel is an m-barrel and every bounded subset of E is m-bounded. Namely, $m\mathcal{B} \equiv \mathcal{B}$ and $m\mathbb{B} \equiv \mathbb{B}$. In this case the diagram is reduced to $(\mathcal{B},0) \longrightarrow (\mathcal{B},\mathbb{B})$. Thus, E is a $(\mathcal{B},\mathbb{B})$-barrelled algebra which is not $(\mathcal{B},0)$-barrelled. For the non-trivial case, consider $E \times F$ with F a Banach algebra or yet a Fréchet locally m-convex algebra.

Example 5.9. *A locally convex algebra that belongs to none of the six classes of barrelled algebras of Definition* 3.1. Let $(E,\|\cdot\|)$ be an infinite-dimensional C^*-algebra. Endow E with the weak topology σ (the weakest topology, induced, as an initial one, from its topological dual E'). Denote by $\tau_{\|\cdot\|}$ the norm-topology on E. Then $\sigma \prec \tau_{\|\cdot\|}$. In that case, any element in E' is still continuous, and $\mathbb{B}\sigma = \mathbb{B}\tau_{\|\cdot\|}$. Furthermore, σ and $\tau_{\|\cdot\|}$ have the same convex closed sets. Besides, the unit ball $B_{\|\cdot\|}$ is then σ-closed. Moreover, it is an absorbent disc and hence a σ-m-barrel, which is not a neighborhood of zero, otherwise, the locally convex algebra (E, σ) would be normable, that is a contradiction. Thus, (E, σ) is neither an $(m\mathcal{B}, \mathbb{B})$-barrelled algebra nor a $(\mathcal{B}, \mathbb{B})$-one. Thus, according to the diagram, it belongs to none of the rest four classes of barrelled-algebras.

For the next example, we remind that a *Q-algebra* is a topological algebra whose the group of its quasi-invertible elements is a neighborhood of zero.

Example 5.10. *A unital commutative complete locally uniformly A-convex algebra, which is locally m-convex, and it does not belong to any class of barrelled algebras, as in Definition* 3.1. Let X be a pseudo-compact, locally compact, non-compact topological space. Consider the algebra $\mathcal{C}_c(X)$ of continuous complex valued functions on X, endowed with the compact-open topology. Then $\mathcal{C}_c(X)$ is *a unital commutative complete locally uniformly A-convex algebra*. The same algebra is *locally m-convex, which is not a Q-algebra* (see [1, p. 245, Example 4.1]). Besides, every element in $\mathcal{C}_c(X)$ is bounded, in the sense that the local spectrum is bounded. So, if it was $(m\mathcal{B}, m\mathbb{B})$-barrelled, it should be, by Proposition 4.1 in [10], a Q-algebra, that is a contradiction. Furthermore, according to the remark after Example 5.5, the considered algebra is not of any kind of barrelledness.

Example 5.11. *An example of a* $(\mathcal{B}, \mathbb{B})$*-barrelled algebra*. Consider the topological algebra $(\mathcal{C}(\mathbb{R}), (|\cdot|_n)_{n\in\mathbb{N}})$, where

$$|f|_n = \sup\{|f(x)|, x \in [-n, n]\}. \tag{5.1}$$

Since $|fg|_n \leq |f|_n|g|_n$, this algebra is locally m-convex and hence a locally A-convex algebra. Take a bornivorous barrel B; this is a disc. Besides, $\mathcal{C}(\mathbb{R})$ is metrizable, therefore bornological (see [20, p. 61, 8.1]), and hence every balanced (: circled) convex subset of the algebra that absorbs every bounded set in it is a neighborhood of zero. The previous shows that B is a neighborhood of zero, and hence the considered topological algebra is a $(\mathcal{B}, \mathbb{B})$-barrelled algebra.

Example 5.12. Let $\mathcal{C}_b(\mathbb{R})$ be the unital algebra of all complex valued bounded functions on the field of reals $\mathbb{R}$ with the pointwise operations. The topological algebra $(\mathcal{C}_b(\mathbb{R}), (|\cdot|_n))$ (see also (5.1), is a pseudo-complete locally uniformly A-convex algebra. Endow $\mathcal{C}_b(\mathbb{R})$ with the topology, denoted by $\tau_{\|\cdot\|_0}$, induced on it by the algebra norm

