Approximation of a Reifenberg-flat set by a smooth surface

Guy David

Résumé

On montre que si l'ensemble $E \subset \mathbb{R}^n$ est bien approché, à l'échelle r_0, par des plans de dimension d, il existe une surface lisse Σ_0 de dimension d, qui est proche de E à l'échelle r_0. Quand E est Reifenberg-plat, ceci permet d'appliquer un résultat de G. David et T. Toro [Memoirs of the AMS 215 (2012), 1012], et de montrer que E est l'image de Σ_0 par un homéomorphisme bi-Höldérien de $\mathbb{R}^n$. Si de plus $d = n - 1$ et E est compact et connexe, alors Σ_0 est orientable, et $\mathbb{R}^n \setminus E$ a exactement deux composantes connexes que la construction ci-dessus permet d'approximer par l'intérieur par des domaines lisses.

Abstract

We show that if the set $E \subset \mathbb{R}^n$ is well approximated at the scale r_0 by planes of dimension d, we can find a smooth surface Σ_0 of dimension d which is close to E at the scale r_0. When E is a Reifenberg flat set, this allows us to apply a result of G. David and T. Toro [Memoirs of the AMS 215 (2012), 1012], and get a bi-Hölder homeomorphism of $\mathbb{R}^n$ that sends Σ_0 to E. If in addition $d = n - 1$ and E is compact and connected, then Σ_0 is orientable, and $\mathbb{R}^n \setminus E$ has exactly two connected components, which we can approximate from the inside by smooth domains.

Received by the editors in December 2012 - In revised form in June 2013.
Communicated by J. Mawhin.
2010 *Mathematics Subject Classification* : 28A75, 49Q20.
Key words and phrases : Reifenberg flat sets, Reifenberg topological disk theorem, smooth approximation, orientability.

1 Introduction

The initial purpose of this paper is to prove that if $E \subset \mathbb{R}^n$ is a set which, in every ball of radius r_0, is sufficiently close to a d-dimensional plane for the Hausdorff distance, we can find a smooth surface Σ_0 of the same dimension, which is close to E (again, for the usual Hausdorff distance) at the scale r_0.

When E is a Reifenberg flat set of dimension d (which means that the property above holds at all scales $r \leq r_0$), this allows us to apply one of the main results of [DT], and get a bi-Hölder homeomorphism of the ambient space $\mathbb{R}^n$ that sends Σ_0 to E. In the present context, the difference between the result of [DT] that we want to use and the classical topological disk theorem of [R] is not large, it only resides in the fact that we authorize E to look like a smooth surface at the scale r_0, rather than just a plane.

So the first main result of this paper can be seen a small preparation lemma that slightly weakens the assumptions in [DT].

In the special case of a compact connected Reifenberg-flat set E of codimension 1, we will deduce from this that $\mathbb{R}^n \setminus E$ has exactly two connected components V_1 and V_2, so that one does not need to mention this separation property as an additional assumption, and we will construct smooth domains that approximate V_1 and V_2 from the inside; see Theorem 1.18. Note that this is a result that concerns the whole set E and its position in $\mathbb{R}^n$, unlike the Reifenberg topological disk theorem, which is often stated locally. Note also that in this statement, E is like a topological manifold with no boundary.

Let us now state the results more precisely. Fix integers $0 < d < n$, let E be a (nonempty) closed set in $\mathbb{R}^n$, and define the (bilateral P. Jones) numbers $\gamma(x,r)$, $x \in E$ and $0 < r < +\infty$, by

$$\gamma(x,r) = \inf\big\{d_{x,r}(E,P)\,;\, P \in \mathcal{P}(x)\big\}, \tag{1.1}$$

where $\mathcal{P}(x)$ denotes the set of d-dimensional affine planes that contain x, and

$$\begin{aligned} d_{x,r}(E,P) = \frac{1}{r}\sup\big\{\, \mathrm{dist}(y,P)\,;\, y \in E \cap B(x,r)\big\} \\ + \frac{1}{r}\sup\big\{\, \mathrm{dist}(y,E)\,;\, y \in P \cap B(x,r)\big\} \end{aligned} \tag{1.2}$$

is a normalized local Hausdorff distance from E to P.

