Existence and multiplicity results for a class of degenerate quasilinear elliptic systems

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Abstract

In this paper, some existence and multiplicity results involving eigenvalues are established for a class of degenerate quasilinear elliptic system by using Ekeland's variational principle, the mountain pass theorem and the critical point theory.

1 Introduction

In this paper, we consider the problem

$$\begin{cases} -\mathrm{div}(h_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + F_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(h_2(x)|\nabla v|^{q-2}\nabla v) = \mu b(x)|v|^{q-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $1 < p, q < N$, λ, μ are non-negative parameters, and $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ satisfies

(F_0) $F(x,0,0) = 0$ for all $x \in \overline{\Omega}$, there exist two constants $s_0, t_0 > 0$ such that $F_s(x,s,t) = F_s(x,s,t_0)$ for all $(x,s,t) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}\backslash[-t_0,t_0])$, $F_t(x,s,t) = F_t(x,s_0,t)$ for all $(x,s,t) \in \overline{\Omega} \times (\mathbb{R}\backslash[-s_0,s_0]) \times \mathbb{R}$,

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Received by the editors in May 2013 - In revised form in September 2014.
Communicated by D. Bonheure.
2010 *Mathematics Subject Classification* : 35J65; 35B30, 34A15.
Key words and phrases : Degenerate quasilinear elliptic systems; Existence and multiplicity results; Mountain pass theorem; Critical point theory.

and the growth condition:

$$\lim_{|s|\to\infty} \frac{F_s(x,s,t)}{a(x)|s|^{p-1}} = 0, \quad \lim_{|t|\to\infty} \frac{F_t(x,s,t)}{b(x)|t|^{q-1}} = 0 \tag{1.2}$$

uniformly in $(x,t) \in \overline{\Omega} \times \mathbb{R}$ and $(x,s) \in \overline{\Omega} \times \mathbb{R}$ respectively, where $\nabla F = (F_s, F_t)$ stands for the gradient of F with respect to $(s,t) \in \mathbb{R}^2$.

We observe that there exists a vast literature on non-uniformly nonlinear elliptic problems in bounded or unbounded domains. Many authors studied the existence of solutions for such problems (equations or systems), for instance see [5, 6, 7, 8, 9, 10, 14, 17, 18, 19, 21, 22]. In a recent paper Caldiroli et al. [5] considered the Dirichlet elliptic problem

$$-\mathrm{div}(h(x)\nabla u) = \lambda u + g(x,u) \text{ in } \Omega, \tag{1.3}$$

where Ω is a (bounded or unbounded) domain in $\mathbb{R}^N$ ($N \geq 2$), and h is a nonnegative measurable weighted function that is allowed to have "essential" zeroes at some points in Ω, i.e., the function h can have at most a finite number of zeroes in Ω.

The results in [5] were used by Zographopoulos [21], Zhang et al. [18] and Chung et al. [7, 8, 9] to study the existence of solutions for a class of degenerate elliptic systems.

In [1], Afrouzi et al. motivated by the paper of Ou and Tang [15], obtained three solutions for problem (1.1) in the case $h_1 = h_2 \equiv 1$ as the parameters λ and μ approach λ_1 and μ_1 from the left, respectively. Inspired by [1, 11, 13, 15, 20] and [22], the goal of this paper is to prove some existence and multiplicity results involving eigenvalues for a class of degenerate elliptic systems.

Let h_1, h_2 be positive weight functions a.e. in Ω such that

$$h_1 \in L^1_{loc}(\Omega), \quad h_1^{-s_1} \in L^1(\Omega), \quad s_1 \in \Big(\frac{N}{p}, \infty\Big) \cap \Big[\frac{1}{p-1}, \infty\Big), \tag{1.4}$$

$$h_2 \in L^1_{loc}(\Omega), \quad h_2^{-s_2} \in L^1(\Omega), \quad s_2 \in \Big(\frac{N}{q}, \infty\Big) \cap \Big[\frac{1}{q-1}, \infty\Big). \tag{1.5}$$

We define the spaces $W_0^{1,p}(\Omega, h_1)$, $W_0^{1,q}(\Omega, h_2)$ as being the completions of $C_0^\infty(\Omega)$ with respect to the norms defined by

$$\|u\|_{h_1,p} = \Big(\int_\Omega h_1(x)|\nabla u|^p\,dx\Big)^{\frac{1}{p}}, \quad \forall u \in C_0^\infty(\Omega),$$

$$\|v\|_{h_2,q} = \Big(\int_\Omega h_2(x)|\nabla v|^q\,dx\Big)^{\frac{1}{q}}, \quad \forall v \in C_0^\infty(\Omega)$$

respectively, and set $H = W_0^{1,p}(\Omega, h_1) \times W_0^{1,q}(\Omega, h_2)$. It is clear that H is a reflexive Banach space under the norm

$$\|w\|_H = \|u\|_{h_1,p} + \|v\|_{h_2,q}$$

for all $w = (u,v) \in H$.

We recall some facts about the homogeneous degenerate eigenvalue problem

$$\begin{cases} -\mathrm{div}(h_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$ and h_1 satisfies (1.4). With the number s given in (1.4) we define

$$p_{s_1} = \frac{ps_1}{s_1+1}, \quad p^*_{s_1} = \frac{Np_{s_1}}{N - p_{s_1}} = \frac{Nps_1}{N(s_1+1) - ps_1} > p.$$

In this paper, we assume that the coefficient function a satisfies

$$\mathrm{meas}\{x \in \Omega : a(x) > 0\} > 0, \quad a \in L^{\frac{r_1}{r_1-p}}(\Omega), \text{ for some } \quad p < r_1 < p^*_{s_1}.$$

The authors in [13] established the existence of sequence of positive eigenvalues $\{\lambda_k\}_{k\in N}$ where λ_k determined by the following way. Let

$$M_1 = \left\{ u \in W_0^{1,p}(\Omega, h_1) : \quad \int_\Omega a(x)|u|^p\,dx = 1 \right\},$$

$$I_1(u) = \int_\Omega h_1(x)|\nabla u|^p\,dx, \quad u \in W_0^{1,p}(\Omega, h_1).$$

Then we get

$$\lambda_k = \inf_{A_1 \in \sum_k} \sup_{u \in A_1} I_1(u), \tag{1.7}$$

where $\sum_k = \{A_1 \subset M_1 :$ there exists a continuous, odd and surjective $\gamma_1 : S^{k-1} \to A_1\}$ and S^{k-1} denotes the unit sphere in $\mathbb{R}^k$, $\lambda_k \to \infty$ as $k \to \infty$. It has been proved in [11, Chapter 3] that the principal eigenvalue λ_1 is simple and isolated and all eigenfunctions corresponding to λ_1 do not change sign in Ω. It is obvious that

$$\lambda_1 = \inf_{u \in M_1} I_1(u),$$

which implies that

$$\int_\Omega h_1(x)|\nabla u|^p\,dx \geq \lambda_1 \int_\Omega a(x)|u|^p\,dx, \quad \forall u \in W_0^{1,p}(\Omega, h_1). \tag{1.8}$$

Besides, the corresponding normalized eigenfunction φ_1 belongs to $W_0^{1,p}(\Omega, h_1)$.