$$\|f\|_0 = \sup\{|f|_n : n\}.$$

Thus, the topology $\tau_{\|\cdot\|_0}$ is stronger that τ induced by $(|\cdot|_n)$. Namely, $|f|_n \leq \|f\|_0$ for all n and all $f \in \mathcal{C}_b(\mathbb{R})$. Remark that $B_{\|\cdot\|_0} = \{f : \|f\|_0 \leq 1\}$ is a τ-barrel and in particular, a τ-m-barrel. Then the m-bounded sets are the same, namely $m\mathbb{B}\tau = m\mathbb{B}\tau_{\|\cdot\|_0}$. Moreover, the topological algebra in question, is metrizable, and thus the previous argument (see also the reasoning in Example 5.11) shows that $(\mathcal{C}_b(\mathbb{R}), \|\cdot\|_0)$ *is a* $(\mathcal{B}, \mathbb{B})$*-barrelled algebra and thus an* $(m\mathcal{B}, \mathbb{B})$*-one, as well. The same topological algebra does not belong to any of* $(\mathcal{B}, m\mathbb{B})$*,* $(m\mathcal{B}, m\mathbb{B})$*,* $(\mathcal{B}, 0)$ *or yet* $(m\mathcal{B}, 0)$*-barrelled algebras*.

Example 5.13. *A* $(\mathcal{B}, m\mathbb{B})$*-barrelled algebra which is not* $(\mathcal{B}, 0)$*-barrelled*. Let (E, d) be a non-barrelled metric space (hence bornological), endowed with the trivial multiplication, and F a Banach algebra. Then the cartesian product of topological algebras $E \times F$ is a $(\mathcal{B}, m\mathbb{B})$-barrelled algebra which is not $(\mathcal{B}, 0)$-barrelled. See also Example 5.8.

6 $(\mathcal{B}, m\mathbb{B})$-barrelledness

As we mentioned above, from what we know, the notion of $(\mathcal{B}, m\mathbb{B})$-barrelledness is new. So one wonders about its usefulness. In which specific context can it be used? It turns out that it is appropriate in the frame of locally uniformly convex algebras. Indeed, in that context, one exhibits a vector space norm which is not necessarily submultiplicative. So, to state similar results, what we exactly need, is the $(\mathcal{B}, m\mathbb{B})$-barrelledness. We first recall the basics on locally uniformly convex algebras. Such complex algebras have been introduced in [19] (see also [9] for the real case). We will exhibit examples of locally uniformly convex algebras which are not uniformly A-convex. But, we first examine the subnormability of such algebras. A locally convex algebra (E, τ) is said to be *subnormable* (*A-subnormable*, *m-subnormable*) if it can be endowed with a vector space norm (A-norm, algebra norm) which induces a topology stronger than τ.

A locally convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$ is said to be a *locally uniformly convex algebra* if,

$$\text{for all } x \text{ and for all } \lambda, \text{there exist } M(x) > 0 \text{ and } \lambda' \text{ such that}$$
$$p_\lambda(xy) \leq M(x)p_{\lambda'}(y), \text{for all } y$$

with $M(x)$ depending only on x (not on λ) and λ' depending only on λ (not on x).

The class of such algebras is different from that of locally uniformly A-convex ones (see Example 6.5). Good properties of the latter algebras are lost, in general, such as the boundedness of every element. The existence of an algebra norm on any unital locally uniformly-A-convex algebra ([17], [18]) provides a setting for a nice spectral theory. Here, the situation is less comfortable as one could expect.

Proposition 6.1. *(i) A unital locally uniformly convex algebra (E, τ) is subnormable. Moreover, if it is barrelled, then it is normable too.*

(ii) A non unital locally uniformly convex algebra (E, τ) is subnormable if and only if, its unitization E_1 is also a locally uniformly convex algebra.

Proof. (i) Let $(p_\lambda)_{\lambda\in\Lambda}$ be a family of seminorms defining τ, such that

$$p_\lambda(xy) \leq M(x)p_{\lambda'}(y); \text{for every } y \in E.$$

In particular, for the unit element e, we get

$$p_\lambda(x) \leq M(x)p_{\lambda'}(e); \text{for every } x \in E.$$

Without loss of generality, we may assume that $p_{\lambda'}(e) \neq 0$; otherwise p_λ being null should be excluded from the beginning. Now, the family $\left([p_{\lambda'}(e)]^{-1} p_\lambda\right)_{\lambda\in\Lambda}$ of seminorms determines exactly the topology τ. And one has

$$\sup_\lambda [p_{\lambda'}(e)]^{-1} p_\lambda(x) \leq M(x); \text{for every } x \in E.$$

But then,

$$x \mapsto \|x\| = \sup_{\lambda} \left[p_{\lambda'}(e)\right]^{-1} p_{\lambda}(x)$$

is a vector space norm that induces a topology stronger than τ and this proves the first assertion. Moreover, as E is subnormable and barrelled, it is normable (see [17, p. 86, Remark 6.1]).

