Hypercyclic behaviour of multiples of composition operators on weighted Banach spaces of holomorphic functions*

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Abstract

Let $S(\mathbb{D})$ be the collection of all the holomorphic self-maps of open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, and C_φ, the composition operator induced by $\varphi \in S(\mathbb{D})$. For $\alpha > 0$, $\lambda \in \mathbb{C}$, we give some sufficient and necessary conditions for the hypercyclicity of multiples of composition operators λC_φ acting on the weighted Banach spaces of entire functions $H^\infty_{\alpha,0}$. Moreover, we obtain a partial characterization for the frequent hypercyclicity of λC_φ on $H^\infty_{\alpha,0}$.

1 Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ the boundary of $\mathbb{D}$. Let $H(\mathbb{D})$ and $S(\mathbb{D})$ denote the set of all holomorphic functions on $\mathbb{D}$ and the collection of all the holomorphic self-maps of $\mathbb{D}$. For $\varphi \in S(\mathbb{D})$, we can define a linear composition operator

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \to f \circ \varphi.$$

The study of composition operators on various spaces of analytic functions has quite a long and rich story. This is because the theory of composition operators

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links quite basic questions, such as the study of commutants of multiplication operators and the theory of dynamical systems. Composition operators have been studied by many authors on various spaces of analytic functions. For general references on the theory of composition operators, see the well-known books [7] by Cowen and MacCluer, [20] by Shapiro and [23] by K. H. Zhu.

We denote the class of automorphisms of $\mathbb{D}$ by $Aut(\mathbb{D})$. Particularly useful automorphisms are the special ones φ_a, defined for each $a \in \mathbb{D}$ by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

The map φ_a interchanges the point a and the origin, and

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}, \quad \varphi_a \circ \varphi_a(z) = z \ \text{ for } \ a, \ z \in \mathbb{D}.$$

Let $\nu : \mathbb{D} \to (0, \infty)$ be a bounded and continuous function (*weight*). Then we define

$$H_\nu^\infty := \{f \in H(\mathbb{D}), \ \|f\|_\nu = \sup_{z \in \mathbb{D}} \nu(z)|f(z)| < \infty\} \ \text{ and}$$
$$H_{\nu,0}^\infty = \{f \in H_\nu^\infty, \ \lim_{|z| \to 1} \nu(z)|f(z)| = 0\}.$$

Endowed with the weighted sup-norm $\|.\|_\nu$, H_ν^∞ and $H_{\nu,0}^\infty$ both are Banach spaces and we refer to them as weighted-type spaces. As we all know the set of polynomials is dense in $H_{\nu,0}^\infty$, so that $H_{\nu,0}^\infty$ is a separable space. In particular, for $\alpha > 0$ and $\nu(z) = (1-|z|^2)^\alpha$, we obtain H_α^∞ and $H_{\alpha,0}^\infty$. Weights which satisfy $\nu(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ are called radial. A typical weight is a radial weight which is non-increasing with respect to $|z|$ such that additionally $\lim_{|z| \to 1} \nu(z) = 0$ holds. If the weight satisfies the following condition (due to Lusky, see [16])

$$(L1) \quad \inf_{n \in \mathbb{N}} \frac{\nu(1-2^{-n-1})}{\nu(1-2^{-n})} > 0,$$

by Theorem 2.3 in [3], all operators $C_\varphi : H_{\nu,0}^\infty \to H_{\nu,0}^\infty$ are bounded. Examples of weights satisfying condition $(L1)$ include the standard weights $\nu_\alpha = (1-|z|^2)^\alpha$ with $\alpha > 0$.

Let $L(X)$ denote the space of linear and continuous operators on a separable, infinite dimensional Banach space X. A continuous linear operator $T \in L(X)$ is said to be hypercyclic if there is an $f \in X$ such that the orbit $orb(f, T) := \{T^n f\}_{n \geq 0}$ is dense in X, and in this case we refer to f as a hypercyclic vector for T.

Recently, the investigation of linear dynamics has become a very active area of research. The hypercyclicity of composition operators on the Hardy space H^2 have been primarily considered by Bourdon and Shapiro in [4, 5, 20]. Very recently, in [2], Bonet showed that the differentiation operator $D : H_\nu \to H_\nu$ is continuous if and only if $D : H_{\nu,0} \to H_{\nu,0}$ is continuous. At the same time, he obtained a beautiful result [2, Theorem 2.3] for the hypercyclicity of $D : H_{\nu,0} \to H_{\nu,0}$. Similarly, here H_ν and $H_{\nu,0}$ are weighted spaces of holomorphic functions on $\mathbb{C}$. We refer the reader to [2, p. 650]. Moreover, In [10], important results about the hypercyclicity of λC_φ acting on weighted Dirichlet spaces S_ν are obtained. In 2010, the authors of [1] have investigated the frequent hypercyclicity of λC_φ also acting on S_ν. Based on the above references, we continue this line and investigate the hypercyclic behavior of $\lambda C_\varphi : H^\infty_{\alpha,0} \to H^\infty_{\alpha,0}$ in this paper, where

$$H^\infty_{\alpha,0} = \{f \in H^\infty_\alpha : \lim_{|z|\to 1}(1-|z|^2)^\alpha |f(z)| = 0\}.$$

The space is a separable, infinite dimensional Banach space when endowed with the norm

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f(z)| < \infty.$$

An important point for our research is the fact that $H^\infty_{\alpha,0}$ is an automorphism-invariant space. Indeed, for any automorphism φ_a $(a \in \mathbb{D})$ and any $f \in H^\infty_{\alpha,0}$, we have that

$$\begin{aligned}
&\lim_{|z|\to 1}(1-|z|^2)^\alpha |f\circ\varphi_a(z)| = \lim_{|w|\to 1}(1-|\varphi_a(w)|^2)^\alpha |f(w)| \\
&= \lim_{|w|\to 1}\frac{(1-|\varphi_a(w)|^2)^\alpha}{(1-|w|^2)^\alpha}(1-|w|^2)^\alpha|f(w)| \\
&= \lim_{|w|\to 1}\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^\alpha (1-|w|^2)^\alpha |f(w)| = 0.
\end{aligned}$$

That is, $f\circ\varphi_a \in H^\infty_{\alpha,0}$ as required.