Similarly, we consider the eigenvalue problem

$$\begin{cases} -\mathrm{div}(h_2(x)|\nabla v|^{q-2}\nabla v) = \mu b(x)|v|^{q-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Where h_2 satisfies condition (1.5), $\mathrm{meas}\{x \in \Omega : b(x) > 0\} > 0, \quad b \in L^{\frac{r_2}{r_2-q}}(\Omega)$ for some $q < r_2 < q^*_{s_2}$ where $q^*_{s_2} = \frac{Nqs_2}{N(s_2+1)-qs_2} > q$. Let

$$M_2 = \left\{ v \in W_0^{1,q}(\Omega, h_2) : \quad \int_\Omega b(x)|v|^q\,dx = 1 \right\}$$

and

$$I_2(v) = \int_\Omega h_2(x)|\nabla v|^q \, dx, \quad v \in W_0^{1,q}(\Omega, h_2).$$

By a standard argument, problem (1.9) has a sequence of eigenvalues with the variational characterization

$$\mu_k = \inf_{A_2 \in \sum_k'} \sup_{v \in A_2} I_2(v), \tag{1.10}$$

where $\sum_k' = \{A_2 \subset M_2 :$ there exists a continuous, odd and surjective $\gamma_2 : S^{k-1} \to A_2\}$ and $\mu_k \to \infty$ as $k \to \infty$. We also have

$$\mu_1 = \inf_{v \in M_2} I_2(v),$$

which implies that

$$\int_\Omega h_2(x)|\nabla v|^q \, dx \geq \mu_1 \int_\Omega b(x)|v|^q \, dx, \quad \forall v \in W_0^{1,q}(\Omega, h_2). \tag{1.11}$$

Besides, the corresponding normalized eigenfunction ψ_1 belongs to $W_0^{1,q}(\Omega, h_2)$.

Let

$$W' = \left\{ w = (u,v) \in H : \quad \int_\Omega a(x)|\varphi_1|^{p-2}\varphi_1 u \, dx = 0 \right.$$
$$\left. \text{and } \int_\Omega b(x)|\psi_1|^{q-2}\psi_1 v \, dx = 0 \right\}.$$

We can easily prove that W' is complementary subspace of $W = \langle \varphi_1 \rangle \times \langle \psi_1 \rangle$. Therefore, we have the following direct sum

$$H = W \oplus W'.$$

Now we are ready to state our main results.

Theorem 1.1. *Suppose that the nonlinearity F satisfies the conditions (F_0), (1.2) and*

$$\lim_{|s|,\, |t| \to \infty} F(x, s\varphi_1, t\psi_1) = \infty \tag{1.12}$$

uniformly in $x \in \Omega$. Then for any $\lambda < \lambda_1$ and $\mu < \mu_1$ sufficiently close to λ_1 and μ_1, problem (1.1) has at least three solutions.

Theorem 1.2. *Suppose that the nonlinearity F satisfies satisfies the conditions (F_0), (1.2) and*

$$\lim_{|(s,t)| \to \infty} \left(\frac{1}{p} F_s(x,s,t)s + \frac{1}{q} F_t(x,s,t)t - F(x,s,t) \right) dx = -\infty. \tag{1.13}$$

Then for $\lambda_k < \lambda < \lambda_{k+1}$, $\mu_k < \mu < \mu_{k+1}$ and also for the case $\lambda = \lambda_k$, $\mu = \mu_k$, problem (1.1) has at least one solution.

2 Preliminaries

For each $\lambda, \mu \in \mathbb{R}$, let $I : H \to \mathbb{R}$ be the functional defined by

$$I_{\lambda,\mu}(u,v) = \frac{1}{p}\int_\Omega h_1(x)|\nabla u|^p\,dx + \frac{1}{q}\int_\Omega h_2(x)|\nabla v|^p\,dx - \frac{\lambda}{p}\int_\Omega a(x)|u|^p\,dx \\ - \frac{\mu}{q}\int_\Omega b(x)|v|^q\,dx - \int_\Omega F(x,u,v)\,dx. \quad (2.1)$$

Since the potential F satisfies (1.2), it follows that $I_{\lambda,\mu} \in C^1(H,\mathbb{R})$ and its derivative is

$$I'_{\lambda,\mu}(u,v)(\eta_1,\eta_2) = \int_\Omega h_1(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\eta_1\,dx + \int_\Omega h_2(x)|\nabla v|^{q-2}\nabla v\cdot\nabla\eta_2\,dx \\ - \lambda\int_\Omega a(x)|u|^{p-2}u\eta_1\,dx - \mu\int_\Omega b(x)|v|^{q-2}v\eta_2\,dx \\ - \int_\Omega F_u(x,u,v)\eta_1\,dx - \int_\Omega F_v(x,u,v)\eta_2\,dx$$

for all $(u,v),(\eta_1,\eta_2) \in H$. In addition, $(u,v) \in H$ is a weak solution of problem (1.1) if and only if (u,v) is a critical point of $I_{\lambda,\mu}$. It is well known that the following lemma holds.

Lemma 2.1 (see [22]). *Assume that Ω is a bounded domain in $\mathbb{R}^N$ and the weight h satisfies*

$$h \in L^1_{loc}(\Omega), \quad h^{-s} \in L^1(\Omega), \quad s \in \Big(\frac{N}{p},\infty\Big) \cap \Big[\frac{1}{p-1},\infty\Big).$$

Then the following embeddings hold true

(i) $W_0^{1,p}(\Omega,h) \hookrightarrow L^{p_s^*}(\Omega)$ *continuously for* $1 < p_s^* < N$;

(ii) $W_0^{1,p}(\Omega,h) \hookrightarrow L^r(\Omega)$ *compactly for* $r \in [1,p_s^*)$.

Putting

$$U := \Big\{u \in W_0^{1,p}(\Omega,h_1) : \quad \int_\Omega a(x)|\varphi_1|^{p-2}\varphi_1 u\,dx = 0\Big\},$$
$$V := \Big\{v \in W_0^{1,q}(\Omega,h_2) : \quad \int_\Omega b(x)|\psi_1|^{q-2}\psi_1 v\,dx = 0\Big\}.$$

Then U and V are closed subspaces and they hold that

$$W_0^{1,p}(\Omega,h_1) = U \oplus \langle\phi_1\rangle, \quad W_0^{1,q}(\Omega,h_2) = V \oplus \langle\psi_1\rangle.$$

Proposition 2.2. *Set*

$$\overline{\lambda} = \inf_{u\in U\setminus\{0\}} \frac{\|u\|^p_{h_1,p}}{\int_\Omega a(x)|u|^p\,dx}, \quad \overline{\mu} = \inf_{v\in V\setminus\{0\}} \frac{\|v\|^q_{h_2,q}}{\int_\Omega b(x)|v|^q\,dx}.$$

Then we have $\lambda_1 < \overline{\lambda}$ and $\mu_1 < \overline{\mu}$, where λ_1 and μ_1 are defined above.

