A new proof for the bornologicity of the space of slowly increasing functions

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Abstract

A. Grothendieck proved at the end of his thesis that the space $\mathcal{O}_\mathrm{M}$ of slowly increasing functions and the space $\mathcal{O}'_\mathrm{C}$ of rapidly decreasing distributions are bornological. Grothendieck's proof relies on the isomorphy of these spaces to a sequence space and we present the first proof that does not utilize this fact by using homological methods and, in particular, the derived projective limit functor.

1 Introduction and notation

In [Sch66, p. 243] L. Schwartz introduced the space of multipliers of temperate distributions, i.e., the space of slowly increasing functions

$$\mathcal{O}_\mathrm{M} = \{f \in \mathcal{C}^\infty(\mathbb{R}^d)\,;\, \forall \alpha \in \mathbb{N}_0^d\ \exists N \in \mathbb{N} :\ \langle x\rangle^{-N}\partial^\alpha f \in L^\infty\},$$

where $\mathcal{C}^\infty(\mathbb{R}^d)$ is the space of complex valued, infinitely differentiable functions on $\mathbb{R}^d$, $\langle x\rangle = 1 + |x|^2$, ∂^α is the partial derivative, and L^∞ is the Lebesgue space of bounded functions. The dual $\mathcal{O}'_M$ of $\mathcal{O}_\mathrm{M}$ is the space of very rapidly decreasing distributions.

Schwartz also introduced the space of convolutors of temperate distributions, i.e., the space $\mathcal{O}'_C$ of rapidly decreasing distributions, which is the dual of the space

$$\mathcal{O}_\mathrm{C} = \{f \in \mathcal{C}^\infty(\mathbb{R}^d)\,;\, \exists N\, \forall \alpha \in \mathbb{N}_0^d :\ \langle x\rangle^{-N}\partial^\alpha f \in L^\infty\}$$

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of very slowly increasing functions. These spaces are related as in the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{C}} & \subseteq & \mathcal{O}_{\mathrm{M}} \\ \wr\Vert & & \wr\Vert \\ \mathcal{O}'_{M} & \subseteq & \mathcal{O}'_{C} \end{array}$$

where in both cases the Fourier transform can be taken as the isomorphism.

It is comparatively easy to see that the four spaces are nuclear and semi-reflexive, that $\mathcal{O}_{\mathrm{M}}$ and $\mathcal{O}'_C$ are complete and that $\mathcal{O}_{\mathrm{C}}$ and $\mathcal{O}'_M$ are (LF)-spaces and hence bornological. But the completeness of $\mathcal{O}_{\mathrm{C}}$ and $\mathcal{O}'_M$ and the bornologicity of $\mathcal{O}_{\mathrm{M}}$ and $\mathcal{O}'_C$ are not trivial (which was even asserted by Grothendieck, [Gro55, Chap. II, p. 130]). Since the dual of a bornological space is complete and the dual of a complete nuclear space is bornological, these two problems are equivalent (for the definitions of these topological properties and relations between them see [Itō87, Section 424]).

Grothendieck proved that $\mathcal{O}_{\mathrm{M}}$ is bornological by showing that it is isomorphic to a complemented subspace of the sequence space $s\hat{\otimes}_\pi s'$ [Gro55, Chap. II, Lemme 18, p. 132] and verified "directly" that the space $s\hat{\otimes}_\pi s'$ is bornological [Gro55, Chap.II, Prop. 15, p. 125, Cor. 2, p. 128]. We will find out more about this isomorphy in Section 2 and also give a homological proof of the bornologicity of $s\hat{\otimes}_\pi s'$.

In [Kuc85], J. Kučera claimed to have presented a new (and simple) proof for the main properties of the space $\mathcal{O}_{\mathrm{M}}$. That Kučera's proof contains severe mistakes and that it is based on incorrect propositions is clarified in [Lar12], where also the lack of a proof of the bornologicity of $\mathcal{O}_{\mathrm{M}}$, that does not use the isomorphy $\mathcal{O}_{\mathrm{M}} \cong s\hat{\otimes}_\pi s'$, is pointed out. In Section 3 we will give such a proof.