We shall assume that there is a radius $r_0 > 0$ such that

$$\gamma(x,r_0) \leq \varepsilon \ \text{ for } x \in E, \tag{1.3}$$

and prove that if ε is small enough, depending only on n and d, there is a smooth d-dimensional surface $\Sigma_0 \subset \mathbb{R}^n$ with no boundary, such that

$$\mathrm{dist}(x,\Sigma_0) \leq C_0\varepsilon r_0 \ \text{ for } x \in E \tag{1.4}$$

and

$$\mathrm{dist}(x,E) \leq C_0\varepsilon r_0 \ \text{ for } x \in \Sigma_0, \tag{1.5}$$

where C_0 depends only on n and d. More precisely, we will show that there is a $\Lambda > 0$, that depends only on n and d, such that for each $y \in \Sigma_0$,

(1.6) Σ_0 coincides with a smooth $\Lambda\varepsilon$-Lipschitz graph in $B(y, 10^{-2}r_0)$.

That is, we can find a d-plane P_y through y and a $\Lambda\varepsilon$-Lipschitz mapping $F_y : P_y \to P_y^\perp$ such that, if

$$\mathcal{G}(F_y) = \{w + F_y(w)\,;\, w \in P_y\} \tag{1.7}$$

denotes the graph of F_y, then

$$\Sigma_0 \cap B(y, 10^{-2}r_0) = \mathcal{G}(F_y) \cap B(y, 10^{-2}r_0). \tag{1.8}$$

In addition, the F_y are smooth, and there exist constants Λ_k, $k \geq 1$, such that

$$||D^k F_y||_\infty \leq \Lambda_k \varepsilon r_0^{1-k}. \tag{1.9}$$

(For $k = 1$ we already knew this, with $\Lambda_1 = \Lambda$.) Let us summarize all this officially.

Theorem 1.10. *There exist constants $\varepsilon_0 > 0$, $C_0 \geq 1$, and Λ_k, $k \geq 1$, that depend only on n and d, such that if $E \subset \mathbb{R}^n$ is a nonempty closed set such that (1.3) holds for some $r_0 > 0$ and some $\varepsilon \in (0, \varepsilon_0)$, then we can find a smooth d-dimensional surface Σ_0 with the properties (1.4)-(1.9).*

Notice that we do not require E to be flat at all scales smaller than r_0 (as will be done for the next two results). By smooth surface, we mean an embedded submanifold, but (1.6) is a little more precise quantitatively.

Theorem 1.10 is designed so that we can apply Theorem 12.1 in [DT], and get the following statement, where we decided to work with $r_0 = 1$ for convenience.

Corollary 1.11. *There exist constants $\varepsilon_1 \leq \varepsilon_0$ and $C_1 > 1$, that depend only on n and d, such that if $E \subset \mathbb{R}^n$ is a nonempty closed set for which*

$$\gamma(x, r) \leq \varepsilon \ \text{ for } x \in E \text{ and } 0 < r \leq 1, \tag{1.12}$$

and if Σ_0 is the smooth surface provided by Theorem 1.10, then there is a bijective mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(x) = x \ \text{ when } \operatorname{dist}(x, E) \geq 10^{-1}, \tag{1.13}$$

$$|g(x) - x| \leq C_1\varepsilon \ \text{ for } x \in \mathbb{R}^n, \tag{1.14}$$

$$C_1^{-1}|x' - x|^{1+C_1\varepsilon} \leq |g(x) - g(x')| \leq C_1|x' - x|^{1-C_1\varepsilon} \tag{1.15}$$

for $x, x' \in \mathbb{R}^n$ such that $|x' - x| \leq 1$, and

$$g(\Sigma_0) = E. \tag{1.16}$$

Let us check this. We want to apply Theorem 12.1 in [DT] to the set $E' = AE$, where we choose $A = 2 \cdot 10^4$, and with the open set

$$U = \{x \in \mathbb{R}^n\,; \mathrm{dist}(x, E') > 3\}. \tag{1.17}$$

The assumption (12.1) of [DT], relative to the smooth set $\Sigma_0' = A\Sigma_0$, is satisfied because of (1.6)-(1.9) (although only with the constant $C\varepsilon$, which does not matter). The proximity condition (12.3) follows from (1.4) and (1.5), and so Theorem 12.1 in [DT] gives a mapping g' that satisfies (12.4)-(12.8) in [DT].

Here we take $g(x) = A^{-1}g'(Ax)$, (1.13) holds because (12.4) in [DT] says that $g'(x) = x$ when $\mathrm{dist}(x, U) \geq 13$, while (1.14) and (1.15) easily follow from (12.6) and (12.7) in [DT]. Finally, (12.8) in [DT] says that $E' \cap U = g'(\Sigma_0' \cap U)$. But $E' = E' \cap U$ by (1.17), and $\Sigma_0' \subset U$ by (1.5), so in fact $E' = g'(\Sigma_0')$ and $E = g(\Sigma_0)$, as needed. So Corollary 1.11 follows from Theorem 1.10 and Theorem 12.1 in [DT]. ∎

We shall also apply Theorem 1.10 in the special case of connected sets of codimension 1, and obtain smooth approximations from the inside for each of the two complementary components.