Proof. Indeed, we argue by contradiction. Since $\lambda_1 \le \frac{\|u\|^p_{h_1,p}}{\int_\Omega a(x)|u|^p\,dx}$ holds for each $u \neq 0$, we assume that $\lambda_1 = \overline{\lambda}$, i.e.

$$\overline{\lambda} = \inf_{u \in W_0^{1,p}(\Omega,h_1)\setminus\{0\}} \frac{\|u\|^p_{h_1,p}}{\int_\Omega a(x)|u|^p\,dx}.$$

Then, we may suppose that there exist a sequence $\{u_n\}_n \subset U$ and $u \in W^{1,p}(\Omega, h_1)$ such that

$$\int_\Omega a(x)|u_n|^p\,dx = 1, \quad \lim_{n\to\infty} \|u_n\|^p_{h_1,p} = \overline{\lambda}, \quad u_n \rightharpoonup u \in W_0^{1,p}(\Omega, h_1),$$

and thus $u_n \to u$ strongly in $L^p(\Omega)$.

Since U is closed and u_n converges strongly to u in $L^p(\Omega)$, we get $u \in U$ and $\int_\Omega a(x)|u|^p dx = 1$ holds. Using weak lower semicontinuity of the norm and the variational characterization of λ_1, we get

$$\lambda_1 \le \|u\|^p_{W_0^{1,p}} \le \lim_{n\to\infty} \inf \|u_n\|^p_{W_0^{1,p}} = \overline{\lambda} = \lambda_1,$$

which implies that $u = \pm\varphi_1$. This contradicts $\pm\varphi_1 \notin U$. Analogously, we can prove that $\mu_1 < \overline{\mu}$. ∎

In what follows, we recall some basic definitions and results to prove our theorems.

Definition 2.3. The functional I is said to satisfy the Palais-Smale condition at level c, $(PS)_c$, if every sequence for which

$$I(w_n) \to c, \quad \|I'(w_n)\|_{H^*} \to 0,$$

possesses a convergent subsequence. When I satisfies $(PS)_c$, for all $c \in \mathbb{R}$, we simply say that I satisfies the (PS) condition.

Definition 2.4. The functional I satisfies $(Ce)_c$ condition at level $c \in R$ if any sequence $\{w_n\} \subset H$ such that

$$I(w_n) \to c, \quad (1 + \|w_n\|_H)\|I'(w_n)\|_{H^*} \to 0 \text{ as } n \to \infty,$$

has a convergent subsequence. The functional I satisfies (Ce) condition if I satisfies $(Ce)_c$ at any $c \in R$. The (Ce) condition was introduced by Cerami, it is a weaker version of the (PS) condition (see [4]).

Definition 2.5 (see [16]). Let Q, Q_0 be submanifolds of a Banach space X with $Q_0 \subset Q$, S be a closed subset of a Banach space Y and $\Gamma \subset C^0(Q_0, Y\setminus S)$. We say S and (Q, Q_0) are $\Gamma-$ linking if for any map $h \in C^0(Q, Y)$ such that $h|_{Q_0} \in \Gamma$ there holds $h(Q) \cap S \neq \emptyset$.

Lemma 2.6 (see [20]). *Suppose that X, Y are Banach spaces. Consider submanifolds $Q, Q_0 \subset X$ with $Q_0 \subset Q$ and a closed subset $S \subset Y$ such that (Q, Q_0) and S are $\Gamma-$ linking. Let $\Gamma^* = \{\gamma \in C^0(Q, Y) : \gamma|_{Q_0} \in \Gamma\}$. For $f \in C^1(Y, R)$ satisfies*

(1) $\exists\gamma_0 \in \Gamma^*$ *such that* $\sup_{x\in Q} f(\gamma_0(x)) < +\infty$*;*

(2) $\exists\beta > \alpha$ *such that* $\inf_{y\in S} f(y) \geq \beta$ *and* $\sup_{x\in Q_0} f(\gamma(x)) \leq \alpha$ *for* $\gamma \in \Gamma^*$*;*

(3) f *satisfies the (Ce) condition.*

Then, the number $c := \inf_{\gamma\in\Gamma^*} \sup_{x\in Q} f(\gamma(x))$ *defines a critical value* $c \geq \beta$ *of* f*.*

For the proof of Lemma 2.6 we refer the readers to [20].

3 Proofs of the main results

3.1 Proof of Theorem 1.1.

We will prove Theorem 1.1 by using Ekeland's variational principal and the mountain pass theorem. More precisely, the proof will be divided into four steps.

Step 1. For $\lambda < \lambda_1$ and $\mu < \mu_1$, the functional $I_{\lambda,\mu}$ is coercive in H, $I_{\lambda,\mu}$ is bounded from below on W' and there is a constant m, independent of λ, μ, such that $\inf_{W'} I_{\lambda,\mu} \geq m$.

From the conditions (F_0), (1.2) and the continuity of the potential F, for any $\varepsilon > 0$, there exists a positive constant $M_\varepsilon = M(\varepsilon)$ such that

$$\left|\frac{\partial F}{\partial s}(x,s,t)\right| \leq \varepsilon a(x)|s|^{p-1} + M_\epsilon, \qquad \left|\frac{\partial F}{\partial t}(x,s,t)\right| \leq \varepsilon b(x)|t|^{q-1} + M_\epsilon$$

for all $(x,s,t) \in \overline{\Omega} \times \mathbb{R}^2$. By Hölder's inequality, we have

$$\begin{aligned} F(x,u,v) &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)\,ds + F(x,0,v) \\ &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)\,ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)\,ds + F(x,0,0) \\ &\leq \int_0^u (\varepsilon a(x)|s|^{p-1} + M_\varepsilon)\,ds + \int_0^v (\varepsilon b(x)|s|^{q-1} + M_\varepsilon)\,ds \\ &= \frac{\varepsilon}{p} a(x)|u|^p + M_\varepsilon u + \frac{\varepsilon}{q} b(x)|v|^q + M_\varepsilon v \end{aligned}$$

for all $(u,v) \in H$. Hence,

$$\begin{aligned} &\left|\int_\Omega F(x,u,v)\,dx\right| \leq \int_\Omega |F(x,u,v)|\,dx \\ &\leq \epsilon\left(\frac{1}{p}\int_\Omega a(x)|u|^p\,dx + \frac{1}{q}\int_\Omega b(x)|v|^q\,dx\right) + M_\epsilon\int_\Omega u\,dx + M_\epsilon\int_\Omega v\,dx \\ &\leq \frac{\epsilon}{p\lambda_1}\int_\Omega h_1(x)|\nabla u|^p\,dx + \frac{\epsilon}{q\mu_1}\int_\Omega h_2(x)|\nabla v|^q\,dx \\ &\qquad + M_\varepsilon|\Omega|^{\frac{p-1}{p}} S_1\left(\int_\Omega h_1(x)|\nabla u|^p\,dx\right)^{\frac{1}{p}} + M_\varepsilon|\Omega|^{\frac{q-1}{q}} S_2\left(\int_\Omega h_2(x)|\nabla v|^q\,dx\right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}&\leq \frac{\varepsilon}{p\lambda_1}\int_\Omega h_1(x)|\nabla u|^p\,dx + \frac{\varepsilon}{q\mu_1}\int_\Omega h_2(x)|\nabla v|^q\,dx\\&\quad + C_1\left[\left(\int_\Omega h_1(x)|\nabla u|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega h_2(x)|\nabla v|^q\,dx\right)^{\frac{1}{q}}\right],\end{aligned}\tag{3.1}$$

where S_1, S_2 are the embedding constants of $W_0^{1,p}(\Omega,h_1) \hookrightarrow L^p(\Omega)$, $W_0^{1,q}(\Omega,h_2) \hookrightarrow L^q(\Omega)$, respectively and $C_1 = \max\{M_\varepsilon|\Omega|^{\frac{p-1}{p}}S_1, M_\varepsilon|\Omega|^{\frac{q-1}{q}}S_2\}$.