2 Projective limits and the space $s\hat{\otimes}_\pi s'$

Since quotients (and, in particular, complemented subspaces) of bornological spaces are bornological, it was sufficient for Grothendieck to prove that $\mathcal{O}_{\mathrm{M}}$ is isomorphic to a complemented subspace of $s\hat{\otimes}_\pi s'$, where s is the space of rapidly decreasing sequences

$$s = \{(x_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} ; \forall k : \sup_{j\in\mathbb{N}} j^k |x_j| < \infty\}$$

and s' is its dual, the space of slowly increasing sequences

$$s' = \{(x_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} ; \exists k : \sup_{j\in\mathbb{N}} j^{-k} |x_j| < \infty\}.$$

By $s\hat{\otimes}_\pi s'$ we denote the completed projective tensor product of these spaces. E.g., by [Bar12, Remark 1, p. 321], this space $s\hat{\otimes}_\pi s'$ is canonically isomorphic to

$$s\hat{\otimes}_\pi s' \cong \{x \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}} ; \forall n\, \exists N : \sup_{i,j} i^n j^{-N} |x_{i,j}| < \infty\}.$$

In [Val81], M. Valdivia proved that $\mathcal{O}_\mathrm{M}$ is even isomorphic to $s\hat{\otimes}_\pi s'$ itself which answered a question posed in [Gro55, Chap. II, p. 134]. C. Bargetz used this fact, the bornologicity of $s\hat{\otimes}_\pi s'$, and methods of the theory of topological tensor products to obtain the isomorphy $\mathcal{O}_\mathrm{C} \cong s\hat{\otimes}_\iota s'$ [Bar12, Prop. 1, p. 318].

The descriptions of the spaces $\mathcal{O}_\mathrm{M}$ and $s\hat{\otimes}_\pi s'$ already indicate how they can be written as projective limits of LB-spaces (countable inductive limits of Banach spaces)

$$\mathcal{O}_\mathrm{M} = \bigcap_{n\in\mathbb{N}} X_n = \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} X_{n,N}, \tag{1}$$

$$s\hat{\otimes}_\pi s' = \bigcap_{n\in\mathbb{N}} Y_n = \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} Y_{n,N}, \tag{2}$$

where $X_{n,N}$ and $Y_{n,N}$ are the Banach spaces

$$X_{n,N} = \{f \in \mathcal{C}^n(\mathbb{R}^d)\,;\, \|f\|_{n,N} = \sup_{x\in\mathbb{R}^d, |\alpha|\le n} \langle x\rangle^{-N} |\partial^\alpha f(x)| < \infty\},$$

$$Y_{n,N} = \{x \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}\,;\, \|x\|_{n,N} = \sup_{i,j} i^n j^{-N} |x_{i,j}| < \infty\}.$$

These representations as projective limits of LB-spaces are not only natural but also extremely useful since there are very good criteria for checking bornologicity. They are related to the derived projective limit functor $\mathrm{Proj}^1 \mathscr{X}$ (which can be defined as the cokernel of the map $\prod X_n \to \prod X_n$, $(x_n)_n \mapsto (x_n - \varrho^n_{n+1}(x_{n+1}))_n$ where ϱ^n_m are the connecting maps of the projective spectrum $\mathscr{X}$, in our cases, ϱ^n_m are just inclusions). Indeed, an unpublished theorem of D. Vogt (his proof reproduced in [Wen03, Th. 3.3.4]) says that $\mathrm{Proj}\,\mathscr{X}$ is bornological whenever $\mathrm{Proj}^1 \mathscr{X} = 0$. Moreover, there is a variety of evaluable conditions ensuring $\mathrm{Proj}^1 \mathscr{X} = 0$. We are going to apply the following results of Palamodov-Retakh [Pal71] and the second named author, respectively:

> A spectrum $\mathscr{X}$ of LB-spaces satisfies $\mathrm{Proj}^1 \mathscr{X} = 0$ if and only if there are Banach discs D_n in X_n with $\varrho^n_m(D_m) \subseteq D_n$ and
>
> $$\forall\, n \in \mathbb{N}\ \exists\, m \ge n\ \forall\, k \ge m:\ \varrho^n_m(X_m) \subseteq \varrho^n_k(X_k) + D_n.$$

The requirement $\varrho^n_m(D_m) \subseteq D_n$ is sometimes very easy to fulfil but in many cases it is very inconvenient. It can be omitted if either all steps X_n are LS-spaces (i.e., the inclusions $X_{n,N} \hookrightarrow X_{n,N+1}$ are compact) or if a slightly stronger condition of Palamodov-Retakh type is required. Denoting by $\varrho^n_\infty : \mathrm{Proj}\,\mathscr{X} \to X_n$ the obvious map we have:

