Some results on AM-compact operators

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Abstract

We characterize Banach lattices under which each AM-compact (resp. b-AM-compact) operator is Dunford-Pettis. Also, we study the AM-compactness of limited completely continuous operators.

1 Introduction

Throughout this paper X, Y will denote Banach spaces, and E, F will denote Banach lattices. The positive cone of E will be denoted by E^+.

The class of AM-compact operators was introduced and studied by Dodds-Fremlin [9]. We say that an operator $T : E \to X$ is called AM-compact if the image of each order bounded subset of E is a relatively compact subset of X.

Following Aliprantis and Burkinshaw we say that an operator $T : X \to Y$ is called a Dunford-Pettis operator if for each weakly null sequence (x_n), we have $\lim\limits_{n\to\infty} \|T(x_n)\| = 0$). Equivalently, T carries relatively weakly compact sets onto relatively compact subsets of Y [1].

Recently, M. Salimi et S. M. Moshtaghioun introduced the class of limited completely continuous operators, and characterized this class of operators and studied some of its properties in [12]. Let us recall that the operator $T : X \longrightarrow Y$ is called limited completely continuous (abb. *lcc*), if T carries limited and weakly null sequences in X to norm null ones. Alternatively, T is *lcc* if, and only if, for each limited set $A \subset X$, the set $T(A)$ is relatively compact, [12].

In [5], the authors studied the AM-compactness of Dunford-Pettis operators. Our goal in the first section of this article is to study the Banach lattice under

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which every AM-compact operator is Dunford-Pettis. In fact,we give necessary conditions under which each AM-compact operator is Dunford-Pettis. More precisely, we show that if any AM-compact operator T from a Banach lattice E such that its norm is order continuous, in a Banach lattice F, is Dunford-Pettis then E admits the positive Schur property or the norm of F is order continuous (Theorem 3.4). Also, we establish some sufficient conditions for each AM-compact operator is Dunford-Pettis (Theorem 3.2) and with an example, we prove that the condition "the norm of E is order continuous" is essential in Theorem 3.2.

In the second section of this article, we will study the AM-compactness of lcc operators. In fact, we give some sufficient conditions under which each AM-compact operator is an lcc (Theorem 3.6). As a consequence of Theorem 3.2 and Theorem 3.6, we give some sufficient conditions under which each Dunford-Pettis operator is lcc.

2 Preliminaries

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice.

A Banach lattice E is said to have the positive Schur property if every weakly convergent sequence to 0 in E^+ is norm convergent to zero. For example, the Banach space ℓ^1 has the positive Schur property but the Banach space ℓ^∞ does not has this property.

A Banach lattice E is called a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space, but the Banach lattice c_0 is not a KB-space.

Recall that a subset A of a Banach lattice E is almost order bounded, if for all $\epsilon > 0$ there exists $x \in E^+$ with $A \subset [-x, x] + \epsilon B_E$.

A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the lattice subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. A subset A of a vector lattice E is called order bounded, if it include in an order interval in E. A linear mapping T from a vector lattice E into another F is order bounded if it carries order bounded set of E into order bounded set of F. We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping, it is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous, If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminologies concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3 Main results

3.1 On the classes of Dunford-Pettis and AM-compact operators

Recall that an operator $T : E \to Y$ is almost Dunford-Pettis if $\|T(x_n)\| \to 0$ for every weakly null sequence (x_n) in E consisting of pairwise disjoint elements [13]. Note that every Dunford-Pettis operator is almost Dunford-Pettis, but the converse is not true in general. Indeed, the identity operator of the Banach lattice $L^1[0,1]$ is almost Dunford-Pettis and fails to be Dunford-Pettis. On the other hand, an AM-compact operator is not necessary a Dunford-Pettis. In fact, the identity operator of the Banach lattice c_0 is AM-compact but fails to be Dunford-Pettis.

Proposition 3.1. *If $T : F \to X$ is an AM-compact operator and $S : E \to F$ is an almost Dunford-Pettis operator, then the product $T \circ S$ is Dunford-Pettis.*

In the following result, we give a sufficient conditions under which each AM-compact operator is Dunford-Pettis.

Theorem 3.2. *Each AM-compact operator $T : E \longrightarrow F$ is Dunford-Pettis if one of the following statements is valid:*

1. *E has the positive Schur property,*
2. *F has the Schur property.*

Proof. (1) Let $T : E \to F$ be an operator and A be a relatively weakly compact subset of E. Since E has the positive Schur property, it follows from Theorem 3.1 of [8] that A is almost order bounded, then there exists $x \in E^+$ with $A \subset [-x, x] + \epsilon B_E$ and hence $T(A) \subset T([-x, x]) + \epsilon\|T\|B_F$.

