Lineability of Nowhere Monotone Measures*

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Abstract

We give a sufficient condition for the set of nowhere monotone measures to be a residual G_δ set in a subspace of signed Radon measures on a locally compact second-countable Hausdorff space with no isolated points. We prove that the set of nowhere monotone signed Radon measures on a d-dimensional real space $\mathbb{R}^d$ is lineable. More specifically, we prove that there exists a vector space of dimension $\mathfrak{c}$ (the cardinality of the continuum) of signed Radon measures on $\mathbb{R}^d$ every non-zero element of which is a nowhere monotone measure that is almost everywhere differentiable with respect to the d-dimensional Lebesgue measure. Using this result we show that the set of these measures is even maximal dense-lineable in the space of bounded signed Radon measures on $\mathbb{R}^d$ that are almost everywhere differentiable with respect to the d-dimensional Lebesgue measure.

1 Introduction

Assume that M is a subset of a topological vector space X and α is a cardinal number. Then M is called *lineable* if $M \cup \{0\}$ contains an infinite-dimensional linear subspace. More specifically, M is called *α-lineable* if $M \cup \{0\}$ contains an α-dimensional linear subspace. If $M \cup \{0\}$ contains a closed infinite-dimensional linear subspace it is called *spaceable*. If the set $M \cup \{0\}$ contains an infinite-dimensional subspace that is dense in X, the set M is called *dense-lineable*. If M is *dim*(X)-(dense)lineable, we call it *maximal (dense)-lineable*. Finally, if $M \cup \{0\}$ contains an infinitely generated algebra we call it *algebrable*.

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The concept of *lineability* was coined by V. I. Gurariy in the early 2000's and it first appeared in print in [Gurariy and Quarta(2004)], [Aron et al.(2005)] and [Seoane-Sepúlveda(2006)]. Note, however, that V. I. Gurariy's interest in linear structures in generally non-linear settings dates as far back as 1966 (see [Gurariy(1966)]). The study of large vector structures in sets of real and complex functions has attracted many mathematicians in the last decade. For example, in [Aron et al.(2005), Theorem 4.3], the authors prove that the set of everywhere surjective functions on $\mathbb{R}$ is $2^{|\mathbb{R}|}$-lineable. This result has been further improved in [Gámez-Merino et al.(2010)], where the authors prove that the set of strongly everywhere surjective functions and the set of perfectly everywhere surjective functions are both $2^{|\mathbb{R}|}$-lineable as well. In the same paper, the authors show that the set $\mathcal{NMD}$ of nowhere monotone everywhere differentiable functions on $\mathbb{R}$ is $|\mathbb{R}|$-lineable (this result has been a motivation for our study of lineability of the set of nowhere monotone measures).

As more and more examples of lineable sets were found, the questions of dense-lineability and spaceability gradually attracted more attention. In [Aron et al.(2009)] the authors show, among other results, that the set of nowhere differentiable functions, the set of non-analytic C^∞ functions, the set of functions in $C^m \setminus C^n$ (with $m < n$) and the set $\mathcal{NMD}$ (all considered on a non-empty closed interval $[a,b]$) are all dense-lineable in $C([a,b])$. In [García et al.(2010)] the authors construct a Banach spaces of non-Riemann-integrable bounded functions that have an antiderivative at each point point of an interval, a Banach space of differentiable functions on $\mathbb{R}^d$ failing the Denjoy-Clarkson property and a Banach space of infinitely differentiable functions that vanish at infinity and are not the Fourier transform of any Lebesgue-integrable function. In [Aron et al.(2006)], it is shown that given any Lebesgue-null subset $J \subset \mathbb{T}$ the set of functions in $C(\mathbb{T})$ whose Fourier series diverges in J contains a dense infinitely generated algebra. For some other recent results concerning algebrability see, for example, [Bayart and Quarta(2007)], [Aron et al.(2010)], or [García-Pacheco et al.(2007)].