(ii) The proof is standard and we omit it (for details see [19]). ∎

Remark 6.2. The norm in (i) in the proof of the previous proposition, is a vector space norm but not necessarily an algebra norm (Example 6.4, below). This example also shows that, unlike to the case of locally uniformly A-convex algebras, not every element here is necessarily bounded.

Under an additional completeness condition, one can say more.

Proposition 6.3. ([17], [19]). *Let (E, τ) be a unital and Mackey complete locally uniformly convex algebra. Then there is a Banach algebra norm the topology of which is stronger than τ with the same bounded structure.*

Proof. See [17, p. 87, Proposition 6.3]. ∎

Example 6.4. Consider the algebra $C[X]$ of complex polynomials and (z_m) a sequence of complex numbers, such that $|z_m| \to +\infty$. Endow $C[X]$ with the topology τ given by the seminorms $P \longmapsto |P|_m = |P(z_m)|$. Then $(C[X], \tau)$ is a unital commutative and metrizable locally m-convex algebra. Since it has a denumerable algebraic basis, it is subnormable by [8, p. 1039, Lemma]. Denote by $\|.\|$ the vector space norm that induces a topology stronger than τ. Thus, for every m, there exists $k_m > 1 : |P|_m \leq k_m \|P\|$; for every $P \in C[X]$. Then

$$|PQ|_m = |P|_m |Q|_m \leq \|P\| k_m |Q|_m ; \text{ for every } Q \in C[X].$$

But the topology τ can also be defined by the family of seminorms $(|.|_m) \cup (\alpha |.|_m)_{\alpha \geq 1}$ which makes E a locally uniformly convex algebra. It can not be a locally uniformly A-convex algebra. Otherwise, it could be endowed with an algebra norm $\|.\|_0$ the topology of which would be stronger than τ. But then, the characters $P \longmapsto P(z_m)$ should be continuous for $\|.\|_0$. Whence $|P(z_m)| \leq \|P\|_0$; for all m which contradicts $|P(z_m)| \to +\infty$.

Here also, not every element is bounded as it is the case in locally uniformly A-convex algebras. Indeed,

$$\sup_m \left(\limsup_l \left| P^l \right|_m^{\frac{1}{l}} \right) = \sup_m |P(z_m)| = +\infty,$$

for every non constant P.

This example shows also that *a unital locally uniformly convex algebra, which is always subnormable, is not necessarily m-subnormable*. Actually, it is not even A-subnormable. Otherwise, it should be m-subnormable.

Example 6.5. Let (E,τ) be an algebra as in Proposition 6.3 and $(C_b(\mathbb{R}),\beta)$ the well known locally uniformly A-convex algebra, described in Example 5.5. Take the algebra $(E,\tau)\times(C_b(\mathbb{R}),\beta)$. It is a locally uniformly convex algebra which is not locally m-convex nor locally uniformly A-convex. It is a locally A-convex algebra with a continuous multiplication.

Example 6.6. Consider the algebra $(C\,[0,1]\,,\|.\|_1)$ where $\|.\|_1$ is the norm in $L^1\,[0,1]$. Then $(E,\tau)\times(C_b(\mathbb{R}),\beta)\times(C\,[0,1]\,,\|.\|_1)$ is a locally uniformly convex algebra which is also a locally A-convex algebra. It is not a locally uniformly A-convex algebra. The multiplication is only separately continuous.

Example 6.7. Let $E = C^1[0,1]$ be the algebra of $\mathbb{C}$-valued continuous functions on $[0,1]$ with continuous derivative also at extreme points. Endow it with the topology τ given by the seminorms $\|f\|_\infty = \sup\{|f(t)| : 0 \le t \le 1\}$ and $|f|_{K_d} = \sup\{|f'(t)| : t \in K_d\}$, where K_d runs over denumerable compact subsets of $[0,1]$. Then (E,τ) is a unital commutative sequentially complete locally m-convex algebra. It is also a locally uniformly convex algebra. Indeed,

$$\begin{aligned} |fg|_{K_d} &= \sup\{|f'(t)g(t)+f(t)g'(t)| : t\in K_d\} \\ &\le M(f)\left[|g|_{K_d}+|g'|_{K_d}\right], \text{with } M(f)=\max\left(\|f\|_\infty, \|f'\|_\infty\right). \end{aligned}$$

It can not be a locally uniformly A-convex algebra. Otherwise, there should be an algebra norm $\|.\|_0$ the topology of which would be stronger than τ, which moreover, is the coarsest norm with such properties (see [17]). But then,

$$\tau_{\|.\|_\infty} \preceq \tau \preceq \tau_{\|.\|_0} \text{ and } \tau_{\|.\|_0} \preceq \tau_{\|.\|_\infty}.$$

So τ should be equivalent to $\tau_{\|.\|_0}$, which is not the case. Here, the norm of Proposition 6.1 is $\|.\|_\infty + \|.\|'_\infty$, where $\|f\|'_\infty = \|f'\|_\infty$.