Now, as T is AM-compact then, $T([-x, x])$ is relatively compact in F and hence $T(A)$ is relatively compact. This show that T is Dunford-Pettis.

(2) In this case each operator is Dunford-Pettis. ∎

Recall from [3] that an operator $T : E \to X$ is said to be b-AM-compact if it carries b-order bounded set of E (i.e., order bounded in E'') into norm relatively compact set of X. Note that a b-AM-compact operator is not necessary Dunford-Pettis. In fact, the identity operator of the Banach lattice ℓ^2 is b-AM-compact (because ℓ^2 is a discrete KB-space) but it is not a Dunford-Pettis operator (because ℓ^2 does not has the Schur property).

As consequence of Theorem 3.2, we obtain the following result,

Corollary 3.3. *Each b-AM-compact operator $T : E \longrightarrow F$ is Dunford-Pettis if one of the following statements is valid:*

1. *E has the positive Schur property,*
2. *F has the Schur property.*

Reciprocally, we give necessary conditions under which each AM-compact operator is Dunford-Pettis,

Theorem 3.4. *Let E and F be two Banach lattices such that the norm of E is order continuous. If each AM-compact operator $T : E \longrightarrow F$ is Dunford-Pettis then one of the following statements is valid:*

1. *E has the positive Schur property,*
2. *F has an order continuous norm.*

Proof. Assume that E does not has the positive Schur property and that the norm of F is not order continuous. Since E does not have the positive Schur property, it follows from Proposition 2.1 of [2] that there is a disjoint weakly null sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n. Hence, by Proposition 2.5 of [2], there exists a positive disjoint sequence (g_n) in E' with $\|g_n\| = g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ if $n \neq m$. $(\star)$

Since the norm of E is order continuous, it follows from Corollary 2.4.3 of [11] that $g_n \to 0$ for $\sigma(E', E)$. Hence, the positive operator $Q : E \to c_0$ defined by

$$Q(x) = (g_n(x))_{n=1}^{\infty} \text{ for all } x \in E,$$

is well defined. On the other hand, since the norm of F is not order continuous, there exists a disjoint sequence (y_n) of F^+ and $y \in F^+$ such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for each n.

Now, we consider the positive operator $S : c_0 \to F$ defined by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \text{ for all } (\lambda_n) \in c_0.$$

The series defining Q is norm convergent for $(\lambda_n) \in c_0$ because the sequence (y_n) is disjoint and order bounded.

Now, we consider the positive operator $T = S \circ Q : E \to c_0 \to F$. It is clear that T is AM-compact but T is not Dunford-Pettis. In fact, note that (x_n) is a weakly null sequence of E^+ and then by $(\star)$ we have

$$T(x_n) = S \circ Q(x_n) = S(e_n) = y_n \text{ for all } n.$$

If T is Dunford-Pettis, then $\lim_{n\to\infty} \|T(x_n)\| = \lim_{n\to\infty} \|y_n\| = 0$, which contradicts with $\|y_n\| = 1$ for all n. ∎

Remark 1. *The assumption "E has an order continuous norm" is essential in Theorem 3.4. In fact, each operator $T : \ell^\infty \to c$ is Dunford-Pettis but neither ℓ^∞ has the positive Schur property nor c has an order continuous norm.*

3.2 On the classes of limited completely continuous and AM-compact operators

To study the AM-compactness of *lcc* operators, we need to recall some definitions.

A norm bounded subset A of X is said limited set if every weak* null sequence (f_n) of X' converges uniformly to zero on A [7], that is, $\lim\limits_{n\to\infty} \sup_{x\in A} |\langle f_n, x\rangle| = 0$.

Note that every relatively compact set is limited but the converse is not true in general. Indeed, the set $\{e_n : n \in \mathbb{N}\}$ of unit coordinate vectors is a limited set in ℓ^∞ which is not relatively compact. If every limited subset of X is relatively compact then, X has the Gelfand-Phillips property (abb. GP-property). Alternatively, X has the GP-property if, and only if, every limited and weakly null sequence (x_n) in X is norm null ones [10]. As an example, the classical Banach spaces c_0 and ℓ^1 has the GP-property but the Banach space ℓ^∞ does not has this property.

Let us recall from Borwein [6] that, X has the Dunford-Pettis* property (abb. DP* property) if every relatively weakly compact subset of X is limited. Also, the lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)$ converges to 0 for the weak* topology $\sigma(E', E)$ whenever the sequence (f_n) converges to 0 for the topology $\sigma(E', E)$. And, the lattice operations of E are weak sequentially continuous if the sequence $(|x_n|)$ converges to 0 for the weak topology $\sigma(E, E')$ whenever the sequence (x_n) converges to 0 for the topology $\sigma(E, E')$.