Even though most of the results on lineability focus on the study of sets of functions, papers concerning lineability of vector measures [Muñoz-Fernández et al.(2008)] and lineability of operators [Puglisi and Seoane-Sepúlveda(2008)] have also been published. Some results concerning general properties of lineable sets have appeared, despite the fact that most literature deals with specific sets of functions or operators. For example, in [Aron et al.(2009), Theorem 2.2] the authors present a sufficient condition on a lineable subset of a separable Banach space to be dense-lineable. This technique allows for short, elegant proofs of several of the theorems mentioned above. These conditions have been further studied in a recent article [Bernal-González and Cabrera(2014)] in which the authors prove an analogous result in the setting of general vector spaces. In another recent article (see [Ciesielski et al.(2014)]), the authors present the concept of *maximal lineability cardinal number* and use it to prove several results concerning lineability of specific sets of real functions. Lastly, we recommend the recently published survey [Bernal-González et al.(2014)] to the interested reader. It provides an exceptionally well arranged list of the most important results in the field, far beyond the scope of this article.

2 Notation

For $d \in \mathbb{N}$ we denote by λ_d the d-dimensional Lebesgue measure (we write λ instead of λ_1). For a function $g : \mathbb{R}^d \mapsto \mathbb{R}$ we denote by $\{g > 0\}$ the set of all $x \in \mathbb{R}^d$ such that $g(x) > 0$. By $\mathfrak{c}$ or $|\mathbb{R}|$ we denote the cardinality of the set of real numbers. Furthermore, for P either $\mathbb{R}$ or a non-empty closed interval of $\mathbb{R}$ we denote by $C(P)$ the set of all real continuous functions on P endowed with the uniform norm $||f||$. For a locally compact Hausdorff space X we denote by $C_c(X)$ the set of all real continuous functions on X with compact support.

For a signed measure μ we denote by μ^+ and μ^- the positive and negative variation of μ, respectively. We denote by $\mathcal{M}(X)$ the set of all signed Radon measures on a locally compact Hausdorff space X endowed with the norm $||\mu|| = \mu^+(X) + \mu^-(X)$. For $\mu \in \mathcal{M}(X)$ we denote the support of μ by *supp* μ. Moreover, if μ and ν are two measures on X such that for any open set $G \subset X$, $\nu(G) = 0$ implies $\mu(G) = 0$, we write $\mu \ll \nu$ and say that μ is *absolutely continuous* with respect to ν. For any given measure $\nu \in \mathcal{M}(X)$ we denote the closed subspace $\{\mu \in \mathcal{M}(X) : \mu \ll \nu\}$ by $\mathcal{AC}_\nu(X)$.

We recall the definitions of density topology and of approximately continuous functions here for the reader's convenience. A measurable set $B \subset \mathbb{R}^d$ is called *density open* if

$$\lim_{r \to 0_+} \frac{\lambda_d(B(x,r) \cap B)}{\lambda_d(B(x,r))} = 1,$$

for every $x \in B$. It is clear that every euclidean open set is also density open, that is, the density topology is finer than the euclidean topology.

A function $F : \mathbb{R}^d \mapsto \mathbb{R}$ is called *approximately continuous* if the sets $\{F > \beta\}$ and $\{F < \beta\}$ are density open for all $\beta \in \mathbb{R}$. There are several mutually equivalent ways how to define approximately continuous functions. The reader interested in the density topology and in the properties of approximately continuous functions should consult [Bruckner(1994)], [Lukeš and Malý(2005)] or [Lukeš et al.(1986)] where these topics are presented in detail.

3 Nowhere Monotone Measures, Spaces with Humps

Let us start with a definition.

Definition 3.0.1. Let X be a locally compact Hausdorff space and μ be a Radon measure on X. We say that μ is *nowhere monotone* if $\mu^+(G) > 0$ and $\mu^-(G) > 0$ for every non-empty open set $G \subset X$.

The concept of nowhere monotone measures has its origins in works concerning Choquet theory (see [McDonald(1971)], [McDonald(1973)]). However, the concept closely relates to that of nowhere monotone functions. Indeed, suppose that f is a nowhere monotone function of bounded variation on $[0, 1]$. Then f is a distribution function of some nowhere monotone Lebesgue-Stieltjes measure. This example can be, in fact, taken as a direct motivation for defining these measures.

