

The problems of n -valence in some classes of analytic functions

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Abstract

In this paper we determine some examples of the domains of n -valence for functions which are analytic with classical normalization and locally univalent. This problem has not been studied yet. It seems to be rather difficult. We formulate a few conjectures concerning some subclasses of typically real functions and odd typically real functions.

1 Introduction

Suppose that \mathbb{T} denotes the well-known family of all typically real functions, i.e. functions which are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and such that $\Im z \Im f(z) \geq 0$ for $z \in \mathbb{U}$. Moreover, let $\mathbb{T}^{(2)}$ be the class which consists of all odd typically real functions, that is, $\mathbb{T}^{(2)} = \{f \in \mathbb{T} : f(z) = -f(-z), z \in \mathbb{U}\}$.

The univalence problems in the class of typically real functions were considered by Golusin in [1]. He proved that the domain of local and global univalence for typically real functions coincide. The problems of univalence for the class of odd typically real functions were investigated by Koczan and Zaprawa in [4] and [5]. They proved that there can be infinitely many domains of univalence for odd typically real functions and they determined some of these domains.

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In this paper we study the problems of n -valence in some classes of analytic functions. This problem has not been studied yet. It seems to be rather difficult. Recall now the definition of n -valent function (see e.g. [2], vol. I, p.89).

Definition 1. A function f analytic in a domain D is said to be n -valent in D , if for each w_0 the equation $f(z) = w_0$ has at most n roots in D (where the roots are counted in accordance with their multiplicity) and if there is some w_1 such that the equation $f(z) = w_1$ has exactly n roots in D .

Now we formulate a few definitions connected with n -valence problems. Let \mathcal{A} be the class of all analytic functions. A domain $D \subset \mathbb{U}$ is called *the domain of n -valence of the class $A \subset \mathcal{A}$* , if every function from the class A is at most n -valent in D and for every domain Ω such that $D \subsetneq \Omega \subset \mathbb{U}$ there exists a function which is at least $(n + 1)$ -valent in Ω . A domain $D \subset \mathbb{U}$ is called *the domain of n -valence of the function $f \in \mathcal{A}$* , if the function f is n -valent in D and for every domain Ω such that $D \subsetneq \Omega \subset \mathbb{U}$ the function f is at least $(n + 1)$ -valent in Ω . We say that $r_n(A)$ is *the radius of n -valence of the class $A \subset \mathcal{A}$* , if all functions of the class A are at most n -valent in the disc $|z| < r_n(A)$ and for every $r \in (r_n(A), 1)$ there exists a function in A which is at least $(n + 1)$ -valent in the disc $|z| < r$. We say that r_n is *the radius of n -valence of the function $f \in \mathcal{A}$* , if the function f is n -valent in the disc $|z| < r_n$ and the function f is at least $(n + 1)$ -valent in every disc $|z| < r$, $r \in (r_n, 1)$.

2 Main results

Let us denote by F a typically real function, called the universal function, which was introduced by Goodman in the paper [3], that is,

$$(1) \quad F(z) = \frac{1}{\pi} \tan \frac{\pi z}{1+z^2}, \quad z \in \mathbb{U}.$$

We determine the radius of n -valence of the function F . We know from [3] that $F \in \mathbb{T}$ and $F'(z) \neq 0$, $z \in \mathbb{U}$. This means that the function F is typically real and locally univalent. One can observe that

$$F(z) = \frac{1}{\pi i} \frac{1 - \exp \left\{ \frac{-2i\pi z}{1+z^2} \right\}}{1 + \exp \left\{ \frac{-2i\pi z}{1+z^2} \right\}} = \frac{1}{\pi i} \frac{1 - e^{-i\omega}}{1 + e^{-i\omega}} \quad \text{for } \omega(z) = \frac{2\pi z}{1+z^2}, \quad z \in \mathbb{U}.$$