> A spectrum $\mathscr{X}$ of LB-spaces satisfies $\mathrm{Proj}^1 \mathscr{X} = 0$ if and only if, for every $n \in \mathbb{N}$, there are a Banach discs D_n in X_n and $m \ge n$ with
>
> $$\varrho^n_m(X_m) \subseteq \varrho^n_\infty(\mathrm{Proj}\,\mathscr{X}) + D_n.$$

We refer to [Wen03] for the proofs of these characterization and much more information about derived functors. Typically, the decompositions required in conditions of Retakh-Palamodov type are quite easy to produce in the case of spaces of sequences (or matrices) since one can write $x = \chi x + (1-\chi)x$ where χ is the indicator function of a suitably chosen set. We want to exemplify this by giving a very short proof for the bornologicity of $s\hat{\otimes}_\pi s'$ (which is similar to Vogt's proof of $\mathrm{Ext}^1(s,s) = 0$ [Vog84, Lemma 2.1, p. 359]).

Proposition 1. *The space $s\hat{\otimes}_\pi s'$ is bornological.*

Proof. We keep the notation $s\hat{\otimes}_\pi s' \cong \bigcap_{n\in\mathbb{N}} Y_n = \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} Y_{n,N}$ from above and we will verify the Palamodov-Retakh condition for the unit balls D_n of $Y_{n,0}$ which trivially satisfy $D_{n+1} \subseteq D_n$. For $n \in \mathbb{N}$ we take $m = n+1$ and fix $x \in Y_n$ as well as $k \geq n+1$. Since $x \in Y_{m,M}$ for some $M \in \mathbb{N}$ we have

$$\|x\|_{m,M} = \sup_{i,j} i^m j^{-M} |x_{i,j}| = c < \infty.$$

We set $y_{i,j} = x_{i,j}$ if $i < cj^M$ and $y_{i,j} = 0$ else, as well as $z = x - y$. For $i < cj^M$ we have $z_{i,j} = 0$ and for $i \geq cj^M$ we estimate

$$i^n j^{-0} |z_{i,j}| = i^m j^{-M} |z_{i,j}|\, j^M / i \leq \|x\|_{m,M}/c = 1$$

which proves $z \in D_n$. It remains to show $y \in Y_{k,K}$ for K sufficiently large. Indeed, for $K = M(k-m+1)$ we have $y_{i,j} = 0$ if $i \geq cj^M$ and if $i < cj^M$ we estimate

$$i^k j^{-K} |y_{i,j}| = i^m j^{-M} |y_{i,j}| i^{k-m} j^{M-K} \leq \|x\|_{m,M} c^{k-m} j^{(k-m)M+M-K} = c^{k-m+1}.$$

This proves $\|y\|_{k,K} < \infty$, as required. ∎

3 The new proof

Now we want to prove $\mathrm{Proj}^1 \mathscr{X} = 0$ for the spectrum $\mathscr{X} = (X_n)_{n\in\mathbb{N}}$ in (1) in order to obtain that $\mathcal{O}_\mathrm{M}$ is bornological. Splitting up a given function $f \in X_m$ as $f = \chi f + (1-\chi) f$ with a cut-off function χ (as in the proof of Proposition 1) does not work in this case. But we will see how f can be "split up" in the following proof of Grothendieck's result.

Proposition 2. *The space $\mathcal{O}_\mathrm{M}$ is bornological.*

Proof. To obtain $\mathrm{Proj}^1 \mathscr{X} = 0$ we will show

$$\forall n\, \exists m, N: \; X_m \subseteq \mathcal{O}_\mathrm{M} + B_{n,N} \tag{3}$$

where $B_{n,N}$ is the unit ball of $X_{n,N}$. This condition means that we have to approximate every $f \in X_m$ with respect to the norm $\|\cdot\|_{n,N}$ by elements of $\mathcal{O}_\mathrm{M}$. To achieve such an approximation we use a kernel $K \in \mathcal{O}_\mathrm{M}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$K \geq 0, \; \int_{\mathbb{R}^d} K(t,x)\, dt = 1 \text{ for all } x \in \mathbb{R}^d, \text{ and}$$

