Existence and multiplicity results for semilinear elliptic equations at resonance*

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Abstract

In this paper, the existence and multiplicity results for semilinear elliptic equations at resonance are obtained by the minimax methods and Morse theory.

1 Introduction and main results

Consider the semilinear elliptic boundary value problem:

$$\begin{cases} -\Delta u = \lambda_k u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_j < \cdots$ are the distinct eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and $g \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies the subcritical growth condition

$$|g(x,t)| \leq C(1 + |t|^{p-1}), \text{ a.e. } x \in \Omega, \ 1 < p < 2^*, \tag{2}$$

where

$$2^* = \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ \infty, & N = 1, 2. \end{cases}$$

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Under the condition that

$$0 \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{k+1} - \lambda_k,$$

uniformly for a.e. $x \in \Omega$, problem (1.1) is called the double resonant problem. The solvability of this problem has been studied by many authors. There are some well-known sufficient conditions, such as the Landesman-Lazer-type condition (see [1, 2]) and the nonquadratic condition at infinity (see [3, 4]). The following theorems were obtained in [3, 4].

Theorem A (see [3]) Suppose that (1.2) holds and that g satisfies

$$0 \leq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \preceq \lambda_{k+1} - \lambda_k, \tag{3}$$

and

$$2G(x,t) - g(x,t)t \to +\infty \tag{4}$$

as $|t| \to \infty$ for a.e. $x \in \Omega$. Then problem (1.1) has a solution.

Here and then, $G(x,t) = \int_0^t g(x,s)ds$ and we write $a(x) \preceq b(x)$ to indicate that $a(x) \leq b(x)$ on Ω, with strict inequality holding on a subset of positive measure.

Theorem B (see [4]) Suppose that (1.2), (1.3), (1.4), $k \geq 2$ and $g(x,0) \equiv 0$ for $x \in \Omega$ hold. Assume that there exists a function $\alpha \in L^1(\Omega)$ such that

$$2G(x,t) - g(x,t)t \geq \alpha(x) \tag{5}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$ and

$$g'(x,0) \preceq \lambda_1 - \lambda_k. \tag{6}$$

Then problem (1.1) has at least three nontrivial solutions in which one is positive and one is negative.

In this paper, we consider the existence and multiplicity of solutions of problem (1.1) under double resonant and a condition weaker than the nonquadratic condition. Our main results are the following theorems. First we consider the existence of solutions.

Theorem 1.1 Suppose that (1.2), (1.3) and (1.5) hold. Assume that there exists a positive measure subset Ω_1 of Ω such that

$$2G(x,t) - g(x,t)t \to +\infty \tag{7}$$

as $|t| \to \infty$ for a.e. $x \in \Omega_1$. Then problem (1.1) has at least one solution.

Remark 1.1 Theorem 1.1 generalizes Theorem A, because conditions (1.5) and (1.7) are weaker than (1.4). Indeed there are functionals satisfying (1.3), (1.5) and (1.7) but not satisfying (1.4). For example, let $\rho(x) : \overline{\Omega} \to [0, \frac{\pi}{2}]$ be continuous with $\rho(x) = 0$ on Ω_1, $\rho(x) = \frac{\pi}{2}$ on Ω_2, where Ω_1 and Ω_2 are two subsets of Ω with positive measures. Define

$$g(x,t) = \frac{3}{2}|t|^{-\frac{1}{2}} t \sin\rho(x).$$

Then it is easy to check that $g(x,t)$ satisfies all the assumptions of Theorem 1.1, but it does not satisfy (1.4) and does not satisfy the conditions in [5].

Now, we assume that $g(x,0) \equiv 0$ for $x \in \Omega$ and $k \geq 2$. Hence problem (1.1) admits a trivial solution $u = 0$. Then we have the following two theorems.

Theorem 1.2 Suppose that (1.2), (1.3), (1.5), (1.6) and (1.7) hold. Then problem (1.1) has at least three nontrivial solutions in which one is positive and one is negative.