A linear fractional transformation is a mapping of the form

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $ad-bc \neq 0$. We will write $LFT(\mathbb{D})$ to refer to the set of all such maps, which are additionally self-maps of the unit disk $\mathbb{D}$. Those maps that take $\mathbb{D}$ onto itself are precisely the members of $Aut(\mathbb{D})$, so that $Aut(\mathbb{D}) \subset LFT(\mathbb{D}) \subset S(\mathbb{D})$.

We classify those maps according to their fixed point behaviour, see [20, p. 5]:

(a) *Parabolic* members of $LFT(\mathbb{D})$ have their fixed point on $\partial\mathbb{D}$.

(b) *Hyperbolic* members of $LFT(\mathbb{D})$ must have an attractive fixed point in $\overline{\mathbb{D}}$, with the other fixed point outside $\mathbb{D}$, and lying on $\partial\mathbb{D}$ if and only if the map is an automorphism of $\mathbb{D}$.

(c) *Loxodromic* and *elliptic* members of $\varphi \in LFT(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside $\overline{\mathbb{D}}$. The elliptic ones are precisely the automorphisms in $LFT(\mathbb{D})$ with this fixed point configuration.

We can find that most our work will focus on maps φ with no fixed point in $\mathbb{D}$. For these we give the following remarkable proposition. We refer the interested reader to [20, p. 78].

Proposition 1.1. *(Denjoy-Wolff) If* $\varphi : \mathbb{D} \to \mathbb{D}$ *is an analytic map with no fixed point in* $\mathbb{D}$*. Then there exists a point* $a \in \partial\mathbb{D}$ *such that* $\varphi^n \to a$ *uniformly on compact subsets of* $\mathbb{D}$*.*

The point is called the Denjoy-Wolff point of φ and satisfies that a is a boundary fixed point of φ, that is, φ has non-tangential limit a at a.

The hypercyclicity of composition operators in one complex variable has been discussed in [4, 5, 13, 14, 18]. Especially from [18], we obtain the following conditions for the hypercyclicity of $C_\varphi : H^\infty_{\nu,0} \to H^\infty_{\nu,0}$ according to the classification of φ.

Theorem 1.2. *Let ν be a weight on $\mathbb{D}$ and $\varphi \in S(\mathbb{D})$. If $C_\varphi : H^\infty_{\nu,0} \to H^\infty_{\nu,0}$ is continuous, then the following holds:*

(1) If $\varphi \in Aut(\mathbb{D})$ fixes no point in $\mathbb{D}$, then C_φ is hypercyclic.

(2) If $\varphi \in LFT(\mathbb{D})$ is a hyperbolic non-automorphism, then C_φ is hypercyclic.

(3) Let $\nu(z) = (1 - |z|^2)^\alpha$ for $\alpha \leq 1$ and $z \in \mathbb{D}$. If $\varphi \in LFT(\mathbb{D})$ is a parabolic non-automorphism, then C_φ is not hypercyclic.

Remark 1.3. In fact, (3) is a generalization of [18, Theorem 3.12]. We can use a similar approach to show that only constant functions can be limit points of C_φ orbits when $\alpha < 1$ and the orbit of f under C_φ is bounded if $\alpha = 1$. Thus C_φ is not hypercyclic in both cases.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \preceq B$, $A \succeq B$ and $A \asymp B$ mean that there may be different positive constants C such that $A \leq CB$, $A \geq CB$ and $B/C \leq A \leq CB$, respectively.

2 Preliminary results

In this section, we give some well-known lemmas which are needed in the proof of main results.

Lemma 2.1. *[24, Lemma 1] Assume $0 < \alpha < \infty$, and let $f \in H^\infty_\alpha$. Then we have that*

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f(z)| \asymp \sup_{z\in\mathbb{D}}(1 - |z|^2)^{\alpha+1}|f'(z)|. \tag{2.1}$$

Recall that two linear continuous operators T and S on a separable infinite dimensional Banach space X are called quasiconjugate, if there exists a continuous map ϕ on X with dense range such that $T \circ \phi = \phi \circ S$. Moreover, if ϕ can be chosen to be a homeomorphism, then T and S are called conjugate. We have the following result:

Lemma 2.2. *[14, Proposition 2.24] Hypercyclicity is preserved under quasiconjugacy and conjugacy.*

Lemma 2.3. *Let m be any positive integer and $a \in \mathbb{C}$. If $|a| \geq 1$, then the subspace of all polynomials that vanish m times at a is dense in $H^\infty_{\alpha,0}$.*

Proof. This Lemma can be proved similarly to [18, Proposition 3.1], so we omit the details. ∎

Lemma 2.4. *[14, Theorem 3.12] (Hypercyclicity Criterion) Let T be a continuous linear operator on a separable Banach space X. If there are two dense subsets X_0 and Y_0 in X, an increasing sequence $\{n_k\}$ of positive integers, and maps $S_{n_k}: Y_0 \to X,\ k \geq 1$, satisfying,*

(a) $T^{n_k}x \to 0$ for any $x \in X_0$, as $k \to \infty$, and

(b) $S_{n_k}y \to 0$ and $T^{n_k}S_{n_k}y \to y$ as $k \to \infty$, for any $y \in Y_0$,

then T is hypercyclic.