For $\lambda < \lambda_1$ and $\mu < \mu_1$, from the definition of λ_1, μ_1 and (3.1), we get

$$\begin{aligned}I_{\lambda,\mu}(u,v) &= \frac{1}{p}\int_\Omega h_1(x)|\nabla u|^p\,dx + \frac{1}{q}\int_\Omega h_2(x)|\nabla v|^q\,dx - \frac{\lambda}{p}\int_\Omega a(x)|u|^p\,dx\\&\quad - \frac{\mu}{q}\int_\Omega b(x)|v|^q\,dx - \int_\Omega F(x,u,v)\,dx\\&\geq \frac{1}{p}\left(1-\frac{\lambda}{\lambda_1}-\frac{\varepsilon}{\lambda_1}\right)\int_\Omega h_1(x)|\nabla u|^p\,dx\\&\quad + \frac{1}{q}\left(1-\frac{\mu}{\mu_1}-\frac{\epsilon}{\mu_1}\right)\int_\Omega h_2(x)|\nabla v|^q\,dx - C_1\|(u,v)\|_H.\end{aligned}$$

Letting $\varepsilon = \min\{\frac{\lambda_1-\lambda}{2}, \frac{\mu_1-\mu}{2}\}$, it follows that $I_{\lambda,\mu}$ is coercive in H.

Similarly, from Proposition 2.2 we obtain

$$\begin{aligned}I_{\lambda,\mu}(u,v) &\geq \frac{1}{p}\left(1-\frac{\lambda}{\overline{\lambda}}-\frac{\varepsilon}{\lambda_1}\right)\int_\Omega h_1(x)|\nabla u|^p\,dx\\&\quad + \frac{1}{q}\left(1-\frac{\mu}{\overline{\mu}}-\frac{\varepsilon}{\mu_1}\right)\int_\Omega h_2(x)|\nabla v|^q\,dx - C_1\|(u,v)\|_H\\&\geq \frac{1}{p}\left(1-\frac{\lambda_1}{\overline{\lambda}}-\frac{\varepsilon}{\lambda_1}\right)\int_\Omega h_1(x)|\nabla u|^p\,dx\\&\quad + \frac{1}{q}\left(1-\frac{\mu_1}{\overline{\mu}}-\frac{\varepsilon}{\mu_1}\right)\int_\Omega h_2(x)|\nabla v|^q\,dx - C_1\|(u,v)\|_H.\end{aligned}$$

Let $\varepsilon = \frac{1}{2}\min\left\{\lambda_1(1-\frac{\lambda_1}{\overline{\lambda}}), \mu_1(1-\frac{\mu_1}{\overline{\mu}})\right\}$. Hence $I_{\lambda,\mu}$ is coercive in W and $I_{\lambda,\mu}$ is bounded from below on W', and moreover, there is a constant m, independent of λ, μ, such that $\inf_{W'} I_{\lambda,\mu} \geq m$.

Step 2. If $\lambda < \lambda_1$ and $\mu < \mu_1$ are sufficiently close to $\lambda_1,\ \mu_1$, we have $t_1^- < 0 < t_1^+$, $t_2^- < 0 < t_2^+$ such that $I_{\lambda,\mu}(t_1^\pm\varphi_1, t_2^\pm\psi_1) < m$.

For $\lambda < \lambda_1$ and $\mu < \mu_1$, we have

$$\begin{aligned} I_{\lambda,\mu}(t_1^+\varphi_1, t_2^+\psi_1) &= \frac{t_1^{+p}}{p}\int_\Omega h_1(x)|\nabla\varphi_1|^p\,dx + \frac{t_2^{+q}}{q}\int_\Omega h_2(x)|\nabla\psi_1|^q\,dx \\ &\quad - \lambda\frac{t_1^{+p}}{p}\int_\Omega a(x)|\varphi_1|^p\,dx \\ &\quad - \mu\frac{t_2^{+p}}{q}\int_\Omega b(x)|\psi_1|^q\,dx - \int_\Omega F(x, t_1^+\varphi_1, t_2^+\psi_1)\,dx \\ &= \frac{t_1^{+p}}{p}\int_\Omega h_1(x)|\nabla\varphi_1|^p\,dx + \frac{t_2^{+q}}{q}\int_\Omega h_2(x)|\nabla\psi_1|^q dx \\ &\quad - \frac{\lambda}{p\lambda_1}t_1^{+p}\int_\Omega h_1(x)|\nabla\varphi_1|^p\,dx \\ &\quad - \frac{\mu t_2^{+q}}{q\mu_1}\int_\Omega h_2(x)|\nabla\psi_1|^q\,dx - \int_\Omega F(x, t_1^+\varphi_1, t_2^+\psi_1)\,dx. \end{aligned} \tag{3.2}$$

From Fatou's Lemma and condition (1.12), we get

$$\int_\Omega F(x, t_1^+\varphi_1, t_2^+\psi_1)\,dx > -m + 1. \tag{3.3}$$

For $\lambda_1 - \frac{p\lambda_1}{2t_1^p} < \lambda < \lambda_1$ and $\mu_1 - \frac{q\mu_1}{2t_2^q} < \mu < \mu_1$, combining (3.2) and (3.3) yields $I_{\lambda,\mu}(t_1^+\varphi_1, t_2^+\psi_1) < m$. A similar condition holds for $t_1^-, t_2^- < 0$.

Step 3. If $\lambda < \lambda_1$ and $\mu < \mu_1$ the functional $I_{\lambda,\mu}$ satisfies the (PS) condition.