$$\operatorname{supp} K(\cdot, x) \subseteq \prod_{j=1}^d [x_j, x_j + \varepsilon\langle x\rangle^{-\mu}] =: A_x \text{ for all } x \in \mathbb{R}^d$$

where we will see later how ε and μ have to be chosen in dependence on $f \in X_m$. We can obtain such a kernel by defining

$$K(t,x) = \varepsilon^{-d}\langle x\rangle^{\mu d}\varphi(\varepsilon^{-1}\langle x\rangle^{\mu}(t-x))$$

for a positive test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with support in $[0,1]^d$ and $\int_{\mathbb{R}^d}\varphi(t)dt = 1$ (the conditions above can be checked easily and $K \in \mathcal{O}_M$ since every derivative of K can be estimated by a polynomial).

We start with the one-dimensional case $d = 1$ where we can take $m = n+1$ and $N = 0$. So let $f \in X_{n+1,M}$ for some $M \in \mathbb{N}$. We want to find $g \in \mathcal{O}_M$ such that $f - g \in B_{n,0}$. At first we set

$$g_n(x) = \int_{\mathbb{R}} f^{(n)}(t)K(t,x)\,dt$$

and show that this is a good approximation to $f^{(n)}$. Since for $l \in \mathbb{N}_0$

$$\left|g_n^{(l)}(x)\right| = \left|\int_{A_x} f^{(n)}(t)\partial_x^l K(t,x)\,dt\right| \le \int_{A_x} |P(t)|\,|Q(t,x)|\,dt \le |R(x)|$$

for some polynomials P, Q, R, the function g_n is contained in $\mathcal{O}_M$. Furthermore we can estimate in virtue of Taylor's formula

$$\left|f^{(n)}(t) - f^{(n)}(x)\right| \le |t-x|\langle\xi(t,x)\rangle^M \|f\|_{n+1,M}$$

with a point $\xi(t,x)$ between t and x. For ε small enough the inequality $\langle\xi(t,x)\rangle \le 2\langle x\rangle$ holds for every $x \in \mathbb{R}$ and $t \in A_x$. We obtain

$$\begin{aligned} |g_n(x) - f^{(n)}(x)| &= \left|\int_{\mathbb{R}}\left(f^{(n)}(t) - f^{(n)}(x)\right)K(t,x)\,dt\right| \\ &\le \int_{A_x}\left|f^{(n)}(t) - f^{(n)}(x)\right|K(t,x)\,dt \\ &\le \int_{A_x}|t-x|\langle\xi(t,x)\rangle^M\|f\|_{n+1,M}K(t,x)\,dt \\ &\le \varepsilon\, 2^M\,\langle x\rangle^{M-\mu}\|f\|_{n+1,M}\int_{A_x}K(t,x)\,dt \\ &= \varepsilon\, 2^M\,\langle x\rangle^{M-\mu}\|f\|_{n+1,M}. \end{aligned} \tag{4}$$

Now if

$$T : \mathcal{O}_M(\mathbb{R}) \to \mathcal{O}_M(\mathbb{R}), h \mapsto \left(x \mapsto \int_0^x h(t)\,dt\right),$$

we can set

$$g(x) = \sum_{j=0}^{n-1}\frac{f^{(j)}(0)}{j!}x^j + (T^n g_n)(x).$$

Then $g \in \mathcal{O}_M$ and since

$$(T^n f^{(n)})(x) = f(x) - \sum_{j=0}^{n-1}\frac{f^{(j)}(0)}{j!}x^j,$$

integrating (4) (the integral starting at 0) yields

$$|g^{(l)}(x) - f^{(l)}(x)| \le 1,\ x \in \mathbb{R}^d, l \le n$$

for ε small enough and μ large enough. Hence $g - f \in B_{n,0}$ and the proof is complete for the one-dimensional case.