Example 6.8. Let $E = C[0,1]$ be the complex algebra of continuous functions on the interval $[0,1]$. Endow it with the vector space norm $\|.\|_1$ namely, $\|f\|_1 = \int_0^1 |f(t)|\,dt$. Then $(E,\|.\|_1)$ is an A-normed algebra. It is pseudo-complete, but not Mackey complete. To obtain a non uniformly A-convex example, take e.g. $C^1[0,1]\times C[0,1]$.

The characterization, via a seminorm, is the next.

Proposition 6.9. *Let (E,τ) be a locally uniformly convex algebra. The following assertions are equivalent:*

(i) (E,τ) is a $(\mathcal{B}, m\mathbb{B})$-barrelled algebra.

(ii) Every seminorm which is m-bounded and the topology of which is locally sequentially complete is automatically continuous.

Proof. As in Proposition 4.1. ∎

The previous result allows a characterization of normed algebras among locally uniformly convex algebras.

Proposition 6.10. *Let (E, τ) be a locally uniformly convex algebra. The following assertions are equivalent:*

(i) (E, τ) is a $(\mathcal{B}, m\mathbb{B})$-barrelled algebra.

(ii) (E, τ) is normable.

Remark 6.11. We do not know if there exists an $(mB, \mathbb{B})$-barrelled algebra which is not a $(B, \mathbb{B})$-one.

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References

[1] M. Akkar, C. Nacir, *Adjunction of unity and invertible elements in uniformly locally A-convex algebra*, Rend. Circ. Mat. Palermo, Serie II, XLIV(1995), 239–248.

[2] G.R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. (3), 15(1965), 399–421.

[3] A.K. Chilana, S. Sharma, *The locally boundedly multiplicatively convex algebras*, Math. Nachr. 77(1977), 139–161.

[4] A.C. Cochran, *Weak A-convex algebras*, Proc. Amer. Math. Soc. 26(1970), 73–77.

[5] A. C. Cochran, *Representation of A-convex algebras*, Proc. Amer. Math. Soc. 41(1973), 473–479.

[6] A.C. Cochran, R. Keown, C.R. Williams, *On a class of topological algebras*, Pacific J. Math. 34(1970), 17–25.

[7] A. El Kinani, *Equicontinuity of power maps in locally pseudo-convex algebras*, Comment. Math. Univ. Carolin. 44(2003), 91–98.

[8] J. Esterle, *Sur la non normabilité de certaines algèbres d'opérateurs,* C. R. Acad. Sc. Paris, t 278, Série A (1974), 1037–1040.

[9] R. Hadjigeorgiou, M. Oudadess, *Strictly real locally convex algebras*, Frontiers Sci. Techno., Hassan II Acad. Sci. Techno. 2 (2011), 1–12.

[10] M. Haralampidou, M. Oudadess, *m-infrabarrelledness and m-convexity*, Bull. Belg. Math. Soc. Simon Stevin, 19(2012), no. 3, 473–483.

[11] H. Hogbé–Nlend, *Théorie des bornologies et applications*, Lecture Notes in Mathematics 213, Springer-Verlag, Berlin-New York, 1971.

[12] J. Horváth, *Topological Vector Spaces and Distributions, Vol. I.*, Addition-Wesley Publ. Co., Reading, Massachusetts, 1966.

[13] B.E. Johnson, *Centralizers on certain topological algebras*, Journal London Math. Soc., 39(1964), 603–614.

[14] A. Mallios, *On the spectra of topological algebras*, J. Functional Analysis, 3(1969), 301–309.

[15] A. Mallios, *Topological algebras. Selected topics*, North-Holland, Mathematics Studies 124, Amsterdam, 1986.

[16] E.A. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. 11, (1952).

[17] M. Oudadess, *Bounded structures in locally A-convex algebras*, ICTAA 2008, 80–88, Math. Stud. (Tartu) 4, Est. Math. Soc., Tartu, 2008.

[18] M. Oudadess, *Remarks on locally A-convex algebras*, Bull. Greek Math. Soc. 56 (2009), 47–55.

[19] M. Oudadess, *Locally uniformly convex algebras* (preprint; Ecole Normale Supérieure de Rabat, 2007).

[20] H.H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York, 1971.

[21] S. Warner, *Inductive limits of normed algebras*, Trans. Amer. Math. Soc. 82(1956), 190–216.

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