Remark 1.2 Theorem 1.2 generalizes Theorem B, because that condition (1.7) is weaker than (1.4). In [1, 6, 7], some theorems of three solutions are obtained. There are functionals $G(x,t)$ satisfying our Theorem 1.2 and not satisfying the conditions in [1, 6, 7]. For example, let

$$G(x,t) = \begin{cases} -\frac{1}{2}t^2 \left(d + \int_1^t s^{\varepsilon-3} ds\right) \sin\rho(x) & |t| > \delta, \\ \frac{1}{2}t^2(\lambda_1 - \lambda_m - \varepsilon) & |t| \leq \delta, \end{cases}$$

where $\varepsilon \in (0,1)$. And $d \in \mathbb{R}$ is such that $d + \int_1^t s^{\varepsilon-3} ds \to 1$ as $t \to +\infty$.

Then, we consider the case $g'(x,0) = \lambda_m - \lambda_k$. We need a local sign condition: there exists $\delta > 0$, such that

$$G(x,t) + \frac{1}{2}\lambda_k t^2 \geq \frac{1}{2}\lambda_m t^2 \tag{8}$$

or

$$G(x,t) + \frac{1}{2}\lambda_k t^2 \leq \frac{1}{2}\lambda_m t^2 \tag{9}$$

for $|t| \leq \delta$.

Theorem 1.3 Suppose that (1.2), (1.3), (1.5), (1.7), $k \geq 2$ and $g'(x,0) = \lambda_m - \lambda_k$ hold. Assume that there exist $t_1 < 0$, $t_2 > 0$ such that $\lambda_k t_1 + g(x,t_1) = \lambda_k t_2 + g(x,t_2) = 0$ for $x \in \Omega$. If g satisfies (1.8), $m \geq 2$ and $k \neq m$ or (1.9), $m > 2$ and $k \neq m-1$. Then problem (1.1) has at least five nontrivial solutions.

There are conjugate results of Theorems 1.1-1.3.

Theorem 1.4 Suppose that (1.2) holds and that g satisfies that

$$0 \preceq \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \leq \lambda_{k+1} - \lambda_k, \tag{10}$$

uniformly for a.e. $x \in \Omega$. Assume that there exist a positive measure subset Ω_1 of Ω and a function $\beta \in L^1(\Omega)$ such that

$$2G(x,t) - g(x,t)t \to -\infty \tag{11}$$

as $|t| \to \infty$ for a.e. $x \in \Omega_1$, and

$$2G(x,t) - g(x,t)t \leq \beta(x) \tag{12}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Then problem (1.1) has at least one solution.

Theorem 1.5 Suppose that (1.2), (1.6), (1.10), (1.11), (1.12) and $k \geq 2$ hold. Then problem (1.1) has at least three nontrivial solutions in which one is positive and one is negative.

Theorem 1.6 Suppose that (1.2), (1.10), (1.11), (1.12), $k > 2$ and $g'(x,0) = \lambda_m - \lambda_k$ hold. Assume that there exist $t_1 < 0$, $t_2 > 0$ such that $\lambda_k t_1 + g(x,t_1) = \lambda_k t_2 + g(x,t_2) = 0$ for $x \in \Omega$. If g satisfies (1.8), $m \geq 2$ and $k \neq m$ or (1.9), $m > 2$ and $k \neq m-1$. Then problem (1.1) has at least five nontrivial solutions.

2 Proofs of main results

As it is known that $u \in H_0^1(\Omega)$ is a solution of problem (1.1) if and only if u is a critical point of the functional $J : H_0^1(\Omega) \to R$ as

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\lambda_k \int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx. \tag{1}$$

Then J is a C^2 functional with derivatives given by

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v dx - \lambda_k \int_\Omega uv dx - \int_\Omega g(x,u)v dx, \text{ for } u,\ v \in H_0^1(\Omega), \tag{2}$$

$$\langle J''(u)v, w\rangle = \int_\Omega \nabla v \cdot \nabla w dx - \lambda_k \int_\Omega vw dx - \int_\Omega g'(x,u)vw dx, \text{ for } u,\ v,\ w \in H_0^1(\Omega).$$

Denote

$$E_k^- = \bigoplus_{j<k} \ker(-\Delta - \lambda_j),\ E_k = \ker(-\Delta - \lambda_k),\ E_k^+ = \overline{\bigoplus_{j>k} \ker(-\Delta - \lambda_j)}.$$

Then $H_0^1(\Omega)$ has the following decomposition:

$$H_0^1(\Omega) = E_k^- \oplus E_k \oplus E_k^+.$$

In order to prove Theorems 1.1-1.3 we require the following lemmas. Then we can prove Theorems 1.4-1.6 in a similar way.