Lemma 2.5. *Let $\alpha > 0, k \in \mathbb{N}$ and $0 \leq x \leq 1$. Let*

$$H_{k,\alpha}(x) = x^k(1-x^2)^\alpha.$$

Then

$$\max_{0\leq x\leq 1} H_{k,\alpha}(x) = H_{k,\alpha}(r_k) = \begin{cases} 1, & k=1; \\ \left(\frac{2\alpha}{k+2\alpha}\right)^\alpha \left(\frac{k}{k+2\alpha}\right)^{k/2}, & k>1. \end{cases}$$

where

$$r_k = \begin{cases} 0, & k=1; \\ \left(\frac{k}{k+2\alpha}\right)^{1/2}, & k>1. \end{cases}$$

Remark 2.6. Using $\lim_{x\to\infty}(1-\frac{1}{x})^x = \frac{1}{e}$, we obtain that

$$\lim_{k\to\infty}\left(\frac{k}{k+2\alpha}\right)^{k/2} = e^{-\alpha}. \tag{2.2}$$

The next statements is due to León and Müller.

Lemma 2.7. *[14, Theorem 6.7] Let T be an operator on a complex Fréchet space X. If $x \in X$ is such that $\{\lambda T^n x,\ \lambda \in \mathbb{C},\ |\lambda| = 1,\ and\ n \in \mathbb{N}_0\}$ is dense in X, then $orb(x, \lambda T)$ is dense in X for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

In particular, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, T and λT have the same hypercyclic vectors, that is,

$$HC(T) = HC(\lambda T).$$

Lemma 2.8. *[14, Proposition 5.1] Let T be a hypercyclic operator on a complex Banach space X. Then we have the orbit of every $x^* \neq 0$ in X^* under the adjoint T^* is unbounded.*

3 Hypercyclicity

3.1 φ with an interior fixed point

In this section, we first show that the hypercyclicity of λC_φ is impossible if φ has an interior fixed point. Although the proof for the following proposition is similar to that of [10, Proposition 2.10], we will include a proof for the sake of completeness.

Proposition 3.1. *Suppose $\varphi \in S(\mathbb{D})$ with an interior fixed point on $\mathbb{D}$. Suppose that $C_\varphi : H^\infty_{\alpha,0} \to H^\infty_{\alpha,0}$ is bounded. Then for each $\lambda \in \mathbb{C}$, the operator λC_φ in not hypercyclic on $H^\infty_{\alpha,0}$.*

Proof. Assume that $p \in \mathbb{D}$ is a fixed point of φ, so that $\varphi(p) = p$. Suppose f is the hypercyclic vector for λC_φ and suppose also that for $g \in H^\infty_{\alpha,0}$ there is a sequence $\{n_k\}$ such that $\lambda^{n_k} C^{n_k}_\varphi f$ tends to g in $H^\infty_{\alpha,0}$. Since

$$|\lambda^{n_k} C^{n_k}_\varphi f(z) - g(z)| \leq \frac{\|\lambda^{n_k} C^{n_k}_\varphi f - g\|_\alpha}{(1-|z|^2)^\alpha} \quad (z \in \mathbb{D},\ k \geq 1),$$

we have that the norm convergence in $H^\infty_{\alpha,0}$ implies pointwise convergence; it follows that

$$g(p) = \lim_{k\to\infty} \lambda^{n_k} C_{\varphi_{n_k}} f(p) = \lim_{k\to\infty} \lambda^{n_k} f(\varphi_{n_k}(p)).$$

Since $f(\varphi_{n_k}(p)) = f(p)$ for any $k \in \mathbb{N}$, we obtain

$$g(p) = \lim_{k\to\infty} \lambda^{n_k} f(p).$$

Thus, $g(p) = 0$ when $|\lambda| < 1$, that is not the case for every function $g \in H^\infty_{\alpha,0}$. If $|\lambda| = 1$, then $|g(p)| = |f(p)|$, that is neither the case for every $g \in H^\infty_{\alpha,0}$. Finally, if $|\lambda| > 1$, then $g(p)$ is not even defined, unless $f(p) = 0$. But $f(p)$ cannot be zero for every hypercyclic vector, because the set of hypercyclic vectors is a residual subset. Thus λC_φ is not hypercyclic on $H^\infty_{\alpha,0}$. This completes the proof. ∎

From Proposition 3.1, we need only to consider the cases where φ is a *hyperbolic*-member of $LFT(\mathbb{D})$ without interior fixed point or a *parabolic* member of $LFT(\mathbb{D})$.

3.2 Hyperbolic non automorphism

In this section we analyze the hypercyclicity of λC_φ in the case that φ is a hyperbolic non automorphism.

Theorem 3.2. *Let $\varphi \in S(\mathbb{D})$ be a hyperbolic non-automorphism and η its boundary fixed point. Then λC_φ is hypercyclic on $H^\infty_{\alpha,0}$ if and only if $|\lambda| > \varphi'(\eta)^\alpha$.*

Proof. Necessity. Since hypercyclicity is invariant under similarity by Lemma 2.2, we may suppose that the boundary fixed point is 1. First, we perform the change of variables

$$\sigma(z) = \frac{i(1+z)}{1-z},$$

that sends the unit disc to the upper half plane. Moreover, σ sends 1 to ∞ and the exterior fixed point to a point p in the lower half plane. Upon conjugating with an appropriate affine map in the upper half plane, we may suppose that p is on the imaginary axis. Finally, coming back to the unit disk, σ^{-1} sends p to a negative number $a < -1$. Therefore, we may suppose that φ has the expression

$$\varphi(z) = \frac{(\mu a - 1)z + a(1-\mu)}{(\mu - 1)z + a - \mu}, \quad 0 < \mu < 1. \tag{3.1}$$

Then we conjugate once more with

$$\frac{az-1}{a-z},$$

that is an automorphism of the unit disk which fixes 1 and sends a to ∞. Therefore, we may suppose that

$$\varphi(z) = \mu z + 1 - \mu, \text{ with } 0 < \mu < 1.$$

Observe that $\varphi'(1) = \mu$. Then the above inequality allows us to obtain the following easy expression for its iterates

$$\varphi_n(z) = \mu^n z + 1 - \mu^n. \tag{3.2}$$

Now let $|\lambda| \leq \mu^\alpha$ and consider any $f \in H^\infty_{\alpha,0}$. By Lemma 2.1, it follows that

$$\begin{aligned}
\|\lambda^n C_{\varphi_n} f\|_\alpha &\asymp \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^n |f'(\varphi_n(z))||\varphi_n'(z)| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^n |f'(\varphi_n(z))|\mu^n \\
&\preceq \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha+1}}{(1-|\varphi_n(z)|^2)^{\alpha+1}} |\lambda|^n \mu^n \\
&\preceq \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\alpha+1}}{(1-|\varphi_n(z)|)^{\alpha+1}} |\lambda|^n \mu^n \\
&\preceq \sup_{z\in\mathbb{D}} \left(\frac{1-|z|}{1-(\mu^n|z|+1-\mu^n)}\right)^{\alpha+1} |\lambda|^n \mu^n \\
&= \left(\frac{|\lambda|}{\mu^\alpha}\right)^n .
\end{aligned}$$