Now we will prove the two-dimensional case $d = 2$. We set $m = 2n+1$ and $N = n-1$ in (3). So let $f \in X_{2n+1,M}$ for some M. With the help of a kernel $K \in \mathcal{O}_\mathrm{M}(\mathbb{R}^2 \times \mathbb{R}^2)$ like above, we set

$$g_n(x) = \int_{\mathbb{R}^2} \partial^{(n,n)} f(t) K(t,x)\, dt$$

in order to approximate $\partial^{(n,n)} f$ by g_n. Similar to the one-dimensional case we have

$$\left|\partial^{(n,n)} f(t) - \partial^{(n,n)} f(x)\right| \le c \cdot |t-x| \langle \xi(t,x)\rangle^M \|f\|_{2n+1,M}$$

and $\langle \xi(t,x)\rangle \le 2\langle x\rangle$ for $t \in A_x$ and ε small enough and thus

$$\begin{aligned} |g_n(x) - \partial^{(n,n)} f(x)| &\le c \int_{A_x} |t-x| \langle \xi(t,x)\rangle^M \|f\|_{2n+1,M} K(t,x)\, dt \\ &\le \tilde{c}\, \varepsilon\, \langle x\rangle^{M-\mu} \|f\|_{2n+1,M}. \end{aligned} \tag{5}$$

Let us denote T_j the integral with respect to the j-th component (the integral starting at 0). Applying $T_1 \circ T_2$ n-times to $\partial^{(n,n)} f(x)$ yields

$$(T_1^n T_2^n f)(x) =$$
$$f(x) + \sum_{\alpha < (n,n)} \partial^\alpha f(0,0) \frac{x^\alpha}{\alpha!} - \sum_{j=0}^{n-1} \partial^{(j,0)} f(0,x_2) \frac{x_1^j}{j!} - \sum_{j=0}^{n-1} \partial^{(0,j)} f(x_1,0) \frac{x_2^j}{j!}.$$

As in the one-dimensional case we can choose $g_0^1, \ldots g_{n-1}^1, g_0^2, \ldots, g_{n-1}^2 \in \mathcal{O}_\mathrm{M}(\mathbb{R})$ such that $\|g_j^1 - \partial^{(0,j)} f(\cdot,0)\|_{n,0} \le \varepsilon$ and $\|g_j^2 - \partial^{(j,0)} f(\cdot,0)\|_{n,0} \le \varepsilon$. Defining

$$g(x) = (T_1^n T_2^n) g_n(x) - \sum_{\alpha<(n,n)} \partial^\alpha f(0,0) \frac{x^\alpha}{\alpha!} + \sum_{j=0}^{n-1} g_j^2(x_2) \frac{x_1^j}{j!} + \sum_{j=0}^{n-1} g_j^1(x_1) \frac{x_2^j}{j!}$$

and applying $T_1^n T_2^n$ to (5) yields

$$|g(x) - f(x)| \le$$
$$\varepsilon + \sum_{j=0}^{n-1} \left(\left| g_j^1(x_1) - \partial^{(0,j)} f(x_1,0) \right| \frac{|x_2|^j}{j!} + \left| g_j^2(x_2) - \partial^{(j,0)} f(0,x_2) \right| \frac{|x_1|^j}{j!} \right)$$

for μ large enough which implies

$$|g(x) - f(x)| \le \varepsilon + \varepsilon \sum_{j=0}^{n-1} \frac{|x_2|^j}{j!} + \varepsilon \sum_{j=0}^{n-1} \frac{|x_1|^j}{j!} \le \varepsilon\, c\, \langle x\rangle^{n-1}$$

for some $c > 1$. Since similar estimates also hold for $|\partial^\alpha g(x) - \partial^\alpha f(x)|$, $|\alpha| \leq n$, we obtain $g - f \in B_{n,n-1}$ and the proof is complete for $d = 2$.

The general case $d \in \mathbb{N}$ is very similar. Inductively we want to show

$$X_{dn+1} \subseteq \mathcal{O}_\mathrm{M} + B_{n,(d-1)(n-1)}$$

and start by approximating $\partial^{(n,\dots,n)} f$ by $g_n(x) := \int_{R^d} \partial^{(n,\dots,n)} f(t) K(t,x)\, dt$. Then we integrate the estimate of $g_n - \partial^{(n,\dots,n)} f$ n-times with respect to each component. The integral $T_1^n \cdots T_d^n \partial^{(n,\dots,n)} f$ contains f as a summand and terms that are the product of a derivative of f that only depends on less than d components and a polynomial in less than d components with exponents less than n. But we can estimate the functions that only depend on less than d variables by the induction hypothesis and hence we can obtain $g \in \mathcal{O}_\mathrm{M}$ with $g - f \in B_{n,(d-1)(n-1)}$. ∎

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