The function $\omega \mapsto e^{i\omega}$ is n -valent function in every curvilinear strip with the width in the direction to the real axis equals to $2n\pi$, i.e. the intersection of this strip with any straight line parallel to the real axis is the segment of the measure $2n\pi$. Hence let us consider vertical strip

$$\{\omega \in \mathbb{C} : |\Re \omega| < n\pi\}, \quad n \in \mathbb{N}.$$

Thus for each $n \in \mathbb{N}$ we determine the subset of \mathbb{U} described by the inequality

$$\left| \Re \frac{2\pi z}{1+z^2} \right| < n\pi,$$

that is, by the inequality

$$(2) \quad \left| \Re \frac{z}{1+z^2} \right| < \frac{n}{2}.$$

Let $z = re^{i\theta}$, $\theta \in [0, \pi/2]$. Then

$$\Re \frac{z}{1+z^2} = \frac{r(1+r^2) \cos \theta}{r^4 + 2r^2 \cos 2\theta + 1} = \frac{r(1+r^2) \cos \theta}{r^4 + 4r^2 \cos^2 \theta + 1 - 2r^2}.$$

From this and (2) we obtain the following inequality

$$\frac{r(1+r^2) \cos \theta}{r^4 + 4r^2 \cos^2 \theta + 1 - 2r^2} < \frac{n}{2}$$

or equivalently, putting $\cos \theta = t$, $|t| \leq 1$, we get

$$\frac{r(1+r^2)t}{r^4 + 4r^2 t^2 + 1 - 2r^2} < \frac{n}{2}.$$

We find the largest r which satisfies the inequality above for every $t \in [-1, 1]$.

Define

$$w(t) = 4r^2 t^2 - \frac{2}{n} r(1+r^2)t + (r^2 - 1)^2.$$

We solve the following inequality

$$(3) \quad \min_{t \in [-1, 1]} w(t) > 0.$$

We obtain

$$\begin{aligned} \min_{t \in [-1, 1]} w(t) &= \begin{cases} w(t_v) & \text{for } t_v \leq 1, \\ w(1) & \text{for } t_v > 1, \end{cases} \\ &= \begin{cases} \left(r^2 - 1 - \frac{1+r^2}{2n}\right) \left(r^2 - 1 + \frac{1+r^2}{2n}\right) & \text{for } 2n - \sqrt{4n^2 - 1} \leq r \leq 1, \\ (r^2 + 1) \left(r^2 + 1 - \frac{nr}{2}\right) & \text{for } 0 \leq r < 2n - \sqrt{4n^2 - 1}, \end{cases} \end{aligned}$$

where $t_v = \frac{1+r^2}{4rn}$ is the vertex of the parabola. It is easy to check that $w(1) > 0$ for $0 \leq r < 2n - \sqrt{4n^2 - 1}$. Thus from $w(t_v) > 0$ we have

$$\begin{cases} \left(r^2 - 1 - \frac{1+r^2}{2n}\right) \left(r^2 - 1 + \frac{1+r^2}{2n}\right) > 0 \\ 2n - \sqrt{4n^2 - 1} \leq r \leq 1 \end{cases}$$

or equivalently

$$\begin{cases} r^2 < \frac{2n-1}{2n+1} \\ 2n - \sqrt{4n^2 - 1} \leq r \leq 1 \end{cases} \Leftrightarrow 2n - \sqrt{4n^2 - 1} \leq r < \sqrt{\frac{2n-1}{2n+1}}.$$

Therefore from the inequality (3) we get $0 \leq r < \sqrt{\frac{2n-1}{2n+1}}$. Hence we have proved the following theorem.

Theorem 1. *The function F given by (1) is n -valent function in the disc $|z| < \sqrt{\frac{2n-1}{2n+1}}$.*

Corrolary 1. *The radius of univalence of the function F given by (1) is equal to $\sqrt{3}/3$.*

The domains of n -valence of a given function could not be unique. There exist functions having infinitely many domains of univalence (see for example [4]). We show that the universal function F has infinitely many domains of n -valence and we determine three of them.