Then, for any $f \in H^\infty_{\alpha,0}$, $\|\lambda^n C_{\varphi_n} f\|_\alpha$ is bounded if $|\lambda| \leq \mu^\alpha$, which is a contradiction. This shows that the condition $|\lambda| > \mu^\alpha$ is necessary.

Sufficiency. Suppose that $|\lambda| > \mu^\alpha$. We will use Lemma 2.4 (the Hypercyclicity Criterion) to show the result. Let X_0 be the set of all polynomials that vanish m times at 1, where m will be determined later. Due to Lemma 2.3 the set X_0 is dense in $H^\infty_{\alpha,0}$.

We fix $p(z) = (1-z)^m q(z) \in X_0$ and denote $p'(z) = (z-1)^{m-1} q_1(z)$, where $q(z)$ and $q_1(z)$ are polynomials. By Lemma 2.1, it follows that

$$\begin{aligned}
\|\lambda^n C_{\varphi_n} p\|_\alpha &\asymp \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^n |p'(\varphi_n(z))||\varphi_n'(z)| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^n |p'(\varphi_n(z))|\mu^n \\
&\leq |\lambda|^n \mu^n \sup_{z\in\overline{\varphi_n(\mathbb{D})}} |p'(z)|.
\end{aligned}$$

Let C_1 be the maximum of $|q_1(z)|$ on $\overline{\mathbb{D}}$. Since $\varphi_n(\mathbb{D})$ is a disk of radius $(1-2\mu^n)/2$ which is interiorly tangent to the unit disk at 1, the maximum of

$|1-z|^{m-1}$ on $\overline{\varphi_n(\mathbb{D})}$ is attained at $\varphi_n(-1) = 1 - 2\mu^n$. Thus, it follows that

$$\sup_{z\in\overline{\varphi_n(\mathbb{D})}} |p'(z)| \leq 2^{m-1} C_1 \mu^{(m-1)n}.$$

Thus we have that

$$\|\lambda^n C_{\varphi_n} p\|_\alpha \preceq |\lambda\mu^m|^n.$$

From the above inequality, we can choose m large enough to get $|\lambda\mu^m| < 1$ (just take $m > -\log|\lambda|/\log\mu$). Thus, for every $\lambda \in \mathbb{C}$, the iterates $\lambda^n C_{\varphi_n}$ tend to zero pointwise on X_0 as $n \to \infty$.

To verify hypothesis (b) of Lemma 2.4 (Hypercyclicity Criterion) we will use (3.1) instead of (3.2). We are free to do so because both formulae induce conjugate composition operators. Thus the iterates of $\varphi_{-1} = \varphi^{-1}$ are

$$\varphi_{-n}(z) = \frac{(\mu^{-n}a - 1)z + a(1-\mu^{-n})}{(\mu^{-n}-1)z + a - \mu^{-n}}, \text{ with } 0 < \mu < 1 \text{ and } a < -1, \tag{3.3}$$

where n ranges through all the non negative integers.

Define $S = \lambda^{-1} C_{\varphi_{-1}}$, that is, $Sf(z) = \lambda^{-1} f(\varphi^{-1}(z))$. Let Y_0 be the set of all polynomials that vanish m times at a, where m will be determined later. The set Y required by the Hypercyclicity Criterion will be

$$Y = \bigcup_{n=1}^{\infty} \lambda^{-n} C^n_{\varphi_{-1}}(Y_0) = \bigcup_{n=1}^{\infty} \lambda^{-n} C_{\varphi_{-n}}(Y_0).$$

Obviously, $\lambda^{-1} C_{\varphi_{-1}}$ takes Y into itself and it is a right inverse of λC_φ on Y. It is obvious that $S^n = \lambda^{-n} C_{\varphi_{-n}}$. Next we show that S^n tends pointwise to zero on Y_0, then so does on Y.

Let Δ denote the disk that touches tangentially the unit disk at 1 and passes through the exterior fixed point a. In fact $\varphi_{-n}(\mathbb{D})$ is contained in Δ for every n and $\varphi_{-n}(\mathbb{D})$ approaches Δ as n tends to ∞. We refer the interested reader to [10, p. 30, Figure 1]. Let us fix $p \in Y_0$. Then $p(z) = (z-a)^m q(z)$, where $q(z)$ is a polynomial. It follows that

$$\|\lambda^{-n} C_{\varphi_{-n}} p\|_\alpha \asymp \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^{-n} |p'(\varphi_{-n}(z))| |\varphi'_{-n}(z)|$$

Since $p'(z) = (z-a)^{m-1} q_1(z)$, where $q_1(z)$ is another polynomial, we obtain that

$$\begin{aligned}
\|\lambda^{-n} C_{\varphi_{-n}} p\|_\alpha &\asymp \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^{-n} |\varphi_{-n}(z) - a|^{m-1} |q_1(\varphi_{-n}(z))| |\varphi'_{-n}(z)| \\
&\leq \sup_{z\in\overline{\varphi_{-n}(\mathbb{D})}} |q_1(z)| \\
&\qquad \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^{-n} |\varphi_{-n}(z) - a|^{m-1} |(\varphi_{-n}(z) - a)'| \\
&\leq C_2 \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+1} |\lambda|^{-n} |\varphi_{-n}(z) - a|^{m-1} |(\varphi_{-n}(z) - a)'| \\
&\asymp \|\lambda^{-n} (\varphi_{-n}(z) - a)^m\|_\alpha
\end{aligned}$$

where $C_2 = \max_{\overline{\Delta}} |q_1(z)|$. The second inequality follows from the fact $\varphi_{-n}(\mathbb{D}) \subset \Delta$ for every n and the last approximation is due to (2.1).