Theorem 1.18. *Suppose $d = n - 1$. There exist constants $\varepsilon_2 \leq \varepsilon_0$, $C_2 > 1$, and Λ_k', $k \geq 1$, that depend only on n, with the following property. If $E \subset \mathbb{R}^n$ is a nonempty compact connected set such that for some choice of $r_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$\gamma(x, r) \leq \varepsilon \ \text{ for } x \in E \text{ and } 0 < r \leq r_0, \tag{1.19}$$

then

$$\mathbb{R}^n \setminus E \text{ has exactly two connected components,} \tag{1.20}$$

which we shall denote by V_1 and V_2. In addition, for $0 < r < r_0$, there exist smooth connected domains $W_{r,1} \subset V_1$ and $W_{r,2} \subset V_2$, with the following properties:

$$\mathbb{R}^n \setminus [W_{r,1} \cup W_{r,2}] \subset \{x \in \mathbb{R}^n\,; \mathrm{dist}(x, E) \leq 2C_2\varepsilon r\} \tag{1.21}$$

and, for $j = 1, 2$,

$$C_2\varepsilon r \leq \mathrm{dist}(x, E) \leq 2C_2\varepsilon r \ \text{ for } x \in \partial W_{r,j} \tag{1.22}$$

and we have a local Lipschitz description of $\partial W_{r,j}$ as in (1.6)-(1.9). That is, for each $y \in \partial W_{r,j}$ we can find a hyperplane $P_{r,j,y}$ through y and a Lipschitz mapping $F_{r,j,y} : P_{r,j,y} \to P_{r,j,y}^{\perp}$ such that

$$\partial W_{r,j} \text{ coincides with the graph of } F_{r,j,y} \text{ inside } B(y, 10^{-3}r) \tag{1.23}$$

and, for $k \geq 1$,

$$||D^k F_{r,j,y}||_\infty \leq \Lambda_k' \varepsilon r^{1-k}. \tag{1.24}$$

Finally, $W_{r,j} \cap B(y, 10^{-3}r)$ lies on only one side of the graph of $F_{r,j,y}$.

Notice that (1.20) is part of the conclusion: as we shall see, the Reifenberg-flatness of E implies that it separates, and the fact that there are at most two complementary components follows from its connectedness.

When E is a little more regular than just Reifenberg-flat, we can expect to find other, more precise ways of saying that the $W_{r,j}$ converge to V_j, but hopefully the main ingredient will be (1.22)-(1.24) and the regularity of E. For instance, if E is also Ahlfors-regular of dimension $n-1$, so are the $\partial W_{r,j}$, with uniform bounds, and the characteristic function of $W_{r,j}$ converges, locally in $BV(\mathbb{R}^n)$, to the characteristic function of V_j.

Compared with Corollary 1.11, what is new here is the existence of the good domains $W_{r,j}$ that approximate the two components of $\mathbb{R}^n \setminus E$. Indeed, as we shall see soon, we can even get (1.20) from Corollary 1.11 and the orientability of the smooth codimension 1 submanifold Σ_0 given by Theorem 1.10. These good domains could be useful for some PDE problems on the domains bounded by E, but this is not the object of the present paper. Observe also that we do not get extra information on the topology of E itself; by Corollary 1.11, it is the same as the topology of the smooth approximation Σ_0 given by Theorem 1.10, which could be a sphere, a torus, or many other things.

Let us give a short proof of (1.20) that relies on Corollary 1.11. We shall see in Section 3 another proof that does not use the parameterization from [DT], but only Theorem 1.10 and some small amount of tracking at different scales.

Let E be as in the statement. By scale invariance and without loss of generality, we may assume that $r_0 = 1$, and this way we will apply Corollary 1.11 without rescaling. Let Σ_0 and g be as in Theorem 1.10 and then Corollary 1.11. We know that Σ_0 is a smooth hypersurface without boundary, and it is compact by (1.5). We shall soon check that

$$\Sigma_0 \text{ is connected;} \tag{1.25}$$

let us conclude from here. By [AH], page 440 (for nonsmooth manifolds), or rather [S] (a shorter and intuitive argument for the smooth case, using transversality), Σ_0 is orientable and $\mathbb{R}^n \setminus \Sigma_0$ has exactly two connected components, which we denote by U_1 and U_2. Thus this result can be seen as a partial generalization of the Jordan curve theorem in $\mathbb{R}^n$. Recall that g is smooth, bijective, and equal to the identity near infinity; then we get that $\mathbb{R}^n \setminus E = g(\mathbb{R}^n \setminus \Sigma_0)$ has two components, namely the $g(U_j)$, $j = 1, 2$.