Lemma 2.1 Suppose that (1.3), (1.5) and (1.7) hold, then the functional J satisfies the (C) condition.

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be such that

$$|J(u_n)| \leq c \text{ and } \langle J'(u_n), u_n \rangle \to 0, \text{ as } n \to +\infty. \tag{3}$$

Here and then, we use c to denote various positive constants. We only need to show that $\{\|u_n\|\}$ is bounded since g has subcritical growth. Suppose, by the way of contradiction, that $\{u_n(x)\}$ possesses a subsequence, still denoted by $\{u_n(x)\}$, satisfies $\|u_n\| \to \infty$ as $n \to \infty$. We claim that $\{u_n(x)\}$ possesses a subsequence, denoted by $\{u_n(x)\}$, satisfies $|u_n(x)| \to \infty$ as $n \to \infty$ for a.e $x \in \Omega$. By (1.4),

$$2G(x,t) - tg(x,t) \to +\infty, \quad \text{as } |t| \to +\infty,$$

on a positive measure subset Ω_1. Therefore by $\Omega_1 \subset \Omega$,

$$2G(x,u_n) - g(x,u_n)u_n \to +\infty, \quad \text{as } n \to +\infty, \text{ uniformly for a.e. } x \in \Omega_1.$$

Fatou's lemma gives

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega [2G(x,u_n) - g(x,u_n)u_n]dx \\
&\geq \int_\Omega \liminf_{n\to+\infty} [2G(x,u_n) - g(x,u_n)u_n]dx \\
&\geq \int_{\Omega_1} \liminf_{n\to+\infty} [2G(x,u_n) - g(x,u_n)u_n]dx + \int_{\Omega\setminus\Omega_1} \alpha(x)dx = +\infty.
\end{aligned}$$

Thus,

$$\int_\Omega [2G(x,u_n) - g(x,u_n)u_n]dx \to +\infty, \qquad \text{as } n \to \infty. \tag{4}$$

From (2.3), we have $J'(u_n)u_n - 2J(u_n)$ is bounded. It follows from (2.1), (2.2) that

$$\lim_{n\to\infty} \int_\Omega [2G(x,u_n) - g(x,u_n)u_n] = \lim_{n\to\infty} [\langle J'(u_n), u_n \rangle - 2J(u_n)],$$

which contradicts (2.4).

It remains to prove the claim.

Denote $z_n = \frac{u_n}{\|u_n\|}$, then $\|z_n\| = 1$. Passing to a subsequence if necessary, we may assume that there exists $z \in H_0^1(\Omega)$ such that

$$\begin{aligned}
z_n &\rightharpoonup z && \text{in } H_0^1(\Omega), \\
z_n &\to z && \text{in } L^2(\Omega), \\
z_n(x) &\to z(x) && \text{a.e. } x \in \Omega.
\end{aligned}$$

By (1.3), for every $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$-\varepsilon \le \frac{g(x,t)}{t} \le \lambda_{k+1} - \lambda_k + \varepsilon$$

for all $|t| \ge M_1$. Choose $\eta \in C(\mathbb{R}, \mathbb{R})$ such that $0 \le \eta \le 1$ and

$$\eta(t) = \begin{cases} 1, & |t| \le M_1, \\ 0, & |t| \ge 2M_1. \end{cases}$$

Set

$$p_n(x) = \begin{cases} \frac{(1-\eta(u_n(x)))g(x,u_n(x))}{u_n(x)}, & u_n(x) \ne 0, \\ 0, & u_n(x) = 0. \end{cases}$$

Then we have $-\varepsilon \le p_n(x) \le \lambda_{k+1} - \lambda_k + \varepsilon$ for a.e. $x \in \Omega$. Thus without loss of generality we may also assume that

$$p_n \rightharpoonup p \quad \text{weakly}^* \text{ in } L^\infty(\Omega)$$

as $n \to \infty$. Note the fact that

$$-\varepsilon \le p(x) \le \lambda_{k+1} - \lambda_k + \varepsilon$$

for a.e. $x \in \Omega$, follows from the weak closedness of convex subset K of $L^2(\Omega)$ given by

$$K = \{s \in L^2(\Omega) | -\varepsilon \le s(x) \le \lambda_{k+1} - \lambda_k + \varepsilon \ \text{ for a.e. } x \in \Omega\}.$$