In the following we compute

$$\|(\varphi_{-n}(z) - a)^m\|_\alpha = \left\| \frac{(a-1)^m (z-a)^m}{((\mu^{-n}-1)z + a - \mu^{-n})^m} \right\|_\alpha . \tag{3.4}$$

Since the operator of multiplication by $z - a$ is bounded on $H^\infty_{\alpha,0}$, so is the operator of multiplication by $(z-a)^m$. Hence we have that

$$\|(\varphi_{-n}(z) - a)^m\|_\alpha \preceq \left\| \frac{1}{((\mu^{-n}-1)z + a - \mu^{-n})^m} \right\|_\alpha . \tag{3.5}$$

Since

$$\frac{1}{(1-x)^m} = \sum_{k=0}^\infty \frac{\Gamma(k+m)}{\Gamma(k+1)\Gamma(m)} x^k,$$

then

$$\begin{aligned} \frac{1}{((\mu^{-n}-1)z + a - \mu^{-n})^m} &= \frac{\mu^{mn}}{(a\mu^n - 1)^m} \frac{1}{\left(1 - \frac{\mu^n - 1}{a\mu^n - 1} z\right)^m} \\ &= \frac{\mu^{mn}}{(a\mu^n - 1)^m} \sum_{k=0}^\infty \frac{\Gamma(k+m)}{\Gamma(k+1)\Gamma(m)} \left(\frac{\mu^n - 1}{a\mu^n - 1}\right)^k z^k, \end{aligned}$$

we obtain

$$\|(\varphi_{-n}(z) - a)^m\|_\alpha \preceq \frac{\mu^{mn}}{|a\mu^n - 1|^m} \sum_{k=0}^\infty \frac{\Gamma(k+m)}{\Gamma(k+1)\Gamma(m)} \left|\frac{\mu^n - 1}{a\mu^n - 1}\right|^k \|z^k\|_\alpha. \tag{3.6}$$

From Lemma 2.5 and (2.2) one gets

$$\begin{aligned} \|z^k\|_\alpha &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |z|^k \\ &= \left(\frac{2\alpha}{k+2\alpha}\right)^\alpha \left(\frac{k}{k+2\alpha}\right)^{\frac{k}{2}} \\ &\preceq \left(\frac{1}{k+1}\right)^\alpha . \end{aligned}$$

Thus it follows that

$$\|(\varphi_{-n}(z) - a)^m\|_\alpha \preceq \frac{\mu^{mn}}{|a\mu^n - 1|^m} \sum_{k=0}^\infty \frac{\Gamma(k+m)}{\Gamma(k+1)\Gamma(m)} \left|\frac{\mu^n - 1}{a\mu^n - 1}\right|^k \frac{1}{(k+1)^\alpha}. \tag{3.7}$$

By Stirling's formula $\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we have that

$$\|(\varphi_{-n}(z) - a)^m\|_\alpha \preceq \frac{\mu^{mn}}{|a\mu^n - 1|^m} \sum_{k=0}^\infty k^{m-\alpha-1} \left|\frac{\mu^n - 1}{a\mu^n - 1}\right|^k .$$

On the other hand, for $l > 0$, Stirling's formula shows that

$$\sum_{k=1}^{\infty} k^{l-1}|z|^k \asymp \frac{1}{(1-|z|)^l} \asymp \frac{1}{(1-|z|^2)^l}.$$

Hence, if we choose $m > \alpha$, we find that

$$\begin{aligned}
\|(\varphi_{-n}(z)-a)^m\|_\alpha &\preceq \frac{\mu^{mn}}{|a\mu^n-1|^m}\left(1-\left|\frac{\mu^n-1}{a\mu^n-1}\right|^2\right)^{\alpha-m} \\
&= \frac{\mu^{mn}|a\mu^n-1|^{m-2\alpha}}{((a^2-1)\mu^n+2(1-a))^{m-\alpha}} \\
&\leq \frac{\mu^{\alpha n}|a\mu^n-1|^{m-2\alpha}}{((a^2-1)\mu^n+2(1-a))^{m-\alpha}}.
\end{aligned}$$

Since $0 < \mu < 1$ and $a < -1$ we have that

$$\|\lambda^{-n}C_{\varphi_{-n}}p\| \preceq |\lambda|^{-n}\mu^{n\alpha},$$

Since $|\lambda|^{-1}\mu^\alpha < 1$, it follows that the iterates of $\lambda^{-1}C_{\varphi_{-1}}$ tend to zero pointwise on Y_0, consequently, so do on Y. Therefore all the hypotheses of the Hypercyclicity Criterion are fulfilled, thus the conditions are also sufficient. This completes the proof. ■

3.3 Parabolic automorphism

In this section we analyze the hypercyclicity of λC_φ in the case that φ is a parabolic automorphism.

Theorem 3.3. *Let φ be a parabolic automorphism of the unit disk. Then λC_φ is hypercyclic on $H^\infty_{\alpha,0}$ if and only if $|\lambda| = 1$.*

Proof. Sufficiency. Suppose that $|\lambda| = 1$, from (1) of Theorem 1.2 and Lemma 2.7 we obtain that λC_φ is hypercyclic on $H^\infty_{\alpha,0}$.