Let us consider a strip bounded by two straight lines $l_1 : \omega = -n\pi + ye^{i\varphi}$ and $l_2 : \omega = n\pi + ye^{i\varphi}$, $n \in \mathbb{N}$, $\varphi \in (0, \pi)$ (the strip includes 0). Obviously this strip has the width in the direction to the real axis equals to $2n\pi$, i.e. the intersection of this strip with any straight line parallel to the real axis is the segment of the measure $2n\pi$. So the inverse images of the strip through the function $z \mapsto \frac{\pi z}{1+z^2}$ are the domains of n -valence of the function F given by (1). Thus we get the one-parameter family of domains of n -valence of the function F ; there are infinitely many such domains.

Assume that E_n is a set which is bounded, symmetric with the respect to both axes of the complex plane and which has the boundary in the first quadrant of the coordinate system given by the polar equation $\rho = r(\theta) e^{i\theta}$, where

$$r(\theta) = \begin{cases} 1 & \text{for } \theta \in \left[0, \arccos \frac{1}{n}\right), \\ \frac{1}{2n} \left(\alpha - \sqrt{\alpha^2 - 4n^2}\right) & \text{for } \theta \in \left[\arccos \frac{1}{n}, \frac{\pi}{2}\right], \end{cases}$$

where $\alpha = \cos \theta + \sqrt{\cos^2 \theta + 4n^2 \sin^2 \theta}$. In Figures 1 and 2 we can see the sets E_1 , E_5 and the discs of univalence $|z| < \sqrt{3}/3$ and of 5-valence $|z| < 3\sqrt{11}/11$ of the function F .

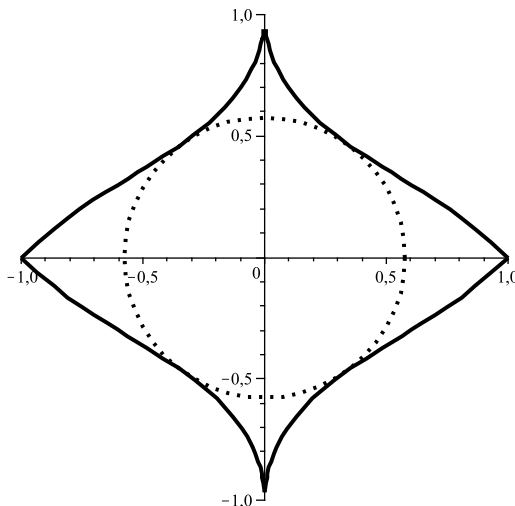


Figure 1: The boundary of the set E_1 (solid line) and the disc of univalence $|z| < \sqrt{3}/3$ (dot line).

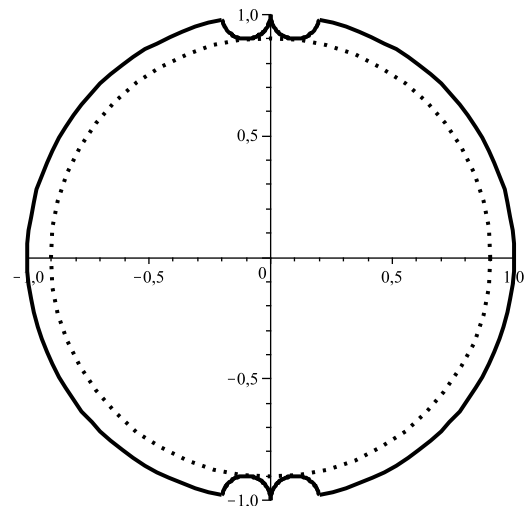


Figure 2: The boundary of the set E_5 (solid line) and the disc of 5-valence $|z| < 3\sqrt{11}/11$ (dot line).

Theorem 2. The set E_n is the domain of n -valence of the function F given by (1).