Once the relation between nowhere monotone functions and nowhere monotone measures has been established, various questions arise. We have already, albeit informally, established that such measures exist. In the following section we will prove that these measures form a residual set in the space of signed Radon measures $\mathcal{M}(X)$ on specific Hausdorff spaces. Note that this is analogous to the density of the set of nowhere monotone functions in $C([a,b])$ (see [Aron et al.(2009), Theorem 3.3]). To prove this result we will introduce the notion of *spaces with humps*.

Definition 3.0.2. Let $\mathcal{A}$ be a closed subspace of $\mathcal{M}(X)$. We say that $\mathcal{A}$ *has humps*, if there exists $q \in (0,1)$ such that for every non-empty open set $G \subset X$ there exists $\mu \in \mathcal{A}$, such that $||\mu|| = 1$ and $\mu(G) \geq q$.

The idea of humps first appeared in a slightly less general form in a Bachelor thesis of M. Kolář (see [Kolář(2009)]).

The following lemma establishes that for certain $\mu \in \mathcal{M}(X)$ the spaces $\mathcal{AC}_\mu(X)$ have humps. This will prove useful later, when we study lineability of nowhere monotone measures.

Lemma 3.0.3. *Suppose that μ_0 is a locally finite measure on X such that $supp\, \mu_0 = X$. Then $\mathcal{AC}_{\mu_0}(X)$ has humps.*

Proof. Let $G \subset X$ be a non-empty open set such that $|\mu_0(G)|$ is finite and $|\mu_0|(G) =: \alpha > 0$. We may assume that $\mu_0^+(G) \geq \frac{\alpha}{2}$ (otherwise consider $-\mu_0$ instead of μ_0). By inner regularity of μ_0^+ and μ_0^- there exist disjoint compact sets $K^+ \subset G$ and $K^- \subset G$ such that $\mu_0 = \mu_0^+$ on K^+, $\mu_0 = \mu_0^-$ on K^- and $\mu_0^+(K^+) + \mu_0^-(K^-) > \frac{3\alpha}{4}$. Since X is Hausdorff, there exist disjoint open sets V and W such that $K^+ \subset V \subset G$ and $K^- \subset W \subset G$. By Urysohn's lemma we can construct functions $f, g \in C_c(X)$, $0 \leq f \leq 1$, $0 \leq g \leq 1$, $f = 1$ on K^+ and $f = 0$ on $X \setminus V$, $g = 1$ on K^- and $g = 0$ on $X \setminus W$. Let $h = f - g$ and $\frac{D\mu}{D\mu_0} = h$. Then $||\mu|| \leq \alpha$ and

$$\mu(G) \geq \mu_0^+(K^+) + \mu_0^-(K^-) - |\mu_0|(G \setminus (K^+ \cup K^-)) \geq \frac{3\alpha}{4} - \frac{\alpha}{4} = \frac{\alpha}{2}$$

It easily follows that the measure $\nu := \frac{\mu}{||\mu||}$ has all the desired properties. ∎

Theorem 3.0.4. *Let X be a locally compact second-countable Hausdorff space with no isolated points and let $\mathcal{A} \subseteq \mathcal{M}(X)$ be a closed subspace that has humps. Then the set of all nowhere monotone Radon signed measures on X is residual in $\mathcal{A}$.*

Proof. Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable basis of open sets in X and denote by $\mathcal{N}(X)$ the set of all nowhere monotone measures in X. Define

$$E_n^+ := \{\mu \in \mathcal{A} : \mu^+(G_n) = 0\} \text{ and } E_n^- := \{\mu \in \mathcal{A} : \mu^-(G_n) = 0\}.$$

Clearly

$$\mathcal{A} \setminus \mathcal{N} = \bigcup_{n\in\mathbb{N}} E_n^+ \cup \bigcup_{n\in\mathbb{N}} E_n^-.$$