By Mazur's Theorem, we only need to prove that the convex set K is closed, which follows from Theorem 3.1.2 in [8]. By

$$\frac{(1-\eta(u_n))g(x,u_n)}{\|u_n\|} = p_n z_n \rightharpoonup pz \quad \text{in } L^2(\Omega)$$

and

$$\frac{\eta(u_n)g(x,u_n)}{\|u_n\|} \to 0 \quad \text{in } L^2(\Omega),$$

we have

$$\frac{g(x,u_n)}{\|u_n\|} \rightharpoonup pz \quad \text{in } L^2(\Omega),$$

where $0 \le p \le \lambda_{k+1} - \lambda_k$ for arbitrariness of ε. By (1.3), there exists a constant $M_2 > 0$, such that

$$G(x,t) + \frac{1}{2}(\lambda_k - \lambda_{k+1})t^2 \le 0 \tag{5}$$

for all $|t| > M_2$ and a.e. $x \in \Omega$. When $|t| \le M_2$, by $G(x,t)$ continuous there exists a constant C such that

$$G(x,t) + \frac{1}{2}(\lambda_k - \lambda_{k+1})t^2 \le C \tag{6}$$

It follows from (2.5) and (2.6) that

$$G(x,t) + \frac{1}{2}(\lambda_k - \lambda_{k+1})t^2 \leq C$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Thus, we obtain

$$\int_\Omega \frac{G(x,u_n) + \frac{1}{2}(\lambda_k - \lambda_{k+1})u_n^2}{\|u_n\|^2} dx \leq \frac{C}{\|u_n\|^2}|\Omega|$$

Therefore, one has

$$\limsup_{\|u_n\|\to\infty} \int_\Omega \frac{G(x,u_n) + \frac{1}{2}(\lambda_k - \lambda_{k+1})u_n^2}{\|u_n\|^2} dx \leq \limsup_{\|u_n\|\to\infty} \frac{C}{\|u_n\|^2}|\Omega| = 0 \tag{7}$$

By (2.1), we have

$$\frac{J(u_n)}{\|u_n\|^2} = \frac{1}{2}(1 - \lambda_{k+1}\|z_n\|_{L^2}^2) - \int_\Omega \frac{G(x,u_n) + \frac{1}{2}(\lambda_k - \lambda_{k+1})u_n^2}{\|u_n\|^2} dx.$$

Let $n \to \infty$, from $|J(u_n)| \leq c$ and (2.7), we have

$$\frac{1}{2}(1 - \lambda_k\|z\|_{L^2}^2) \leq 0.$$

Thus, $z \neq 0$. For every $v \in H_0^1(\Omega)$

$$\frac{\langle J'(u_n), v\rangle}{\|u_n\|} = \int_\Omega \left(\nabla z_n \cdot \nabla v - \lambda_k z_n v - \frac{g(x,u_n)}{\|u_n\|} v\right) dx.$$

Let $n \to \infty$, then

$$\int_\Omega (\nabla z \cdot \nabla v - \lambda_k z v - p z v) dx = 0.$$

In other words, we verify that z, satisfies

$$\begin{cases} -\Delta z = (\lambda_k + p)z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Hence 1 is an eigenvalue of problem (2.8). If $0 \preceq p \preceq \lambda_{k+1} - \lambda_k$, by strict monotonicity we have

$$\lambda_k(\lambda_k + p) < \lambda_k(\lambda_k) = 1 \text{ and } 1 = \lambda_{k+1}(\lambda_{k+1}) < \lambda_{k+1}(\lambda_k + p)$$

Where $\lambda_k(\lambda_k + p)$ is the eigenvalue of (2.8). This contradicts that 1 is an eigenvalue of problem (2.8). So either $p \equiv \lambda_k$ or $p \equiv \lambda_{k+1}$. By the unique continuation property, $z(x) \neq 0$ a.e. $x \in \Omega$. And then $|u_n(x)| = |z_n(x)|\|u_n\| \to \infty$ a.e. $x \in \Omega$. ∎

Lemma 2.2 Let g satisfy (1.3), then J is coercive on E_k^+, i.e., $J(u) \to +\infty$, as $\|u\| \to \infty$, $u \in E_k^+$.