If $\{w_n\} = \{(u_n, v_n)\}$ is a (PS) sequence of $I_{\lambda,\mu}$, $\{(u_n, v_n)\}$ must be bounded. Then passing to a subsequence if necessary, there exists $w = (u, v) \in H$ such that

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in} \quad H, \\ &(u_n, v_n) \to (u, v) \quad \text{strongly in} \quad L^p(\Omega) \times L^q(\Omega). \end{aligned}$$

So there exists a strictly decreasing subsequence ε_n, $\lim_{n\to\infty}\varepsilon_n = 0$ such that

$$|I'_{\lambda,\mu}(u_n, v_n)(u_n - u, 0)| \le \varepsilon_n \|(u_n - u, 0)\|_H.$$

In particular,

$$\begin{aligned} \Big| \int_\Omega h_1(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\,dx - \lambda\int_\Omega a(x)|u_n|^{p-2}u_n(u_n - u)\,dx \\ - \int_\Omega F_u(x, u_n, v_n)(u_n - u)\,dx \Big| \le \varepsilon_n\|(u_n - u, 0)\|_H. \end{aligned} \tag{3.4}$$

Since $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$, we have

$$\int_\Omega a(x)|u_n|^{p-2}u_n(u_n - u)\,dx \le \|a\|_{\frac{r_1}{r_1 - p}}\left(\int_\Omega |u_n|^{r_1}\,dx\right)^{\frac{p-1}{r_1}}\left(\int_\Omega |u_n - u|^{r_1}\,dx\right)^{\frac{1}{r_1}}, \tag{3.5}$$

which approaches 0 as $n \to \infty$.

Since the potential F satisfies (1.2) we have

$$\left|\int_\Omega F_u(x,u_n,v_n)(u_n-u)\,dx\right| \le \left|\int_\Omega (\varepsilon a(x)|u_n|^{p-1}+M_\varepsilon)(u_n-u)\,dx\right|$$
$$\le \varepsilon\|a\|_{\frac{r_1}{r_1-p}}\|u_n\|_{r_1}^{p-1}\|u_n-u\|_{r_1}\,dx + M_\varepsilon|\Omega|^{\frac{p-1}{p}}\|u_n-u\|_p, \quad (3.6)$$

which approaches 0 as $n\to\infty$.

Combining (3.4) with (3.5) and (3.6) we get

$$\lim_{n\to\infty}\int_\Omega h_1(x)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n-\nabla u)\,dx = 0.$$

Subtracting

$$\int_\Omega h_1(x)|\nabla u|^{p-2}\nabla u(\nabla u_n-\nabla u)\,dx,$$

which converges to zero as n tends to infinity, we conclude that

$$\lim_{n\to\infty}\int_\Omega h_1(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)(\nabla u_n-\nabla u)\,dx = 0. \quad (3.7)$$

Next, we recall the following useful inequalities:

$$\begin{cases} (|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta)\ge C_2(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2, & \text{if } 1<p<2,\\ (|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta)\ge C_2|\xi-\eta|^p, & \text{if } p\ge 2 \end{cases} \quad (3.8)$$

for all $\xi,\eta\in\mathbb{R}^N$, where C_2 is a positive constant and $(.,.)$ denotes the usual product in $\mathbb{R}^N$.

As in [2], if $1<p<2$, by Hölder's inequality and substituting $z_n = h_1^{\frac{1}{p}}u_n$, $z = h_1^{\frac{1}{p}}u$ in system (3.8), there exists $C_3, C_4>0$ such that

$$\begin{aligned} 0 &\le \int_\Omega |\nabla z_n-\nabla z|^p\,dx \\ &= \int_\Omega |\nabla z_n-\nabla z|^p(|\nabla z_n|+|\nabla z|)^{\frac{p(p-2)}{2}}(|\nabla z_n|+|\nabla z|)^{\frac{p(2-p)}{2}}\,dx \\ &\le \left(\int_\Omega |\nabla z_n-\nabla z|^2(|\nabla z_n|+|\nabla z|)^{p-2}\,dx\right)^{\frac{p}{2}}\left(\int_\Omega(|\nabla z_n|+|\nabla z|)^p\,dx\right)^{\frac{2-p}{2}} \\ &\le C_3\left(\int_\Omega(|\nabla z_n|^{p-2}\nabla z_n-|\nabla z|^{p-2}\nabla z,(\nabla z_n-\nabla z)\,dx\right)^{\frac{p}{2}} \\ &\qquad\qquad \left(\int_\Omega(|\nabla z_n|+|\nabla z|)^p\,dx\right)^{\frac{2-p}{2}} \\ &\le C_4\left(\int_\Omega(|\nabla z_n|^{p-2}\nabla z_n-|\nabla z|^{p-2}\nabla z,(\nabla z_n-\nabla z)\,dx\right)^{\frac{p}{2}} \end{aligned}$$

which implies $\|u_n-u\|_{h_1,p}\to 0$ as $n\to\infty$.

If $p \geq 2$, by (3.8), there exists $C_5 > 0$ such that

$$0 \leq \|u_n - u\|_{h_1,p} \leq C_5 \left(\int_\Omega (|\nabla z_n|^{p-2}\nabla z_n - |\nabla z|^{p-2}\nabla z, (\nabla z_n - \nabla z)\, dx \right),$$

so we get $\|u_n - u\|_{h_1,p} \to 0$ as $n \to \infty$. Therefore, $\|u_n - u\|_{h_1,p} \to 0$ for $p > 1$ as $n \to \infty$, that is, $u_n \to u$ in $W_0^{1,p}(\Omega, h_1)$ as $n \to \infty$. Similarly, we obtain $v_n \to v$ in $W_0^{1,q}(\Omega, h_2)$ as $n \to \infty$. Consequently, the functional $I_{\lambda,\mu}$ satisfies the (PS) condition for all $\lambda < \lambda_1,\ \mu < \mu_1$.

In addition, let

$$\sum_\pm = \left\{ w \in H : w = \pm(t_1\varphi_1, t_2\psi_1) + w' \text{ with } t_1, t_2 > 0 \text{ and } w' \in W' \right\}.$$

Then $I_{\lambda,\mu}$ satisfies $(PS)_{c,\sum_+}$ and $(PS)_{c,\sum_-}$ for all $c < m$.

Let $\{w_n\} \subset \sum_+$ satisfies $I_{\lambda,\mu}(w_n) \to c < m$ and $I'_{\lambda,\mu}(w_n) \to 0$ an $n \to \infty$. Since $I_{\lambda,\mu}$ is coercive and the potential F satisfies (2), there is $w \in H$ such that $\|w_n\|_H \to \|w\|_H$ strongly in H. If $w \in \partial \sum_+ = W'$, since $\inf_{W'} I_{\lambda,\mu} \geq m$, we get $I_{\lambda,\mu}(w_n) \to c \geq m$, which is impossible. Hence $w \in \sum_+$ and $I_{\lambda,\mu}$ satisfies the $(PS)_{c,\sum_+}$ condition. Similarly we have that $(PS)_{c,\sum_-}$ holds for all $c < m$.

Step 4. If $\lambda < \lambda_1$ is sufficiently close to λ_1 and $\mu < \mu_1$ is sufficiently close to μ_1, we get

$$-\infty < \inf_{\sum_\pm} I_{\lambda,\mu} < m,$$

which implies that $I_{\lambda,\mu}$ is bounded below in $\sum_+$. Consequently, from Ekeland's variational principle, there exists $\{w_n\} \subset \sum_+$ such that $I_{\lambda,\mu}(w_n) \to \inf_{\sum_\pm} I_{\lambda,\mu}$ and $I'_{\lambda,\mu}(w_n) \to 0$ as $n \to \infty$. Since $I_{\lambda,\mu}$ satisfies $(PS)_{c,\sum_+}$ for all $c < m$, there is $z^+ \in \sum_+$ such that $I_{\lambda,\mu}(w^+) = \inf_{\sum_+} I_{\lambda,\mu}$, that is, the infimum is attained in $\sum_+$. A similar conclusion holds in $\sum_-$. So $I_{\lambda,\mu}$ has two distinct critical points, denoted by w^+, w^-. As in [15], we can obtain the third critical point z of $I_{\lambda,\mu}$ by applying mountain pass theorem such that $I_{\lambda,\mu}(w) = c \geq m$.