Necessity. As usual we may suppose that the fixed point is 1. The change of variables

$$\sigma(z) = \frac{i(1+z)}{1-z}$$

takes the unit disk onto the upper half plane. Thus φ satisfies the following formula

$$\varphi(z) = \frac{(2-a)z+a}{-az+2+a} \quad \text{with} \quad a \neq 0 \quad \text{and} \quad Rea = 0. \tag{3.8}$$

It follows that

$$\varphi_n(z) = \frac{(2-na)z+na}{-naz+2+na}, \ n \in \mathbb{N}. \tag{3.9}$$

Now suppose that $|\lambda| < 1$. Let $\delta_0 \in (H^\infty_{\alpha,0})^*$ be the point evaluation functional, that is $\delta_0(f) = f(0)$, it follows that

$$\begin{aligned} |\langle(\lambda^n C_{\varphi_n})^*\delta_0, f\rangle| &= |\lambda^n f(\varphi_n(0))| = \left|\lambda^n f\left(\frac{na}{2+na}\right)\right| \\ &\preceq \frac{|\lambda|^n}{\left(1-|\frac{na}{2+na}|^2\right)^\alpha}. \end{aligned}$$

Since Re $a = 0$, it follows that $|2+na|^2 = 4 + |na|^2$, so $|2+na|^2 - |na|^2 = 4$, so we have that

$$|\langle(\lambda^n C_{\varphi_n})^*\delta_0, f\rangle| \preceq \frac{|\lambda|^n |2+na|^{2\alpha}}{4^\alpha} \to 0, \quad n \to \infty. \tag{3.10}$$

Combining (3.10) with Lemma 2.8, it follows that T is not hypercyclic. Similarly, if $|\lambda| > 1$, then $\lambda^{-1}C_{\varphi_{-1}}$ is not hypercyclic and, therefore, neither is λC_φ. Thus the condition are necessary. This completes the proof. ■

3.4 Hyperbolic automorphism

Theorem 3.4. *Let φ be a hyperbolic automorphism of the unit disk and η its attractive fixed point. Then λC_φ is hypercyclic if and only if $\varphi'(\eta)^\alpha < |\lambda| < \varphi'(\eta)^{-\alpha}$.*

Proof. First, we obtain the expression for the iterates of φ. Without loss of generality we may suppose that φ has -1 and 1 as its fixed points. Moreover, we may assume that 1 is the attractive fixed point. Employing again the change of variables

$$\sigma(z) = \frac{i(1-z)}{1+z}$$

that sends the unit disk onto the upper half plane, the fixed points 1 and -1 to 0 and ∞, respectively, and φ to the contraction map $\varphi(w) = \mu w,\ 0 < \mu < 1$. Coming back to the unit disk we have that

$$\varphi(z) = \frac{(1+\mu)z + 1 - \mu}{(1-\mu)z + 1 + \mu} \quad \text{with } 0 < \mu < 1,$$

from which we can easily obtain the following formula for the iterates

$$\varphi_n(z) = \frac{(1+\mu^n)z + 1 - \mu^n}{(1-\mu^n)z + 1 + \mu^n}, \quad n \in \mathbb{N}.$$

Observe that the derivative at the attractive fixed point 1 is $\varphi'(1) = \mu$.

Necessity. Now for any $f \in H^\infty_{\alpha,0}$, we have the following estimate,

$$\begin{aligned} \|\lambda^n C_{\varphi_n} f\|_\alpha &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\lambda|^n |f(\varphi_n(z))| \\ &\preceq \sup_{z\in\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\right)^\alpha |\lambda|^n \\ &= \sup_{z\in\overline{\varphi_n(\mathbb{D})}} \left(\frac{1-|\varphi_{-n}(z)|^2}{1-|z|^2}\right)^\alpha |\lambda|^n, \end{aligned}$$

By easy computation it follows that

$$1-|\varphi_{-n}(z)|^2=\frac{4\mu^n(1-|z|^2)}{|(\mu^n-1)z+\mu^n+1|^2}, \tag{3.11}$$

where

$$\varphi_{-n}(z)=\frac{(1+\mu^{-n})z+1-\mu^{-n}}{(1-\mu^{-n})z+1+\mu^{-n}}=\frac{(\mu^n+1)z+\mu^n-1}{(\mu^n-1)z+\mu^n+1}.$$

Since $|(\mu^n-1)z+\mu^n+1|\geq(\mu^n+1)-(1-\mu^n)$, we have that

$$\begin{aligned}\|\lambda^n C_{\varphi_n}f\|_\alpha &\leq \sup_{z\in\overline{\varphi_n(\mathbb{D})}}\left(\frac{4\mu^n}{|(\mu^n-1)z+\mu^n+1|^2}\right)^\alpha|\lambda|^n\\ &\leq \left(\frac{4\mu^n}{|\mu^n+1-(1-\mu^n)|^2}\right)^\alpha|\lambda|^n\\ &= \frac{|\lambda|^n}{\mu^{n\alpha}},\end{aligned}$$

that remains bounded for $|\lambda|\mu^{-\alpha}\leq 1$. Therefore, if λC_φ is hypercyclic, then $|\lambda|>\mu^\alpha$. In addition, the inverse operator $\lambda^{-1}C_{\varphi^{-1}}$ must also be hypercyclic. The attractive fixed point of φ_{-1} is -1 and $\varphi'_{-1}(-1)=\mu$. Thus we must also have $|\lambda^{-1}|>\mu^\alpha$. Therefore, we obtain that $\mu^\alpha<|\lambda|<\mu^{-\alpha}$.

Sufficiency. We suppose that $\mu^\alpha<|\lambda|<\mu^{-\alpha}$. Let X_0 be the set of all holomorphic functions on a neighborhood of $\overline{\mathbb{D}}$ that vanish m times at 1, where m is to be determined later on. By Lemma 2.3, the set X_0 is dense in $H^\infty_{\alpha,0}$. Let $f\in X_0$ be fixed. We have that

$$\|\lambda^n C_{\varphi_n}f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\lambda|^n|f(\varphi_n(z))|.$$