Fix $n \in \mathbb{N}$. To prove that E_n^+ is closed, consider a sequence $\{\mu_k\}_{k\in\mathbb{N}}$ in E_n^+ such that $||\mu_k - \mu|| \to 0$ for some $\mu \in \mathcal{A}$. Then also $\mu_k \overset{\omega^*}{\to} \mu$. Therefore (see [Lukeš and Malý(2005), Theorem 17.4])

$$\mu^+(G_n) \leq \liminf_k \mu_k^+(G_n) = 0.$$

To prove that the interior of E_n^+ is empty fix $\mu \in E_n^+$, $\varepsilon > 0$ and $q \in (0,1)$ from Definition 3.0.2. Since X is Hausdorff and has no isolated points, G_n is infinite. Therefore there exists $z \in G_n$ such that $\mu^-(\{z\}) < \frac{q\varepsilon}{2}$. By regularity of μ there exists an open set $G \subset G_n$ such that $z \in G$ and $\mu^-(G) < \frac{q\varepsilon}{2}$. Since $\mathcal{A}$ has humps, we can find $\nu \in \mathcal{A}$ such that $||\nu|| = \varepsilon$ and $\nu(G) \geq q\varepsilon$. Set $\gamma := \mu + \nu$. Then $||\mu - \gamma|| = \varepsilon$ and $\gamma^+(G_n) \geq \gamma^+(G) \geq q\varepsilon - \frac{q\varepsilon}{2} > 0$, which means that $\gamma \notin E_n^+$.

The sets E_n^+ are closed and their interiors are empty, therefore they are nowhere dense. Using a similar argument we can prove that the sets E_n^- have the same properties. ∎

Remark 3.0.5. Letting $\mathcal{A} = \mathcal{M}(X)$ in the previous theorem answers the question of existence of nowhere monotone measures in $\mathcal{M}(X)$ for suitable choices of X. We will, however, use Theorem 3.0.4 to prove the existence and lineability of a more specific type of nowhere monotone measures.

A question one might ask at this point is under what assumptions can we expect $\mathcal{A}$ to have humps. It follows immediately from definition that no finite-dimensional subspace of $\mathcal{M}(X)$ can have this property. It is also not difficult to come up with an example of a subspace generated by a countably infinite sequence of measures that does (for example, for $X = \mathbb{R}$ consider the subspace $\mathcal{A} = \overline{span}\{\varepsilon_q,\ q \in \mathbb{Q}\}$). However, not every infinitely generated closed subspace of $\mathcal{M}(X)$ has to have humps. Take for example $\mathcal{A} = \overline{span}\{\mu_1, \mu_2, \dots\}$ such that $\bigcup_n \operatorname{supp} \mu_n \subset X \setminus G$, where G is a non-empty open subset of X. But even when the the sets $\operatorname{supp} \mu_n$ form a covering of X the space $\mathcal{A}$ need not have humps. We will formulate this as a separate result.

Theorem 3.0.6. *There exists an infinite-dimensional space* $\mathcal{A} \subset \mathcal{M}([0,1])$, $\mathcal{A} = \overline{span}\{\mu_1, \mu_2, \dots\}$ *such that*

$$\bigcup_n supp\, \mu_n = [0,1], \tag{1}$$

$\mathcal{A}$ *does not have humps and the set of all nowhere monotone measures on* $[0,1]$ *is residual in* $\mathcal{A}$.

Proof. Define the sequence of generating measures as follows:

$$\frac{D\mu_n}{D\lambda} := 2^n \chi_{I_n}, \quad n \in \mathbb{N},$$

where $I_n := [1 - 2^{n-1}, 1 - 2^n]$. It is obvious that (1) holds. To prove that $\mathcal{A}$ does not have humps, pick $0 < \varepsilon < \frac{1}{2}$ and find an open interval $J \subset I_1$ such that $\lambda(J) < \varepsilon$. For any $\nu \in span\{\mu_1, \mu_2, \dots\}$, $||\nu|| = 1$ we have

$$|\nu(J)| < \mu_1(J) < 2\varepsilon.$$

For a general $\overline{\nu} \in \mathcal{A}$, $||\overline{\nu}|| = 1$ find a sequence

$$\{\nu_k\}_{k=1}^{\infty} \subset span\{\mu_1, \mu_2, \dots\}, \quad ||\nu_k|| = 1,\ k \in \mathbb{N},$$

such that $||\overline{\nu} - \nu_k|| \to 0$. Then also $\nu_k \xrightarrow{\omega^*} \overline{\nu}$. Using [Lukeš and Malý(2005), Theorem 17.4] and the previous paragraph we get

$$\overline{\nu}^+(J) \leq \liminf_k \nu_k^+(J) < 2\varepsilon.$$

Using the same argument for $-\overline{\nu}$ instead of $\overline{\nu}$ yields the same inequality for $\overline{\nu}^-(J)$. Hence we have

$$|\overline{\nu}(J)| < 2\varepsilon.$$

This shows that $\mathcal{A}$ does not have humps.