3.2 Proof of Theorem 1.2.

We will verify the functional $I_{\lambda,\mu}$ satisfying the condition of Lemma 2.6. We first verify the Cerami condition.

Let $\{w_n\} = \{(u_n, v_n)\} \subset H$ be a (Ce) sequence. we first prove that $\{w_n\}$ is bounded in H, then by a standard argument, $\{w_n\}$ has a convergent subsequence. Supposing by contradiction that $\|w_n\|_H \to \infty$, and defining $\widehat{u}_n = \frac{u_n}{\|w_n\|_H}$, $\widehat{v}_n = \frac{v_n}{\|w_n\|_H}$, then $\widehat{w}_n = (\widehat{u}_n, \widehat{v}_n)$ is bounded in H and

$$\|\widehat{u}_n\|_{h_1,p} + \|\widehat{v}_n\|_{h_2,q} = 1$$

for every $n \in \mathbb{N}$. Going if necessary to a subsequence, also denoted by $\{(\widehat{u}_n, \widehat{v}_n)\}$, there is $\widehat{w} = (\widehat{u}, \widehat{v}) \in H$ such that

$$(\widehat{u}_n, \widehat{v}_n) \rightharpoonup (\widehat{u}, \widehat{v}) \text{ weakly in } H,$$
$$(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v}) \text{ strongly in } L^p(\Omega) \times L^q(\Omega).$$

We have

$$|\langle I'_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n),(\widehat{u}_n-\widehat{u},0)\rangle| \le \|I'_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)\|_{H^*}\|\widehat{u}_n-\widehat{u}\|_{h_1,p} \le \|I'_{\lambda}(\widehat{u}_n,\widehat{v}_n)\|_{H^*}(\|\widehat{u}_n\|_{h_1,p}+\|\widehat{u}\|_{h_1,p}), \tag{3.9}$$

which approaches 0 as $n\to\infty$. Since $\widehat{u}_n\to\widehat{u}$ in $L^p(\Omega)$, $\widehat{v}_n\to\widehat{v}$ in $L^q\Omega)$, we have

$$\lim_{n\to\infty}\int_\Omega a(x)|\widehat{u}_n|^{p-2}\widehat{u}_n(\widehat{u}_n-\widehat{u})\,dx=0 \tag{3.10}$$

and

$$\lim_{n\to\infty}\int_\Omega F_u(x,\widehat{u}_n,\widehat{v}_n)(\widehat{u}_n-\widehat{u})\,dx=0. \tag{3.11}$$

By using

$$\langle I'_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n),(\widehat{u}_n-\widehat{u},0)\rangle=\int_\Omega h_1(x)|\nabla\widehat{u}_n|^{p-2}\nabla\widehat{u}_n\nabla(\widehat{u}_n-\widehat{u})\,dx -\lambda\int_\Omega a(x)|\widehat{u}_n|^{p-2}\widehat{u}_n(\widehat{u}_n-\widehat{u})\,dx-\int_\Omega F_u(x,\widehat{u}_n,\widehat{v}_n)(\widehat{u}_n-\widehat{u})\,dx$$

and relations (3.9), (3.10) and (3.11), we obtain

$$\lim_{n\to\infty}\int h_1(x)(|\nabla\widehat{u}_n|^{p-2}\nabla u_n-|\nabla\widehat{u}|^{p-2}\nabla\widehat{u})(\nabla\widehat{u}_n-\nabla\widehat{u})\,dx=0.$$

Substituting $\xi=h_1^{\frac{1}{p}}\widehat{u}_n,\eta=h_1^{\frac{1}{p}}\widehat{u}$ in system (3.8), we get $\|\widehat{u}_n-\widehat{u}\|_{h_1,p}\to 0$ as $n\to\infty$, then $\|\widehat{u}_n\|_{h_1,p}\to\|\widehat{u}\|_{h_1,p}$ as $n\to\infty$. In a similar way, we get $\|\widehat{v}_n\|_{h_2,q}\to\|\widehat{v}\|_{h_2,q}$ as $n\to\infty$. It follows that $\|\widehat{u}\|_{h_1,p}+\|\widehat{v}\|_{h_2,q}=1$. Hence $(\widehat{u},\widehat{v})\neq(0,0)$. Consequently, we conclude that $|(u_n(x),v_n(x))|\to\infty$ as $n\to\infty$ for a.e. $x\in\Omega$. On the other hand, we have

$$I_{\lambda,\mu}(u_n,v_n)-I'_{\lambda,\mu}\Big(\frac{u_n}{p},\frac{v_n}{q}\Big)= \int_\Omega\Big(\frac{1}{p}F_u(x,u_n,v_n)u_n+\frac{1}{q}F_v(x,u_n,v_n)v_n-F(x,u_n,v_n)\Big)\,dx\to c$$

as $n\to\infty$, which contradicts (1.13). Hence, $\{(u_n,v_n)\}$ is bounded in H. By a standard argument, $\{(u_n,v_n)\}$ has a convergent subsequence.

Now, we are in the position to prove Theorem 1.2. We will split the proof into two cases according to the positions of λ,μ.

Case 1: $\lambda_k<\lambda<\lambda_{k+1}$, $\mu_k<\mu<\mu_{k+1}$. In this case we will apply Lemma 2.6 to get solutions of problem (1.1). It follows from definition of λ_k and μ_k there exist $A_1\in\sum_k$ and $A_2\in\sum'_k$ such that $\sup_{u\in A_1}I_1(u)=m_1\in(\lambda_k,\lambda)$ and $\sup_{v\in A_2}I_2(v)=m_2\in(\mu_k,\mu)$ respectively.