Denote $f(z)=(z-1)^m g(z)$, where $g(z)$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}$. From (3.11) we get

$$\begin{aligned}\|\lambda^n C_{\varphi_n}f\|_\alpha &\leq \max_{z\in\overline{\mathbb{D}}}|g(z)|\sup_{z\in\mathbb{D}}|\lambda|^n(1-|z|^2)^\alpha|\varphi_n(z)-1|^m\\ &\preceq |\lambda|^n\sup_{z\in\overline{\varphi_n(\mathbb{D})}}(1-|\varphi_{-n}(z)|^2)^\alpha|z-1|^m\\ &\leq |\lambda|^n\sup_{z\in\overline{\varphi_n(\mathbb{D})}}\frac{\mu^{n\alpha}4^\alpha(1-|z|^2)^\alpha|1-z|^m}{|(\mu^n-1)z+\mu^n+1|^{2\alpha}}.\end{aligned}$$

Since $0<\mu<1$ and $\alpha>0$, an easy calculation shows that, for every positive integer n,

$$\frac{|1-z|^{2\alpha}}{|(\mu^n-1)z+\mu^n+1|^{2\alpha}}\leq\frac{1}{(1-\mu)^{2\alpha}}\quad(z\in\mathbb{D}).$$

Since $1-|z| \leq |1-z|$, it follows that

$$\begin{aligned}
\|\lambda^n C_{\varphi_n} f\|_\alpha &\preceq |\lambda^n| \mu^{n\alpha} \sup_{z\in\varphi_n(\mathbb{D})} \frac{(1-|z|^2)^\alpha |1-z|^m}{(1-\mu)^{2\alpha}|1-z|^{2\alpha}} \\
&\preceq \frac{|\lambda^n|\mu^{n\alpha}}{(1-\mu)^{2\alpha}} \sup_{z\in\varphi_n(\mathbb{D})} \frac{(1-|z|)^\alpha |1-z|^m}{|1-z|^{2\alpha}} \\
&\preceq \frac{|\lambda^n|\mu^{n\alpha}}{(1-\mu)^{2\alpha}} \sup_{z\in\varphi_n(\mathbb{D})} \frac{|1-z|^m}{|1-z|^\alpha}.
\end{aligned}$$

If we choose $m \geq \alpha$, we have that

$$\begin{aligned}
\|\lambda^n C_{\varphi_n} f\|_\alpha &\preceq \frac{|\lambda|^n \mu^{n\alpha}}{(1-\mu)^{2\alpha}} \sup_{z\in\varphi_n(\mathbb{D})} |1-z|^{m-\alpha} \\
&\leq \frac{|\lambda|^n \mu^{n\alpha} 2^{m-\alpha}}{(1-\mu)^{2\alpha}} \preceq |\lambda|^n \mu^{n\alpha} \to 0,\ n\to\infty.
\end{aligned}$$

The last inequality holds due to $|\lambda| < \mu^{-\alpha}$.

For the right inverse we take $S = \lambda^{-1} C_\varphi^{-1} = \lambda^{-1} C_{\varphi^{-1}}$ and the set Y_0 will be the set of functions that are holomorphic on a neighborhood of $\overline{\mathbb{D}}$ and that vanish m times at -1. It is clear that Y_0 is taken into itself by $\lambda^{-1}C_{\varphi^{-1}}$. As above, we can show that $\lambda^{-n} C_{\varphi_{-n}}$ tends pointwise to zero on Y_0 whenever the hypothesis on λ is satisfied. Thus the hypercyclicity of λC_φ follows from the Hypercyclicity Criterion. This completes the proof. ■

4 Frequent hypercyclicity

The lower density of a subset $A \subset \mathbb{N}_0$ is defined as

$$\underline{dens}(A) = \liminf_{N\to\infty} \frac{card\{0 \leq n \leq N,\ n \in A\}}{N+1}.$$

An operator T on a Fréchet space X is called frequently hypercyclic if there is some $x \in X$ such that, for any nonempty open subset U of X,

$$\underline{dens}\{n \in \mathbb{N}_0,\ T^n x \in U\} > 0.$$

In this case, x is called a frequently hypercyclic vector for T. We recall the following Frequent Hypercyclic Criterion for an operator T.

Lemma 4.1. *[14, Theorem 9.9] (Frequent Hypercyclic Criterion) Let X be a separable F- space and $\|.\|$ a complete F-norm on X defining its topology. Assume that T is an operator on X satisfying the following property: there exist a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that*

(i) $\sum\limits_{n=1}^{\infty} \|T^n x\|$ converges for all $x \in X_0$,
(ii) $\sum_{n=1}^{\infty} \|S^n x\|$ converges for all $x \in X_0$,
(iii) $TSx = x$ for all $x \in X_0$.
Then T is frequently hypercyclic.

Here, we recall that a series $\sum_{n=1}^{\infty} x_n$ in a Fréchet space is called unconditionally convergent if for any bijection $\pi : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\pi_{(n)}}$ converges.

An operator T on an F-space X is said to satisfy the Frequent Hypercyclicity Criterion (in short, FHCC) provided that it possesses the property assumed in above theorem.

Lemma 4.2. *Let X be a separable F-space. Assume that T satisfies the FHCC, and that R is an invertible operator on X. Then the operator RTR^{-1} is also frequently hypercyclic.*

Before the main result in this section, we list some definitions. An operator T on an F-space X turns to be hypercyclic if and only if it is topologically transitive, that is, for any pair of non-empty open subsets $U,\ V$ of X there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Moreover, T is said to be topologically mixing if for any pair of non-empty open subsets $U,\ V$ of X there exist some $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. It is obvious that every topologically mixing operator is hypercyclic, but the converse is not true, see [8]. Besides, T is said to be chaotic if it is hypercyclic and it has a dense set of periodic points (vectors $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$). For the above definitions, we refer the interested reader to [14]. Moreover, we have the the following proposition.

Proposition 4.3. *[14, Proposition 9.11] An operator on a separable Fréchet space that satisfies the FHCC is also chaotic and mixing.*

In the following, we only consider the frequent hypercyclicity of λC_φ acting on $H^\infty_{\alpha,0}$, where φ is a *hyperbolic* member of $LFT(\mathbb{D})$ without interior fixed point.