Since the proof of residuality of nowhere monotone measures in $\mathcal{A}$ is very similar to the proof of Theorem 3.0.4 we will make free use of its notation. In fact, the proof of closedness of the sets E_n^+ is exactly the same. To prove that these sets have empty interior, pick $\mu \in E_n^+$ and $\varepsilon > 0$. Since the support sets of generating measures form a covering of $[0,1]$, there exists an interval I_m such that $G_m \cap I_m \neq \emptyset$. Set $\kappa = \mu + \varepsilon\mu_m$. Then $||\mu - \kappa|| = \varepsilon$ and $\kappa^+(G_m) = \varepsilon\mu_n(G_m) > 0$, which means that $\kappa \notin E_n^+$. This finishes the proof. ■

4 Lineability of Nowhere Monotone λ_d-differentiable Measures

In this section we will prove that not only there exists a nowhere monotone measure on $\mathbb{R}^d$ that is a.e. differentiable with respect to the d-dimensional Lebesgue measure but also that the set of all such measures is $|\mathbb{R}|$-lineable.

Corollary 4.0.1. *There exists a nowhere monotone Radon signed measure μ on $\mathbb{R}^d$ that is absolutely continuous with respect to Lebesgue measure λ_d.*

Proof. According to Lemma 3.0.3 the set $\{\mu \in \mathcal{M}(\mathbb{R}^d) : \mu \ll \lambda_d\}$ has humps. It remains to use Theorem 3.0.4. ■

To prove the existence of an everywhere differentiable nowhere monotone measure we need the following theorem (for details see [Lukeš et al.(1986), Chapter 3]).

Theorem 4.0.2. *Let $E \subset \mathbb{R}^d$ be a density open F_σ set. Then there exists an approximately continuous function ϕ, $0 \leq \phi \leq 1$, such that $\{\phi > 0\} = E$.*

Theorem 4.0.3. *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a measurable function. There exists a Radon measure μ on $\mathbb{R}^d$ that is absolutely continuous with respect to λ_d and such that $\frac{D\mu}{D\lambda_d}$ exists everywhere on $\mathbb{R}^d$. Moreover,*

$$P := \left\{\frac{D\mu}{D\lambda_d} > 0\right\} \subset \{f > 0\},\ N := \left\{\frac{D\mu}{D\lambda_d} < 0\right\} \subset \{f < 0\}$$

and

$$\lambda_d(\{f > 0\} \setminus P) = \lambda_d(\{f < 0\} \setminus N) = 0.$$

Proof. Let $\widetilde{P}$ be the interior of the set $\{f > 0\}$ in the density topology. By the Lebesgue density theorem, $\lambda_d(\{f > 0\} \setminus \widetilde{P}) = 0$. By regularity of Lebesgue measure, there exists an F_σ set $P \subset \widetilde{P}$ such that

$$\lambda_d(\{f > 0\} \setminus P) = \lambda_d(\widetilde{P} \setminus P) = 0.$$

It is obvious that P is also open in the density topology. By Theorem 4.0.2 there exists an approximately continuous function ϕ such that $0 \leq \phi \leq 1$, $\{\phi > 0\} = P$. We may assume that $\phi \in L^1(\mathbb{R}^d)$ (otherwise consider the function $\widetilde{\phi}(x) := \phi(x) \cdot e^{-||x||^2}$ instead).