Proof Following from Proposition 2 in [9] and (1.3), there exists $\delta > 0$ such that

$$\|u\|^2 - \int_\Omega (u^2 \limsup_{|t|\to\infty} \frac{2G(x,t)}{t^2} + \lambda_k u^2)dx \geq \delta \|u\|^2$$

for all $u \in E_k^+$. From (1.3), we obtain that for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\begin{aligned} &G(x,t) \leq t^2 \limsup_{|t|\to\infty} \frac{G(x,t)}{t^2} + \frac{1}{2}\varepsilon t^2, \quad &&\text{for } |t| \geq M, \\ &G(x,t) \leq C, &&\text{for } |t| < M. \end{aligned}$$

Thus, we have

$$G(x,t) \leq t^2 \limsup_{|t|\to\infty} \frac{G(x,t)}{t^2} + \frac{1}{2}\varepsilon t^2 + C.$$

Therefore, we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_k \int_\Omega u^2 dx - \int_\Omega G(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_k \int_\Omega u^2 dx - \int_\Omega \left(u^2 \limsup_{|u|\to\infty} \frac{G(x,u)}{u^2} + \frac{1}{2}\varepsilon u^2 + C \right) dx \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega \left(u^2 \limsup_{|u|\to\infty} \frac{G(x,u)}{u^2} + \frac{1}{2}\lambda_k u^2 \right) dx - \frac{\varepsilon}{2}\|u\|_{L^2}^2 + C|\Omega| \\ &\geq \frac{1}{2}(\delta - \frac{\varepsilon}{\lambda_{k+1}})\|u\|^2 + C|\Omega| \to +\infty. \end{aligned}$$

Then J is coercive on E_k^+. ∎

Lemma 2.3 Assume that g satisfies (1.3), (1.5) and (1.7), then J is anti-coercive on $E_k^- \oplus E_k$, i.e. $J(u) \to -\infty$, as $\|u\| \to \infty$, $u \in E_k^- \oplus E_k$.

Proof Set $u \in E_k^- \oplus E_k$, write $u = u^- + u_0$, where $u^- \in E_k^-$, $u_0 \in E_k$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_k \|u\|_{L^2}^2 - \int_\Omega G(x,u)dx \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right) \|u^-\|^2 - \int_\Omega G(x,u)dx. \end{aligned} \tag{9}$$

We claim that there exist a positive measure subset Ω_0 of Ω and a function $C(x) \in L^1(\Omega)$ such that

$$G(x,t) \to +\infty$$

as $|t| \to \infty$ for a.e. $x \in \Omega_0$, and

$$G(x,t) \geq C(x) \tag{10}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

By $\|u\| \to \infty$, we have two situations: (a) $\|u^-\| \to \infty$; or (b) $\|u^-\|$ is bounded, $\|u_0\| \to \infty$.

(a) If $\|u^-\| \to \infty$, by (2.9), (2.10) we have

$$J(u) \le \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|u^-\|^2 - C \to -\infty.$$

(b) If $\|u^-\|$ is bounded, then $\|u_0\| \to \infty$, we have

$$J(u) \le \frac{1}{2}C - \int_\Omega G(x,u)dx. \tag{11}$$

Then we only need to show that $\int_\Omega G(x,u)dx \to +\infty$.

Setting $\overline{u}_0 = \frac{u_0}{\|u\|}$, $\overline{u}^- = \frac{u^-}{\|u\|}$, $\overline{u} = \overline{u}^- + \overline{u}_0$, then $\|\overline{u}^-\| \to 0$, $\|u_0\| \to 1$ and there exists $\hat{u} \in E_k^- \oplus E_k$, such that

$$\overline{u} \rightharpoonup \hat{u}, \quad \text{in } H_0^1(\Omega),$$

$$\overline{u} \to \hat{u}, \quad \text{in } L^2(\Omega).$$

Therefore $\|\hat{u}\| = 1$, $\hat{u} \in E_k$. Thus $|\hat{u}|$ is a normalized λ_k eigenfunction. By the unique continuation property, we have

$$\hat{u}(x) \ne 0, \text{ a.e. } x \in \Omega,$$

so

$$|u(x)| = \|u\||\hat{u}(x)| \to \infty, \text{ a.e. } x \in \Omega.$$

By Fatou's lemma, we have

$$\int_{\Omega_0} G(x,u)dx \to +\infty, \quad \|u\| \to \infty. \tag{12}$$

Then

$$\int_\Omega G(x,u)dx = \int_{\Omega\setminus\Omega_0} G(x,u)dx + \int_{\Omega_0} G(x,u)dx \to +\infty, \quad \|u\| \to \infty.$$

From (2.11) and (2.12), we have $J(u) \to -\infty$.