Proof. The universal function F is n -valent when

$$-n\pi < \Im \frac{2\pi iz}{1+z^2} < n\pi, \quad \text{that is, when} \quad -n\pi < \Re \frac{2\pi z}{1+z^2} < n\pi.$$

Thus to determine the domain of n -valence, it is sufficient to solve the following equation

$$\Re \frac{z}{1+z^2} = \frac{n}{2}.$$

Let $z = re^{i\theta}$, $\theta \in [0, \pi/2]$. Therefore the equation above takes the form

$$\frac{r(1+r^2)\cos\theta}{r^4+4r^2\cos^2\theta+1-2r^2} = \frac{n}{2} \Leftrightarrow r^4 - \frac{2}{n}r(r^2+1)\cos\theta + 4r^2\cos^2\theta - 2r^2 + 1 = 0.$$

Putting $r + \frac{1}{r} = x$, $x \geq 2$, we obtain the following equation

$$(4) \quad x^2 - \frac{2}{n}x\cos\theta - 4\sin^2\theta = 0,$$

which has two solutions of the form:

$$\begin{aligned} x_1 &= \frac{1}{n} \left(\cos\theta + \sqrt{\cos^2\theta + 4n^2\sin^2\theta} \right), \\ x_2 &= \frac{1}{n} \left(\cos\theta - \sqrt{\cos^2\theta + 4n^2\sin^2\theta} \right). \end{aligned}$$

Notice that $x_2 \leq 0$. The condition $x_1 \geq 2$ implies

$$\begin{aligned} \frac{1}{n} \left(\cos\theta + \sqrt{\cos^2\theta + 4n^2\sin^2\theta} \right) \geq 2 &\Leftrightarrow \cos\theta \left(\cos\theta - \frac{1}{n} \right) \leq 0 \\ &\Leftrightarrow 0 \leq \cos\theta \leq \frac{1}{n} \Leftrightarrow \theta \in \left[\arccos \frac{1}{n}, \frac{\pi}{2} \right]. \end{aligned}$$

Hence $\theta \in \left[\arccos \frac{1}{n}, \frac{\pi}{2} \right]$ and $r(\theta)$ is the solution of the equation

$$r^2 - \frac{r}{n} \left(\cos\theta + \sqrt{\cos^2\theta + 4n^2\sin^2\theta} \right) + 1 = 0$$

which belongs to the interval $[0, 1]$, that is,

$$\begin{aligned} r(\theta) &= \frac{1}{2n} \left(\cos\theta + \sqrt{\cos^2\theta + 4n^2\sin^2\theta} \right) + \\ &\quad - \frac{1}{2n} \sqrt{\left(\cos\theta + \sqrt{\cos^2\theta + 4n^2\sin^2\theta} \right)^2 - 4n^2}. \end{aligned}$$

Therefore, the proof is complete. ■

Suppose that B_n is a set which is bounded, symmetric with the respect of the real axis and which the boundary coincides with the segment $[-i, i]$ and the arc given by the polar equation $\rho = r(\theta)e^{i\theta}$, where

$$r(\theta) = \begin{cases} 1 & \text{for } \theta \in \left(-\arccos \frac{1}{2n}, \arccos \frac{1}{2n}\right), \\ \frac{1}{4n} \left(\beta - \sqrt{\beta^2 - 16n^2}\right) & \text{for } \theta \in \left[-\frac{\pi}{2}, -\arccos \frac{1}{2n}\right] \cup \left[\arccos \frac{1}{2n}, \frac{\pi}{2}\right], \end{cases}$$

where $\beta = \cos \theta + \sqrt{\cos^2 \theta + 16n^2 \sin^2 \theta}$. Moreover, let \tilde{B}_n denote set which is symmetric to the set B_n with the respect of the imaginary axis. In Figure 3 we can see the sets B_1 and \tilde{B}_1 .