Theorem 4.4. *Let $C_\varphi : H^\infty_{\alpha,0} \to H^\infty_{\alpha,0}$ be the composition operator induced by φ which is a hyperbolic member of $LFT(\mathbb{D})$ without interior fixed point. Then the following statements are equivalent:*

(a) λC_φ is frequently hypercyclic.
(b) λC_φ is topologically mixing.
(c) λC_φ is chaotic.
(d) λC_φ is hypercyclic.

Proof. The implications $(a) \Rightarrow (b)$, $(b) \Rightarrow (d)$, and $(c) \Rightarrow (d)$ are trivial. Now we denote $T = \lambda C_\varphi$ and suppose that T is hypercyclic. At this point we distinguish two cases.

Case1: φ is a hyperbolic non-automorphism. In this case, φ has two fixed points, one on $\partial\mathbb{D}$ and the other outside $\overline{\mathbb{D}}$. Choose an automorphism σ of $\mathbb{D}$ sending those points, respectively to 1 and $a \in (-\infty, -1)$. Then by Lemma 4.2, we can assume the fixed points of φ are 1 and a. Thus the explicit expression of φ is

$$\varphi(z) = \frac{(\mu a - 1)z + a(1-\mu)}{(\mu - 1)z + a - \mu},$$

where $a \in (-\infty, -1),\ \mu \in (0,1)$ and, in fact, $\varphi'(1) = \mu$. By Theorem 3.2, we obtain that $T = \lambda C_\varphi$ is hypercyclic on $H^\infty_{\alpha,0}$ if and only if $|\lambda| > \mu^\alpha$. Choose $m \in \mathbb{N}$ satisfying $m > -\log|\lambda| / \log \mu$. Denote by X_0 (Y_0) the set of all polynomials that vanish at least m times at 1 (at a, resp.). Besides, we also define the inverse map $S = \lambda^{-1}C_{\varphi^{-1}}$. Similarly to the proof of Theorem 3.2, we have that

$$\|T^n f\|_\alpha \preceq |\lambda\mu^m|^n \quad \text{for all} \quad f \in X_0$$

and

$$\|S^n f\|_\alpha \preceq \left(\frac{\mu^\alpha}{|\lambda|}\right)^n \quad \text{for all} \quad f \in Y_0.$$

By the conditions $m > -\log|\lambda|/\log\mu$ and $|\lambda| > \mu^\alpha$, we obtain that

$$\sum_{n=1}^{\infty} \|T^n f\|_\alpha < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|S^n f\|_\alpha < \infty \quad \text{for all} \quad f \in X_0 \cap Y_0. \tag{4.1}$$

Then we define $Y := \bigcup_{n=0}^{\infty} S^n(X_0 \cap Y_0)$. Note that S is well-defined on Y, the set Y is dense in $H^\infty_{\alpha,0}$ and S-invariant, TS is the identity on Y and (4.1) is satisfied for all $f \in Y$. Thus T satisfies the FHCC.

Case 2: φ is a hyperbolic automorphism. In this case, φ has two fixed points η, η' on $\partial\mathbb{D}$. Without loss of generality, we assume that η is the attractive one. Take any automorphism σ of $\mathbb{D}$ satisfying $\sigma(\eta) = 1$ and $\sigma(\eta') = -1$. Then $\varphi_0 := \sigma \circ \varphi \circ \sigma^{-1}$ is a hyperbolic automorphism of $\mathbb{D}$ with fixed points at $1, -1$ such that 1 is the attractive point. By Lemma 4.2 we need only to show that λC_{φ_0} satisfies the FHCC. Thus we suppose that 1 and -1 are the fixed points of φ, the point 1 being attractive. By the proof of Theorem 3.4, we have that the explicit expression of φ is

$$\varphi(z) = \frac{(1+\mu)z + 1 - \mu}{(1-\mu)z + 1 + \mu},$$

where $\mu \in (0,1)$ and, in fact $\varphi'(1) = \mu$. Besides, $T = \lambda C_\varphi$ is hypercyclic on $H^\infty_{\alpha,0}$ if and only if $\mu^\alpha < |\lambda| < \mu^{-\alpha}$. Let us fix $m \geq \alpha$. Denote by X_0 the set of all holomorphic functions on a neighborhood of the closed disk $\overline{\mathbb{D}}$ that vanish at least m times at 1. Fix $f \in X_0$, by Theorem 3.4 we obtain that

$$\|T^n f\| \preceq |\lambda|^n \mu^{n\alpha}, \quad \text{for all} \quad f \in X_0.$$

Since $|\lambda| < \mu^{-\alpha}$, it follows that

$$\sum_{n=1}^{\infty} \|T^n f\| < \infty \quad \text{for all} \quad f \in X_0. \tag{4.2}$$

On the other hand, take $S := T^{-1} = \lambda^{-1}C_\varphi^{-1} = \lambda^{-1}C_{\varphi^{-1}}$ and consider the set Y_0 of all holomorphic functions on a neighborhood of $\overline{\mathbb{D}}$ that vanish at least m times at -1. Observe that -1 is the attractive fixed point of φ^{-1} with $(\varphi^{-1})'(-1) = 1/\varphi'(-1) = \mu$ and that $|\lambda| > \mu^\alpha$. Thus we have that

$$\sum_{n=1}^{\infty} \|S^n f\| < \infty \quad \text{for all} \quad f \in Y_0. \tag{4.3}$$

If we set $Y := X_0 \cap Y_0$, then Y is dense in $H_{\alpha,0}$. It is obvious that (4.2) and (4.3) hold for all $f \in Y$. In addition, TS is the identity and Y is S-invariant, because φ^{-1} is conformal and fixes the points 1 and -1. Thus we obtain that T satisfies the FHCC.

In the above two cases, $T = \lambda C_\varphi$ satisfies the FHCC, so by Proposition 4.3 we obtain that T is topologically mixing and chaotic. That is, $(d) \Rightarrow (a)$, $(d) \Rightarrow (b)$, and $(d) \Rightarrow (c)$ hold. This completes the proof. ∎

Open question: Assuming that φ is a parabolic automorphism, when is C_φ frequent hypercyclic on $H^\infty_{\alpha,0}$?

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