It remains to prove the claim.

By (1.7), for any given $M_3 > 0$, there exists $\delta > 0$ such that

$$2G(x,t) - g(x,t)t \ge M_3, \quad |t| \ge \delta, \text{ a.e. } x \in \Omega_0.$$

Setting $[t,T] \subset [\delta, +\infty]$, by

$$\frac{d}{dt}\left[\frac{G(x,t)}{t^2}\right] = \frac{g(x,t)t - 2G(x,t)}{t^3},$$

we have

$$\begin{aligned}\frac{G(x,T)}{T^2} - \frac{G(x,t)}{t^2} &= \int_t^T \frac{g(x,s)s - 2G(x,s)}{s^3} ds \\ &\leq -M_3 \int_t^T \frac{1}{s^3} ds \\ &= \frac{M_3}{2}\left(\frac{1}{T^2} - \frac{1}{t^2}\right).\end{aligned}$$

From

$$\liminf_{T\to\infty} \frac{G(x,T)}{T^2} \geq 0,$$

we obtain

$$-\frac{G(x,t)}{t^2} \leq -\frac{M_3}{2t^2}.$$

Therefore

$$G(x,t) \geq \frac{M_3}{2}, \quad |t| \geq \delta, \text{ a.e. } x \in \Omega_0.$$

Since $M_3 > 0$ is arbitrary, we have

$$G(x,t) \to +\infty, \quad |t| \to \infty, \text{ a.e. } x \in \Omega_0.$$

From (1.5), we can prove that there exists a function $C(x) \in L^1(\Omega)$ such that

$$G(x,t) \geq \frac{C(x)}{2}, \ t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega$$

in a similar way. ■

Lemma 2.4 (See [6]) Assume that g satisfies (1.8) (or (1.9)), then J has the local linking at $u = 0$ with respect to $H_0^1(\Omega) = H^- \oplus H^+$, where $H^- = E_m^- \oplus E_m$, $H^+ = E_m^+$ (or $H^- = E_m^-$, $H^+ = E_m \oplus E_m^+$).

Lemma 2.5 (See [2]) Let function $p \in C(\bar{\Omega} \times \mathbb{R})$ satisfy $p(x,t) = 0$ for $t < 0$ and all $x \in \Omega$. Assume that

$$\lambda_k \leq \liminf_{t\to+\infty} \frac{p(x,t)}{t} \leq \limsup_{t\to+\infty} \frac{p(x,t)}{t} \leq \lambda_{k+1}, \quad \text{uniformly for } x \in \Omega,$$

for $k \geq 2$. Then the functional

$$\hat{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega P(x,u)dx,$$

satisfies the (PS) condition, where $P(t) = \int_0^t p(x,s)ds$.

Proof of Theorem 1.1 When g satisfies (1.3), (1.5) and (1.7), we have that J is anti-coercive on $E_k^- \oplus E_k$ and coercive on E_k^+. Letting $B_R = \{u \in E_k^- \oplus E_k | \|u\| \leq R\}$, for $R > 0$, and ∂B_R be its boundary on $E_k^- \oplus E_k$. So we have that

$$\max_{u\in\partial B_R} J(u) < \inf_{u\in E_k^+} J(u).$$

In view of Lemma 2.1 and the Saddle Point Theorem, we obtain that the functional J has a critical value $c \geq \inf_{u \in E_k^+} J(u)$. Therefore, problem (1.1) has a solution. ∎

In order to prove Theorem 1.2, we need the following proposition.