Note that there exists a *lcc* operator which is not AM-compact. Indeed, the identity operator of the Banach space $L^2[0,1]$ is *lcc* (because $L^2[0,1]$ has the GP-property) but fails to be AM-compact (because $L^2[0,1]$ is not discrete).

To establish our first result in this section, we will need the following Lemma, which gives a characterization of limited order intervals .

Lemma 3.5. *Let E be a Banach lattice. Then the following assertions are equivalent*

1. *E' has weak* sequentially continuous lattice operations,*
2. *for each $x \in E^+$, the order interval $[-x, x]$ is limited.*

Proof. $[-x, x]$ is limited if, and only if, $\sup\{|f_n(z)|; z \in [-x, x]\} \longrightarrow 0$.

As $|f_n|(x) = \sup\{|f_n(z)|; z \in [-x, x]\}$, we conclude that $|f_n|$ converge weak* to 0, i.e. E' has weak* sequentially continuous lattice operations. ∎

The following result gives some sufficient conditions under which each *lcc* operator from E into X is AM-compact,

Theorem 3.6. *Each lcc operator $T : E \to X$ is AM-compact if one of the following assertions is valid:*

1. *E has an order continuous norm and has the DP* property,*
2. *E' has weak* sequentially continuous lattice operations,*
3. *F is a discrete with an order continuous norm,*

Proof. (1) Let $x \in E^+$, since E has an order continuous norm then, the order interval $[-x, x]$ is weakly relatively compact.

On the other hand, since E has the DP* property then, $[-x, x]$ is a limited subset of X. Now, since T is *lcc* then $T([-x, x])$ is relatively compact and hence T is AM-compact.

(2) Let $x \in E^+$ and E' has weak* sequentially continuous lattice operations then, it follows from Lemma 3.5 that the order interval is a limited subset of E. Now, since T is *lcc* then $T([-x, x])$ is relatively compact and hence T is AM-compact.

(3) In this case, each operator $T : E \to X$ is AM-compact. ∎

Note that each Dunford-Pettis operator is *lcc*, but the converse is not true in general. Indeed, the identity operator of the Banach lattice ℓ^2 is *lcc* but fails to be a Dunford-Pettis operator. However, as a consequence of Theorem 3.2 and Theorem 3.6, we obtain sufficient conditions under which each Dunford-Pettis operator is *lcc*,

Corollary 3.7. *Let E and F two Banach lattices. Then, each lcc operator $T : E \to X$ is Dunford-Pettis if one of the following conditions is valid:*

1. *E admits the positive Shur property and has the DP* property,*
2. *E admits the positive Shur property and E' has a sequentially continuous lattice operations,*
3. *E admits the positive Shur property and F is discreet with order continuous norm.*

The following Proposition gives some properties whenever each *lcc* operator from a Dedekind σ-complete Banach lattice E into F is AM-compact,

Proposition 3.8. *Let E and F be a two Banach lattices such that E is Dedekind σ-complete and the lattice operations of F are weakly sequentially continuous. If each lcc operator $T : E \to X$ is AM-compact then one of the following assertions is valid:*

1. *E has an order continuous norm,*
2. *F is a discrete with an order continuous norm.*

Proof. Assume that the norm of E is not order continuous and that F is not discrete with order continuous norm.

Since E is Dedekind σ-complete, it follows from Corollary 2.4.3 of [11] that E contains a sub-lattice which is isomorphic to ℓ^∞ and there exists a positive projection $P : E \to \ell^\infty$.

On the other hand, as the lattice operations of F are weakly sequentially continuous and F is not discrete with order continuous norm, it follows from Theorem 3.7 [4] that there exists a regular Dunford-Pettis operator $S : \ell^\infty \to F$ which is not AM-compact.

Since $S : \ell^\infty \to F$ is Dunford-Pettis, then it is order weakly compact and and Since ℓ^∞ is an AM-space with unit, then $S : \ell^\infty \to F$ is weakly compact. It follows from Corollary 2.5 of [12] that the operator S is *lcc*.

Now we consider the operator product $T = S \circ P : E \to F$. Since the operator S is lcc then operator T is lcc because the class of lcc operators is a two-sided ideal. But it is not AM-compact. Otherwise, the operator $T \circ \imath = S$ would be AM-compact (where $\imath : \ell^\infty \to F$ is the natural embedding). This presents a contradiction. ■

As consequence of Proposition 3.8, we have the following result,

Corollary 3.9. *Let F be a Banach lattice with weakly sequentially continuous lattice operations. Then the following assertions are equivalent:*

1. *each lcc operator $T : \ell^\infty \to F$ is AM-compact,*

2. *the norm of F is order continuous.*

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