From (3.1) and the definition of the functional $I_{\lambda,\mu}$, we deduce for any

$(u,v) \in A_1 \times A_2$ and $t > 0$ that

$$\begin{aligned} I_{\lambda,\mu}(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) &\le \frac{t}{p}\int_\Omega h_1(x)|\nabla u|^p\,dx + \frac{t}{q}\int_\Omega h_2(x)|\nabla v|^q\,dx - \frac{t\lambda}{p}\int_\Omega a(x)|u|^p\,dx \\ &\quad - \frac{t\mu}{q}\int_\Omega b(x)|v|^q\,dx + \left|\int_\Omega F(x, t^{\frac{1}{p}}u, t^{\frac{1}{q}}v)\,dx\right| \\ &\le \frac{t}{p}(m_1 - \lambda) + \frac{t}{q}(m_2 - \mu) + t\left(\frac{\varepsilon}{p\lambda_1}m_1 + \frac{\varepsilon}{q\mu_1}m_2\right) \\ &\quad + C_1 t^{\frac{1}{p}} m_1^{\frac{1}{p}} + C_1 t^{\frac{1}{q}} m_2^{\frac{1}{q}}. \end{aligned} \tag{3.12}$$

For each k, we set

$$F_{k+1} = \left\{u \in W_0^{1,p}(\Omega, h_1) : \int_\Omega h_1(x)|\nabla u|^p\,dx \ge \lambda_{k+1}\int_\Omega a(x)|u|^p\,dx\right\},$$

$$F'_{k+1} = \left\{v \in W_0^{1,q}(\Omega, h_2) : \int_\Omega h_2(x)|\nabla v|^q\,dx \ge \mu_{k+1}\int_\Omega b(x)|v|^q\,dx\right\}.$$

For $(u,v) \in F_{k+1} \times F'_{k+1}$, we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &\ge \left(\frac{1}{p} - \frac{\lambda}{p\lambda_{k+1}} - \frac{\epsilon}{p\lambda_1}\right)\int_\Omega h_1(x)|\nabla u|^p\,dx \\ &\quad + \left(\frac{1}{q} - \frac{\mu}{q\mu_{k+1}} - \frac{\varepsilon}{q\mu_1}\right)\int_\Omega h_2(x)|\nabla v|^q\,dx \\ &\quad - C_1\left(\int_\Omega h_1(x)|\nabla u|^p\,dx\right)^{\frac{1}{p}} - C_1\left(\int_\Omega h_2(x)|\nabla v|^q\,dx\right)^{\frac{1}{q}}. \end{aligned} \tag{3.13}$$

Taking

$$\varepsilon = \frac{1}{2}\min\left\{\frac{\lambda_1(\lambda_{k+1} - \lambda)}{\lambda_{k+1}}, \frac{\mu_1(\mu_{k+1} - \mu)}{\mu_{k+1}}, \frac{(\lambda - m_1)\lambda_1}{m_1}, \frac{(\mu - m_2)\mu_1}{m_2}\right\},$$

we get from (3.12) and (3.13) that

$$\beta := \inf_{(u,v) \in F_{k+1} \times F'_{k+1}} I_{\lambda,\mu}(u,v) \tag{3.14}$$

and there exists $T > 0$ such that

$$\alpha := \max_{(u,v) \in A_1 \times A_2:\ t \ge T} I_{\lambda,\mu}(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) < \beta. \tag{3.15}$$

Now let

$$TA := \{(tu, tv) : (u,v) \in A_1 \times A_2,\ t \ge T\}.$$

Set $Q = B_k$ (B_k represents the closed unit ball in $\mathbb{R}^k$), $\partial Q = S^{k-1}$, and

$$\Gamma = \{\gamma \in C^0(S^{k-1}, H) : \gamma \text{ is odd and } \gamma(S^{k-1}) \subset TA\}.$$

For any $\gamma \in \Gamma$, by (3.14), (3.15) we obtain

$$\gamma(S^{k-1}) \cap (F_{k+1} \times F'_{k+1}) = \emptyset,$$

which shows that

$$\Gamma \subset C(S^{k-1}, H \backslash (F_{k+1} \times F'_{k+1})).$$

Let $\Gamma^* = \{\gamma \in C^0(B_k, H) : \gamma|_{S^{k-1}} \in \Gamma\}$. Then Γ^* is nonempty.
We prove that if $\gamma \in \Gamma^*$ then $\gamma(B_k) \cap (F_{k+1} \times F'_{k+1}) \neq \emptyset$. In fact, by the definition of $\sum_k$, $\sum'_k$ there exist continuous odd surjections $\gamma_1 : S^{k-1} \to A_1$, $\gamma_2 : S^{k-1} \to A_2$. So we can define $\gamma : S^{k-1} \to A_1 \times A_2$ by $\gamma = (\gamma_1, \gamma_2)$. Define $\overline{\gamma} : B_k \to H$ by $\overline{\gamma}(ts) = tT\gamma(s)$ for any $s \in S^{k-1}$ and any $t \in [0,1]$. Thus $\overline{\gamma} \in \Gamma^*$. If there exists $(u, v) \in \gamma(B_k)$ such that

$$\int_\Omega a(x)|u|^p dx = 0, \quad \int_\Omega b(x)|v|^q \, dx = 0,$$

then we get $\gamma(B_k) \cap (F_{k+1} \times F'_{k+1}) \neq \emptyset$.

Otherwise, we consider the map $\widehat{\gamma} : S^k \to E$ by

$$\widehat{\gamma}(x_1, ..., x_{k+1}) = \begin{cases} \pi \circ \gamma(x_1, ..., x_k), & \text{for } x_{k+1} \geq 0, \\ -\pi \circ \gamma(-x_1, ..., -x_k), & \text{for } x_{k+1} < 0, \end{cases} \tag{3.16}$$

where

$$\pi(u, v) = \left(\frac{u}{\int_\Omega a(x)|u|^p dx}, \frac{v}{\int_\Omega b(x)|v|^q \, dx} \right).$$

We can easily show that $\widehat{\gamma}$ is odd. Hence $\widehat{\gamma}(S^k) = (\widehat{\gamma}_1(S^k), \widehat{\gamma}_2(S^k)) \in \sum_k \times \sum'_k$. On the other hand, we have,

$$\lambda_{k+1} = \inf_{A_1 \in \sum_k} \sup_{u \in A_1} I_1(u)$$

then

$$\lambda_{k+1} \leq \sup_{u \in \widehat{\gamma}_1(S^k)} I_1(u).$$

Hence for $u \in \widehat{\gamma}_1(S^k)$, that is, for some $x \in S^{k-1}$ such that $u = \widehat{\gamma}_1(x)$ we have $\lambda_{k+1} \leq I_1(u)$. This implies that $\widehat{\gamma}_1(x) \in F_{k+1}$. Using the definition of $\widehat{\gamma}_1$, we Obtain that $\gamma_1(x) \in F_{k+1}$. In a similar way we Obtain that $\gamma_2(x) \in F'_{k+1}$. So $\gamma(B_k) \cap \{F_{k+1} \times F'_{k+1}\} \neq \emptyset$. Hence S^k and $F_{k+1} \times F'_{k+1}$ are $\Gamma-$ linking. The conditions of Lemma 2.6 are satisfied. So Theorem 1.2 holds for any $\lambda_k < \lambda < \lambda_{k+1}$, $\mu_k < \mu < \mu_{k+1}$ with the critical value

$$c := \inf_{\gamma \in \Gamma^*} \sup_{x \in B_k} I_{\lambda,\mu}(\gamma(x)).$$

Case 2: $\lambda = \lambda_k, \mu = \mu_k$. Let $\delta_1 \in (0, \lambda_{k+1} - \lambda_k), \delta_2 \in (0, \mu_{k+1} - \mu_k)$, we assume $\kappa_n \in (\lambda_k, \lambda_k + \delta_1), \tau_n \in (\mu_k, \mu_k + \delta_2)$ and $\kappa_n \to \lambda_k, \tau_n \to \mu_k$. It follows from the case $\lambda_k < \lambda < \lambda_{k+1}$, $\mu_k < \mu < \mu_{k+1}$ that there exists c_n to be the critical value of I_{κ_n, τ_n} and (u_n, v_n) be the critical point corresponding to c_n satisfying