Proposition 2.1 (See [10]) Assume that $H = H^- \oplus H^+$, J is bounded from below on H^+ and $J(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in H^-$. Then

$$C_k(J, \infty) \ncong 0, \text{ if } k = \dim H^- < \infty.$$

Proof of Theorem 1.2 By (1.3), (1.5), (1.7) and Lemma 2.1, the functional J satisfies the (C) condition. Since J is weakly lower semi-continuous, and coercive on E_k^+ by Lemma 2.2. Then J is bounded from below on E_k^+. By Lemma 2.3, J is anti-coercive on $E_k^- \oplus E_k$. Then by Proposition 2.1, we have

$$C_\mu(J, \infty) \ncong 0, \tag{13}$$

where $\mu = \dim(E_k^- \oplus E_k)$. By $g'(x, 0) \preceq \lambda_1 - \lambda_k$, $u = 0$ is a local minimum of J, hence

$$C_q(J, 0) \cong \delta_{q,0} Z. \tag{14}$$

By (2.13), (2.14) and $k \geqslant 2$, $C_\mu(J, \infty) \ncong C_\mu(J, 0)$. Thus J has a critical point $u_0 \neq 0$ with

$$C_\mu(J, u_0) \ncong 0. \tag{15}$$

We define the following functional $J_\pm : H_0^1(\Omega) \to R$ as

$$J_\pm(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \lambda_k \int_\Omega |u|^2 dx - \int_\Omega G(x, u^\pm) dx,$$

where $u^+ = \max\{u(x), 0\}$, $u^- = \min\{u(x), 0\}$. By Lemma 2.5 and $k \geq 2$, $J_\pm$ satisfies the (PS) condition. From (2.12) and $k \geq 2$, we have

$$J_\pm(t\varphi_1) \to -\infty, \quad \text{as } t \to \pm\infty,$$

where φ_1 is the first eigenfunction of $-\Delta$. Then there exists $t_0 \in \mathbb{R}$ such that $t_0 > 0$ and $J_\pm(\pm t_0 \varphi_1) \leq 0$. Then by the Mountain Pass Lemma and the Maximum Principle, we can obtain a positive critical point u_+ of J_+ and a negative critical point u_- of J_-. By Corollary 8.5 in [11] the critical group of $J_\pm$ at $u_\pm$ is

$$C_q(J_\pm, u_\pm) \cong \delta_{q,1} Z.$$

By the results in [12],

$$C_q(J, u_\pm) \cong C_q(J|_{C_0^1(\Omega)}, u_\pm) \cong C_q(J_\pm|_{C_0^1(\Omega)}, u_\pm) \cong C_q(J_\pm, u_\pm) \cong \delta_{q,1} Z. \tag{16}$$

From (2.15) and (2.16), we get $u_+ > 0$, $u_- < 0$ and u_0 are three nontrivial critical points of J. ∎

So as to prove Theorem 1.3, we need another proposition.

Proposition 2.2 (See [13, 14]) Let 0 be an isolated point of $J \in C^2(H, \mathbb{R})$. Assume that J has a local linking at 0 with respect to a direct sum decomposition $H = H^- \oplus H^+$, $k = \dim H^- < \infty$, i.e. there exists $r > 0$ small such that

$$\begin{aligned} J(u) &> 0, \quad \text{for } u \in H^+, \ 0 < \|u\| \le r, \\ J(u) &\le 0, \quad \text{for } u \in H^-, \ \|u\| \le r. \end{aligned}$$

Then

$$C_q(J, 0) \cong \delta_{q,k} Z, \qquad \text{for } k = \dim H^-.$$

Proof of Theorem 1.3 From the proof of Theorem 1.2, we have $C_\mu(J, \infty) \not\cong 0$. By Lemma 2.4 and (1.8), J has a local linking at 0 with respect to $(E_m^- \oplus E_m) \oplus E_m^+$. Therefore by Proposition 2.2, we have

$$C_q(J, 0) \cong \delta_{q,\mu_0} Z,$$

where $\mu_0 = \dim(E_m^- \oplus E_m)$. From $m \ne k$, we have $C_q(J, \infty) \not\cong C_q(J, 0)$, for $q = \mu$. Thus J has a critical point $u_0 \ne 0$ with

$$C_\mu(J, u_0) \not\cong 0. \tag{17}$$

Because $f(x, t_1) = 0$ for $t_1 < 0$, we can define cut-off functional $\hat{J} : H_0^1(\Omega) \to R$ as

$$\hat{J}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \lambda_k \int_\Omega |u|^2 dx - \int_\Omega \hat{G}(x, u) dx,$$

where $\hat{G}(x, t) = \int_0^t \hat{g}(x, s) ds$,

$$\hat{g}(x, t) = \begin{cases} g(x, t), & t \in [t_1, 0], \\ 0, & t \notin [t_1, 0]. \end{cases}$$