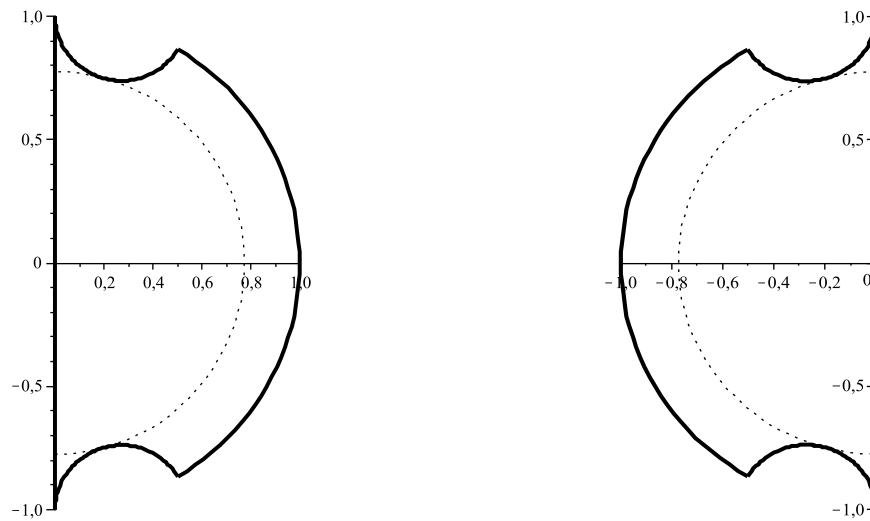


Figure 3: The boundary of the set B_1 and of the set \tilde{B}_1 .

Theorem 3. *The sets B_n and \tilde{B}_n are the domain of n -valence of the function F given by (1).*

The proof is similar to the proof of Theorem 2, but we consider vertical strip

$$\{\omega \in \mathbb{C} : 0 < \Re \omega < 2n\pi\}, \quad n \in \mathbb{N}$$

and we solve the following equation

$$\Re \frac{z}{1+z^2} = n.$$

In each domain of n -valence of a given function, there exist domains of n -valence of a given class. We know that all functions from a given class $A \subset \mathcal{A}$ are n -valent in the disc $|z| < r_n(A)$. It may suggest that the intersection of the domains of n -valence of a given class A includes the disc $|z| < r_n(A)$. Nevertheless, let us notice that in the class $\{F\}$ (i.e. the class which consists of one function F) the intersection of the domains of n -valence is included in the intersection of the sets B_n and \tilde{B}_n , that is, in the set $B_n \cap \tilde{B}_n$, which is the empty set. Thus there

exist classes for which the intersection of the domains of n -valence is the empty set.

And finally, we formulate the following conjectures. Let \mathcal{A}_{LU} be the subclass of \mathcal{A} which consists of functions with classical normalization locally univalent in \mathbb{U} , i.e.

$$\mathcal{A}_{LU} = \left\{ f \in \mathcal{A} : f(z) = z + a_2z^2 + \dots, f'(z) \neq 0, z \in \mathbb{U} \right\}.$$

Conjecture 1.

- (i) The radius of n -valence of the function F given by (1) is the radius of n -valence of the class $\mathbb{T}^{(2)} \cap \mathcal{A}_{LU}$ and of the class $\mathbb{T} \cap \mathcal{A}_{LU}$.
- (ii) The domains of n -valence of the function F given by (1) are the domains of n -valence of the class $\mathbb{T}^{(2)} \cap \mathcal{A}_{LU}$ and of the class $\mathbb{T} \cap \mathcal{A}_{LU}$.

3 Some examples

Similar problems one can consider in the classes generated by functions which are not locally univalent. First, let us introduce some notation: $\mathbb{U}(p, r) = \{z \in \mathbb{C} : |z - p| < r\}$ (so $\mathbb{U}(0, 1) = \mathbb{U}$), $\mathbb{U}^+ = \{z \in \mathbb{U} : \Im z > 0\}$, $\mathbb{U}^- = \{z \in \mathbb{U} : \Im z < 0\}$, $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ and for a set A let \bar{A} be a closure of a set A .