Similarly, there exists a density open F_σ set N such that $N \subset \{f < 0\}$,

$$\lambda_d(\{f < 0\} \setminus N) = 0$$

and an approximately continuous function $\psi \in L^1(\mathbb{R}^d)$ such that $0 \leq \psi \leq 1$ and $\{\phi > 0\} = N$. Let μ be a measure on $\mathbb{R}^d$ such that $\frac{D\mu}{D\lambda_d} = \psi - \phi$. It is easy to check that the sets P, N and the measure μ have all the desired properties. ■

Theorem 4.0.4. *There exists a nowhere monotone Radon signed measure μ on $\mathbb{R}^d$ that is everywhere differentiable with respect to λ_d.*

Proof. Let ν be a nowhere monotone measure that is absolutely continuous with respect to λ_d (the existence of such a measure is guaranteed by Corollary 4.0.1) and let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a representative of the Radon-Nikodym derivative $\frac{D\nu}{D\lambda_d}$. It remains to use the previous theorem to finish the proof. ■

Remark 4.0.5. We obtain an even stronger result in the case $d = 1$. The distribution function of the measure constructed in the previous theorem is then even everywhere differentiable in the classical sense.

Theorem 4.0.6. *The set of all nowhere monotone almost everywhere differentiable Radon signed measures on $\mathbb{R}^d$ is $|\mathbb{R}|$-lineable.*

Proof. First, put $f_\alpha(x) := e^{-\frac{\alpha}{1+||x||^2}}$, $x \in \mathbb{R}^d$, for $\alpha \in \mathbb{R}$. According to Theorem 4.0.4, there exists a nowhere monotone measure μ such that $\frac{D\mu}{D\lambda_d} = f$ exists everywhere on $\mathbb{R}^d$. Define the measures μ_α for $\alpha \in \mathbb{R}$ as follows

$$\frac{D\mu_\alpha}{D\lambda_d} := f \cdot f_\alpha. \tag{2}$$

Since

$$\{(f \cdot f_\alpha) > 0\} = \{f > 0\} \cap \{f_\alpha \neq 0\},$$
$$\{(f \cdot f_\alpha) < 0\} = \{f < 0\} \cap \{f_\alpha \neq 0\},$$

the measures μ_α are nowhere monotone. Consider a measure

$$\nu := \sum_{i=1}^{n} b_i \mu_{\alpha_i}$$

for some $n \in \mathbb{N}$ and some b_i not all zero. Then

$$\frac{D\nu}{D\lambda_d} = f \sum_{i=1}^{n} b_i f_{\alpha_i}$$

λ_d-almost everywhere. Denote

$$g(x) := \sum_{i=1}^{n} b_i f_{\alpha_i}(x), \; x \in \mathbb{R}^d.$$

The function g is constant on every sphere centered in the origin. A simple observation yields that its restriction on any one-dimensional subspace of $\mathbb{R}^d$ takes zero value in at most countably many points. Thus we get

$$\lambda_d(\{g = 0\}) = 0.$$

Hence ν is non-trivial. Furthermore, since

$$\left\{ \frac{D\nu}{D\lambda_d} > 0 \right\} = (\{f > 0\} \cap \{g > 0\}) \cup (\{f < 0\} \cap \{g < 0\}),$$
$$\left\{ \frac{D\nu}{D\lambda_d} < 0 \right\} = (\{f > 0\} \cap \{g < 0\}) \cup (\{f < 0\} \cap \{g > 0\}),$$

the measure ν is nowhere monotone. To finish the proof consider the set

$$span\{\mu_\alpha : \; \alpha \in \mathbb{R}\}.$$

∎

Remark 4.0.7. The idea of using the functions $e^{-\frac{\alpha}{1+||x||^2}}$ to produce the set $\{\mu_\alpha : \; \alpha \in \mathbb{R}\}$ was first used in [Gámez-Merino et al.(2010)], where it was used to prove that the set of nowhere monotone everywhere differentiable functions on $\mathbb{R}$ is $|\mathbb{R}|$-lineable. The fact that this method could have been used in our proof is not too surprising taking in mind our previous commentary on the analogy between nowhere monotone functions and nowhere monotone measures.

5 Maximal Dense-lineability of Nowhere Monotone Measures in $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$

In this section we aim to prove that $\mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$ is even maximal dense-lineable in $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$. Let us first recall the notion of *strong sets* (see [Aron et al.(2009)]): if A and B are subsets of a vector space X, then A is said to be *stronger* than B if $A + B \subseteq A$. The following recent result (see [Bernal-González and Cabrera(2014), Theorem 2.3 (c)]) will play a crucial role in our proof.