It is obvious that $\hat{J}(u)$ is bounded below and satisfies the (PS) condition. Hence there is a minimizer u_1 of $\hat{J}(u)$. In the view of the Maximum Principle, $u_1 = 0$ or $t_1 < u_1 < 0$ for all $x \in \Omega$ and $\frac{\partial u_1}{\partial n} |_{\partial\Omega} < 0$. By $g'(x, 0) = \lambda_m - \lambda_k$ and $m \ge 2$, 0 is not a minimizer. In addition, u_1 is a local minimizer of $\hat{J}$ in the $C_0^1(\Omega)$. And by the result in [15], we have u_1 is a local minimizer of J in $H_0^1(\Omega)$ topology and

$$C_q(J, u_1) \cong \delta_{q,0} Z. \tag{18}$$

For $\lambda_k t_2 + g(x, t_2) = 0$, in a similar way J has local minimizer with $0 < u_2 < t_2$ and

$$C_q(J, u_2) \cong \delta_{q,0} Z. \tag{19}$$

Starting with u_1 to define the functionals

$$\tilde{g}(x, t) = g(x, t + u_1) - g(x, u_1), \quad x \in \Omega, \ t \in R.$$

Now we consider the problem

$$\begin{cases} -\Delta v = \lambda_k v + \tilde{g}(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with energy functional

$$\tilde{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{2}\lambda_k \int_\Omega |v|^2 dx - \int_\Omega \tilde{G}(x,v)dx, \quad v \in H_0^1(\Omega),$$

where $\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)ds$. By u_1 is a solution of problem (1.1), we have

$$\begin{cases} -\Delta(v+u_1) = \lambda_k(v+u_1) + g(x, v+u_1) & \text{in } \Omega, \\ v+u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

So if v is a critical point of $\tilde{J}$, $v+u_1$ is a critical point of J. And

$$C_q(\tilde{J}, v) = C_q(J, v+u_1)$$

Moreover, define

$$\tilde{g}^+(x,t) = \begin{cases} \tilde{g}(x,t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and consider the equation

$$\begin{cases} -\Delta v = \lambda_k v + \tilde{g}^+(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The corresponding energy functional is

$$\tilde{J}^+(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{2}\lambda_k \int_\Omega |v|^2 dx - \int_\Omega \tilde{G}^+(x,v)dx,$$

where $\tilde{G}^+(x,t) = \int_0^t \tilde{g}^+(x,s)ds$. By (1.3), we have that $\tilde{g}^+$ satisfies

$$0 \leq \liminf_{t\to+\infty} \frac{\tilde{g}^+(x,t)}{t} \leq \limsup_{t\to+\infty} \frac{\tilde{g}^+(x,t)}{t} \leq \lambda_{k+1} - \lambda_k.$$

By Lemma 2.5, we have that $\tilde{J}^+$ satisfies the (PS) condition. Since $u_1 < 0$ is a local minimizer of J. So $v = 0$ is a strictly local minimizer of $\tilde{J}^+$. Moreover,

$$\tilde{J}^+(t\varphi_1) \to -\infty, \quad \text{as } t \to +\infty.$$

By the Mountain Pass Lemma and the Maximum Principle, $\tilde{J}^+$ has a critical point $v^+ > 0$ and it is a critical points of $\tilde{J}$. In the same way, $\tilde{J}$ has a critical point $v^- < 0$. So $u_1^\pm = u_1 + v^\pm$ are two critical point of J, and $u_1^- < u_1 < u_1^+$. Thus applying Corollary 8.5 in [11] and Theorem 1 in [12], the critical groups of J at $u_1^\pm$ are

$$C_q(J, u_1^\pm) \cong C_q(\tilde{J}^\pm, u_1^\pm) \cong \delta_{q,1}Z. \tag{20}$$

Starting with u_2 we can show that J has two more critical points $u_2^\pm$ such that $u_2^- \leq u_2 \leq u_2^+$ and

$$C_q(J, u_2^\pm) \cong \delta_{q,1}Z. \tag{21}$$

Comparing the critical groups (2.17)-(2.21), we have see that u_0, u_1, u_2, u_1^-, u_2^+ are five different nontrivial critical points of J. ■

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