$I'_{\kappa_n,\tau_n}(u_n,v_n)=0$ for any n. Thus, for any (τ_n,κ_n) there exists a corresponding set Γ_n^* such that the critical value c_n is characterized by

$$c_n := \inf_{\gamma\in\Gamma_n^*}\sup_{x\in B_k} I_{\kappa_n,\tau_n}(\gamma(x)).$$

For $(u,v)\in F_{k+1}\times F'_{k+1}$, let

$$\varepsilon=\min\left\{\frac{(\lambda_{k+1}-\lambda_k-\delta_1)\lambda_1}{2\lambda_{k+1}},\frac{(\mu_{k+1}-\mu_k-\delta_2)\mu_1}{2\mu_{k+1}}\right\},$$

we obtain

$$\begin{aligned}I_{\kappa_n,\tau_n}(u,v) &\geq \frac{1}{p}\left(1-\frac{\kappa_n}{\lambda_{k+1}}-\frac{\varepsilon}{\lambda_1}\right)\|u\|^p_{h_1,p}+\frac{1}{q}\left(1-\frac{\tau_n}{\mu_{k+1}}-\frac{\varepsilon}{\mu_1}\right)\|v\|^q_{h_2,q}\\ &\quad -C_1\Big(\|u\|_{h_1,p}+\|v\|_{h_2,q}\Big)\\ &\geq \frac{1}{p}\left(1-\frac{\lambda_k+\delta_1}{\lambda_{k+1}}-\frac{\varepsilon}{\lambda_1}\right)\|u\|^p_{h_1,p}+\frac{1}{q}\left(1-\frac{\mu_k+\delta_2}{\mu_{k+1}}-\frac{\varepsilon}{\mu_1}\right)\|v\|^q_{h_2,q}\\ &\quad -C_1\Big(\|u\|_{h_1,p}+\|v\|_{h_2,q}\Big),\end{aligned}$$

which implies that there exists $\beta_0>0$ such that

$$\begin{aligned}\sup_{x\in B_k} I_{\kappa_n,\tau_n}(\gamma(x)) &\geq \sup_{(u,v)\in\gamma(B_k)\cap(F_{k+1}\times F'_{k+1})} I_{\kappa_n,\tau_n}(\gamma(x))\\ &\geq \inf_{(u,v)\in F_{k+1}\times F'_{k+1}} I_{\kappa_n,\tau_n}(\gamma(x))\\ &\geq \beta_0\end{aligned}$$

for all $\gamma\in\Gamma_n^*$. Then one has $c_n\geq\beta_0$, we get that there exists a subsequence of critical points (u_n,v_n) which converges to the desired critical point of I_{λ_k,μ_k}.

Acknowledgments

The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript. The second author was supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED).

References

[1] G.A. AFROUZI, S. MAHDAVI AND Z. NAGHIZADEH, Existence of multiple solutions for a class of (p,q)-Laplacian systems, *Nonlinear Anal.*, **72** (2010), 2243-2250.

[2] G.A. AFROUZI, S. MAHDAVI AND N.B. ZOGRAPHOPOULOS, Existence of solutions for non-uniformly nonlinear elliptic systems, *Electron. J. Diff. Equ.*, **167** (2011), 1-9.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381.

[4] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and application to some nonlinear problems with strong resonance at infinity, *Nonlinear Anal.*, **7** (1983), 981-1012.

[5] P. Caldiroli and R. Musina, On a variational degenerate elliptic problem, *Nonlinear Differ. Eqns. Appl. (NoDEA)*, **7** (2000), 189-199.

[6] N.T. Chung, Existence of weak solutions for a nonuniformly elliptic nonlinear system in $\mathbb{R}^N$, *Electron. J. Diff. Equ.*, **119** (2008), 1-10.

[7] N.T. Chung, On the existence of weak solutions for a degenerate and singular elliptic systems in $\mathbb{R}^N$, *Acta. Appl. Math.*, **110** (2010), 47-56.

[8] N.T. Chung, Existence of infinitely many solutions for degenerate and singular elliptic systems with indefinite concave nonlinearities, *Electron. J. Diff. Equ.*, **30** (2011), 1-12.

[9] N. T. Chung and H. Q. Toan, On a class of degenerate and singular elliptic systems in bounded domains, *J. Math. Anal. Appl.*, **360** (2009) 422-431.

[10] M. Cuesta and H.R. Quoirin, A weighted eigenvalue problem for the p-Laplacian plus a potential, *Nonlinear Differential Equations and Applications (NoDEA)*, **16** (2009), 469-491.

[11] P. Drábek, A. Kufner and F. Niclosi, *Quasilinear elliptic equations with degenerate and singularities*, Vol. 5 of de Gruyter Series in Nonlinear Analysis and Applications, Walter and Gruyter and Co. Berlin, 1997.

[12] P. Drábek and J. Milota, *Methods of Nonlinear Analysis, Applications to Differential Equations*, Birkhäuser Verlag AG, Basel-Boston-Berlin, 2007.

[13] A. Lê and K. Schmitt, Variational eigenvalues of degenerate eigenvalue problem for weighted p-Laplacian, *Advanced Nonlinear Studies*, **5**(2005), 553-565.

[14] O.H. Miyagaki and R.S. Rodrigues, On positive solution for a class of degenerate quasilinear elliptic positone / semipositone systems, *Nonlinear Anal.*, **70** (2009), 99-116.

[15] Z.Q. Ou and C.L. Tang, Existence and multiplicity results for some elliptic system at resonance, *Nonlinear Anal.*, **71** (2009), 2660-2666.

[16] S. Z. Song and C. L. Tang, Existence theorem for quasilinear elliptic resonance problems, *J. Southwest China Normal Univ. (Natural science)*, **30** (1) (2005), 1-6.

[17] H. Q. Toan and N. T. Chung, Existence of weak solution for a class of nonuniformly nonlinear elliptic equation in unbounded domains, *Nonlinear Anal.*, **70** (2009), 3987-3996.

[18] J. ZHANG AND Z. ZHANG, Existence results for some nonlinear elliptic systems, Nonlinear Anal., **71** (2009), 2840-2846.

[19] G. ZHANG AND Y. WANG, Some existence results for a class of degenerate semilinear elliptic systems, *J. Math. Anal. Appl.*, **333** (2007), 904 - 918.

[20] X. X. ZHAO AND C. L. TANG, Resonance problems for (p, q)-Laplacian systems, *Nonlinear Anal.*, **72** (2010), 1019-1030.

[21] N.B. ZOGRAPHOPOULOS, On a class of degenerate potential elliptic systems, *Nonlinear Differ. Equ. Appl.*, **11** (2004), 191-199.

[22] N.B. ZOGRAPHOPOULOS, On the principal eigenvalue of degenerate quasilinear elliptic systems, *Math. Nachr.*, **281**(9) (2008), 1351-1365.

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