Now let us consider the function

$$h(z) = z + z^2.$$

Notice that the function h is 2-valent but it is not neither typically-real nor locally univalent. Let us observe that $h(\mathbb{U}^+)$ is the set which is bounded by the curve $\Gamma_1 \cup \Gamma_2$, where

$$\begin{aligned} \Gamma_1 & : \begin{cases} x(\theta) = \cos \theta + \cos 2\theta \\ y(\theta) = \sin \theta + \sin 2\theta \end{cases}, \theta \in [0, \pi] \\ \Gamma_2 & = [0, 2]. \end{aligned}$$

Since the function h has real coefficients, so $h(\mathbb{U}^-)$ is the set which is symmetric to $h(\mathbb{U}^+)$ with the respect to the real axis of the complex plane. Thus, the set $h(\mathbb{U})$ can be presented as the union of two disjoint sets G_1 and G_2 such that all points of the set G_1 are taken by the function h only once, and all points of the set G_2 are taken by h twice. Hence, the set G_2 is the interior of the curve $\{h(e^{i\theta}) : \theta \in [2\pi/3, 4\pi/3]\}$. In Figure 5 we have the set $h(\mathbb{U})$.

Suppose that $D_1 = \mathbb{U} \setminus (\overline{\mathbb{U}(-1, 1)} \cap \overline{\mathbb{C}^+})$, D_2 is the set which is symmetric to D_1 with respect to the real axis of the complex plane and $D_3 = \mathbb{U} \cap \{z \in \mathbb{C} : \Re z > -1/2\}$. In Figure 4 we can see the sets D_1, D_2 and D_3 .

Theorem 4. The sets D_1, D_2, D_3 are the domains of univalence of the function $h(z) = z + z^2$.

Proof. The function h is univalent in \mathbb{U}^- , because for all $z_1, z_2 \in \mathbb{U}^-$ we have $h(z_1) = h(z_2) \Leftrightarrow z_1 + z_1^2 = z_2 + z_2^2$. Therefore, $z_1 = z_2$ or $z_1 + z_2 = 1$. The second equality is not true for $z_1, z_2 \in \mathbb{U}^-$. Thus, h is univalent also in \mathbb{U}^+ and in each subset of \mathbb{U}^+ .

First we prove that D_1 is the domain of univalence of the function h . Denote by E the subset of \mathbb{U}^+ such that $h(E) = h(\mathbb{U}) \setminus \overline{h(\mathbb{U}^-)}$. To this end, we solve the following equation

$$h(z) = h(e^{i\theta}), \theta \in [2\pi/3, \pi]$$

$$z^2 + z = e^{2i\theta} + e^{i\theta} \Leftrightarrow (z - e^{i\theta})(z + e^{i\theta} + 1) = 0 \Leftrightarrow z = e^{i\theta} \vee z = -e^{-i\theta} - 1.$$

Since

$$\{h(e^{i\theta}) : \theta \in (0, 2\pi/3)\} = \partial h(\mathbb{U}) \cap \mathbb{C}^+,$$

$$\{h(-e^{-i\theta} - 1) : \theta \in (2\pi/3, \pi)\} = \partial h(\mathbb{U}^-) \cap \mathbb{C}^+$$

and

$$\{h(x) : x \in [0, 1)\} = [0, 2),$$

then $h(E) = G_1 \cap \mathbb{C}^+$. Hence, $h(E) \cap h(\mathbb{U}^-) = \emptyset$ and $h(E) \cup h(\mathbb{U}^-) \cup [0, 2) = h(\mathbb{U})$.

Analogously, we can prove that D_2 is the domain of univalence of the function h .

Now we prove that D_3 is the domain of univalence of the function h . Because the function h is univalent in \mathbb{U}^+ , so it is univalent also in $H^+ = D_3 \cap \mathbb{C}^+$. Since we have:

$$\{h(e^{i\theta}) : \theta \in (0, 2\pi/3)\} = \partial h(\mathbb{U}) \cap \mathbb{C}^+,$$

$$\{h(it - 1/2) : t \in (0, \sqrt{3}/2)\} = (-1, -1/4),$$

$$\{h(x) : x \in (-1/2, 1)\} = (-1/4, 2)$$

and the function h is locally univalent for $z \neq -1/2$, thus $h(H^+) = h(\mathbb{U}) \cap \mathbb{C}^+$.