Theorem 5.0.1. *Assume that X is a topological vector space. Let $A \subset X$. Suppose that there exists a subset $B \subset X$ such that A is stronger than B and B is dense-lineable. If the origin possesses a fundamental system $\mathfrak{U}$ of neighborhoods with $\mathrm{card}(\mathfrak{U}) \leq \dim(X)$, A is maximal lineable and $A \cap B = \emptyset$, then A is maximal dense-lineable. In particular, the same conclusion follows if X is metrizable, A is maximal lineable and $A \cap B = \emptyset$.*

Note that the above theorem strengthens the result in [Aron et al.(2009), Theorem 2.2]. Neither separability, nor metrizability of X are needed as a general assumption. Moreover, if the sets A and B are disjoint, the above theorem even provides an estimate of the dimension of the obtained subspaces.

Some additional notation is required in the following: We denote by

$$C_k^1 \subset [0,1], \quad k \in \mathbb{N},$$

the *fat Cantor set* constructed by the well-known inductive process in which in the n-th step we remove 2^{n-1} intervals of length $4^{-(k+n)}$. For any $k, d \in \mathbb{N}$, $d > 1$ we denote by $C_k^d \subset [0,1]^d$ the set

$$C_k^d := \underbrace{C_k^1 \times \cdots \times C_k^1}_{d}.$$

The sets $\{C_k^d\}_{d,k \in \mathbb{N}}$ are closed and nowhere dense in the respective spaces. They also satisfy the following property: For any $d \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\lambda_d(C_k^d) \geq 1 - \varepsilon.$$

We also denote

$$I_n^d := [-n, n]^d, \quad n, d \in \mathbb{N}.$$

Each I_n^d can be written as a union of $(2n)^d$ cubes of λ_d-measure 1. We denote the set of these cubes $\{I_{n,k}^d\}_{k=1}^{(2n)^d}$.

Lemma 5.0.2. *Let us denote for $d \in \mathbb{N}$*

$$B_d := \left\{ \mu \in \mathcal{AC}_{\lambda_d}(\mathbb{R}^d) \setminus \{0\} : \overline{spt} \left\{ \frac{D\mu}{D\lambda} \right\} \text{ is nowhere dense} \right\}.$$

Then B_d has the following properties:

(i) B_d is nonempty,

(ii) $B_d \cup \{0\}$ is closed under linear combinations,

(iii) $\mathcal{NM}_{\lambda_d}(\mathbb{R}^d) \cap B_d = \emptyset$,

(iv) $\mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$ is stronger than B_d.

Proof. To prove (i) consider $\nu \in \mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$ defined as

$$\frac{D\nu}{D\lambda_d} := \chi_{C_1^d}.$$

Clearly $\nu \in B_d$. Properties (ii) and (iii) follow trivially from the definition of B_d. To prove (iv), let $a \in \mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$, $b \in B_d$, and $I \subset \mathbb{R}^d$ open. Since $\overline{spt}\left\{\frac{Db}{D\lambda}\right\}$ is nowhere dense, there exists a nonempty open set $I' \subset I$ such that $|b|(I') = 0$. Thus

$$(a+b)^+(I) \geq (a+b)^+(I') = a^+(I') > 0,$$
$$(a+b)^-(I) \geq (a+b)^-(I') = a^-(I') > 0$$

and $a + b \in \mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$. ■

Lemma 5.0.3. *The set B_d defined in Lemma 5.0.2 is dense in $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$ for any $d \in \mathbb{N}$.*

Proof. Let $\mu \in \mathcal{AC}_{\lambda_d}(\mathbb{R}^d) \setminus B_d$, $||\mu|| = 1$ and $\varepsilon > 0$ be given. Find $n \in \mathbb{N}$ such that $|\mu|(\mathbb{R}^d \setminus I_n^d) \leq \frac{\varepsilon}{2}$. Denote

$$c := \lambda_d \left(\overline{spt} \left\{ \frac{D\mu}{D\lambda_d} \right\} \cap I_n^d \right)$$

and let $\varepsilon' := \min \left\{ \frac{\varepsilon}{2}, \frac{c}{2} \right\}$. Find $k \in \mathbb{N}$ such that

$$\lambda_d(C_k^d) \geq 1 - \frac{\varepsilon'}{(2n)^d}.$$

For every $j \in \{1, \ldots, (2n)^d\}$ denote by $C_{k,j}^d$ the copy of C_k^d in $I_{n,j}^d$ and let

$$C_{\varepsilon'} := \bigcup_{j=1}^{(2n)^d} C_{k,j}^d.$$

Finally, set

$$\frac{D\nu}{D\lambda_d} := \frac{D\mu}{D\lambda_d} \chi_{C_{\varepsilon'}}.$$

We claim that $\nu \in B_d$. Indeed, since

$$\overline{spt} \left\{ \frac{D\nu}{D\lambda_d} \right\} \subseteq C_{\varepsilon'},$$

we have $\nu \in B_d \cup \{0\}$. But $\nu = 0$ would imply

$$\begin{aligned} \lambda_d \left(\overline{spt} \left\{ \frac{D\mu}{D\lambda_d} \right\} \right) &= \lambda_d \left(\overline{spt} \left\{ \frac{D\mu}{D\lambda_d} \right\} \cap (I_n^d \setminus C_{\varepsilon'}) \right) \\ &\leq \lambda_d \left(I_n^d \setminus C_{\varepsilon'} \right) \leq (2n)^d \frac{\varepsilon'}{(2n)^d} \leq \frac{c}{2}, \end{aligned}$$

which is a contradiction. Thus, $\nu \in B_d$. Furthermore,

$$\begin{aligned} ||\mu - \nu|| &= \int\limits_{R^d} \left| \frac{D\mu}{D\lambda_d} - \frac{D\nu}{D\lambda_d} \right| d\lambda_d \\ &\leq \int\limits_{I_n^d \setminus C_{\varepsilon'}} \left| \frac{D\mu}{D\lambda_d} \right| d\lambda_d + \frac{\varepsilon}{2} \\ &\leq \lambda_d(I_n^d \setminus C_{\varepsilon'}) + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

This finishes the proof. ∎

Theorem 5.0.4. *The set $\mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$ is maximal dense-lineable in $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$ for any $d \in \mathbb{N}$.*

Proof. As mentioned at the beginning of this section, we aim to use Theorem 5.0.1. If follows from Riesz' theorem (see [Rudin(1987), Theorem 6.19]) that the cardinality of $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$ is $\mathfrak{c}$. Thus, by Theorem 4.0.6, $\mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$ is maximal-lineable in $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$. Furthermore, it follows from Lemma 5.0.2(ii) and Lemma 5.0.3 that B_d is dense-lineable in $\mathcal{AC}_{\lambda_d}(\mathbb{R}^d)$. Disjointness of B_d and $\mathcal{NM}_{\lambda_d}(\mathbb{R}^d)$ follows from Lemma 5.0.2(iii) and by Lemma 5.0.2(iv) $\mathcal{NM}_{\lambda_d}(\mathbb{R})$ is stronger than B_d. The result thus follows from Theorem 5.0.1. ∎

6 Final Remarks, Open Problems

Further inspection of spaces with humps could provide some useful results. The author would be interested in finding out whether some useful characterizations of these spaces can be found. One could also ask whether restricting the values of the coefficient q to some strictly smaller interval would yield some non-trivial (and perhaps interesting) classes of spaces. Lastly, the space constructed in Theorem 3.0.6 has the following property: for every open set $I \subset [0,1]$, there exist only finitely many measures in $\{\mu_n\}_{n\in\mathbb{N}}$ such that I intersects the supports of these measures. This leads to the following question: Suppose we have a closed infinitely-dimensional subspace $\mathcal{A} \subset \mathcal{M}(X)$, $\mathcal{A} = \overline{span}\{\mu_\gamma,\ \gamma \in \Gamma\}$ such that for every non-empty open set $G \subset X$ there exist infinitely many measures in the set $\{\mu_\gamma\}_{\gamma\in\Gamma}$ such that the supports of these measures intersect G. Does $\mathcal{A}$ then have to have humps?

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