Analogously, the function h is univalent in $H^- = D_3 \cap \mathbb{C}^-$ and $h(H^-) = h(\mathbb{U}) \cap \mathbb{C}^-$. ■

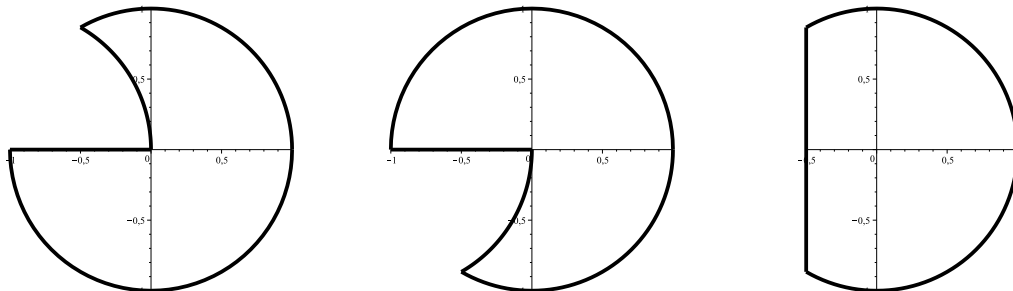


Figure 4: The boundaries of the sets D_1, D_2 and D_3 .

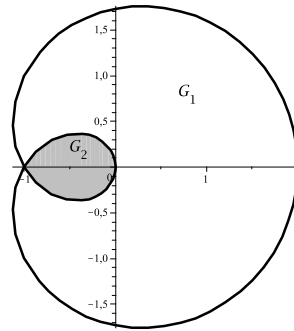


Figure 5: The boundary of the set $h(\mathbb{U})$.

Finally, we present (without proof) one more example of the function with their domains of univalence. Namely, the function

$$g(z) = z + z^2 + \frac{1}{3}z^3$$

which is locally univalent but it is not typically-real. Let $H_1 = \mathbb{U} \setminus \overline{\mathbb{U} \left(-\frac{3}{2} + \frac{\sqrt{3}}{2}i, 1 \right)}$, $H_2 = \mathbb{U} \setminus \overline{\mathbb{U} \left(-\frac{3}{2} - \frac{\sqrt{3}}{2}i, 1 \right)}$ and $H_3 = \mathbb{U} \cap \{z : |\arg(z + 1)| < \frac{\pi}{3}\}$. In Figure 6 we can see the sets H_1, H_2, H_3 and in Figure 7 - the set $g(\mathbb{U})$.

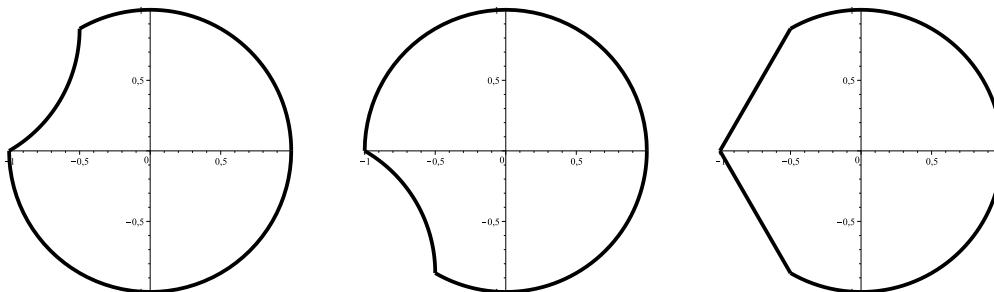


Figure 6: The boundaries of the sets H_1, H_2 and H_3 .

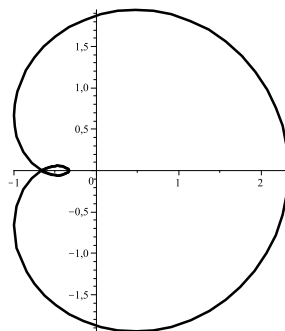


Figure 7: The boundary of the set $g(\mathbb{U})$.

Theorem 5. *The sets H_1, H_2, H_3 are the domains of univalence of the function $g(z) = z + z^2 + \frac{1}{3}z^3$.*

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