We still need to check that Σ_0 is connected. Let $a_1, a_2 \in \Sigma_0$ be given. By (1.5), we can find $b_1, b_2 \in E$ such that $|b_i - a_i| \leq C\varepsilon$. Because E is connected, we can find a chain of points $w_i \in E$, $0 \leq i \leq m$, with $w_0 = b_1$, $w_m = b_2$, and $|w_i - w_{i-1}| \leq C\varepsilon$ for $1 \leq i \leq m$. (Otherwise, the set of points of E that can be connected to b_1 by such a chain, which is open and closed in E, would contain b_1 but not b_2.) By (1.4), we can find $y_i \in \Sigma_0$ such that $|y_i - w_i| \leq C\varepsilon$. By definition of b_1 and b_2, we can take $y_0 = a_1$ and $y_m = a_2$. Finally, by (1.6)-(1.8), applied to the points y_i, we get that for $1 \leq i \leq m$, there is a curve in Σ_0 that connects y_{i-1} to y_i. Thus Σ_0 is connected, and (1.20) follows. ∎

Remark 1.26. Instead of assuming that E is connected, it is enough to assume that it is $\frac{r_0}{20}$-connected. That is, that any two points b_1, b_2 in E can be connected by a chain $\{w_i\}$ in E as above, with $|w_i - w_{i-1}| \leq \frac{r_0}{20}$ for $1 \leq i \leq m$. Indeed, our proof of (1.20) shows that Σ_0 is connected, and then $E = g(\Sigma_0)$ is connected too. We can even prove that E is connected without using Corollary 1.11; see the proof of (3.1) below.

If we do not suppose that E is connected (or $\frac{r_0}{20}$-connected) we only get that $\mathbb{R}^n \setminus E$ has at least two components, but we still have the approximation result for each component. This is not hard: let V be any component of $\mathbb{R}^n \setminus E$, pick any $x_0 \in E$ such that $\mathrm{dist}(x_0, V) < r_0/100$, and apply the result to the set E' of points $x \in E$ that can be $\frac{1}{20}$-connected to x_0. By Theorem 1.18, E' separates $\mathbb{R}^n$ into two components, and our approximation result holds for those. It is easy to see that one of them contains V, and this gives a good description of V near x_0.

Remark 1.27. We do not really need to assume that E is bounded either; Theorem 1.18 is still valid when E is closed (instead of compact) and we keep the same other assumptions. The only place where we need to change the proof is when we apply [S], to show that smooth compact sets are orientable and have exactly two complementary components, but the transversality argument still works for the surfaces that we construct. It is probable that the results of this paper can also be extended to unbounded sets, even when the scale r_0 under which (1.19) is assumed to hold depends gently on on x. But we shall not try to do this.

Remark 1.28. In [DT] the authors also consider situations where instead of requiring the bilateral control (1.12), one merely assumes that the points of E lie close to d-planes $P(x,r)$, with some control on how fast they depend on (x,r). Here we can try to construct Σ_0 with similar data; our construction will only give a smooth surface Σ_0, but with a boundary, and we shall not try to see whether it is contained in a smooth surface without boundary. See Remark 2.31.

Theorem 1.10 will be proved in Section 2, by a simple construction where we start from a net of points and use a covering to fill the holes in a finite number of steps.

For Theorem 1.18 (proved in Section 3), we will first apply Theorem 1.10 at all intermediate scales $0 < r \leq r_0$ to get central surfaces Σ_r, then move in the normal direction (in both directions, and until we are far enough from E) to get two surfaces $\Sigma_{r,1}$ and $\Sigma_{r,2}$, and show that these surfaces bound domains $W_{r,1}$ and $W_{r,2}$ that satisfy the desired properties. The control of the connected components, if not surprising, will be the most unpleasant part of the argument. It will be taken care of by a top-down compatibility argument.

The author wishes to thank T. D. Luu for discussions about Theorem 1.10 (we thought at some time that a result like this would be needed for his PhD thesis, but it turned out not to be the case), T. Toro for suggesting that Theorem 1.10 be used to approximate Reifenberg-flat domains from inside, and A. Lemenant and A. Chambolle, in particular for discussions about (1.20) and orientability. He wishes to acknowledge support from the Institut Universitaire de France, and the grant ANR-12-BS01-0014-01 ”GEOMETRYA”.

2 A proof of Theorem 1.10

In this section we prove Theorem 1.10. Since the statement is invariant under dilations, it will be enough to prove the theorem when $r_0 = 1$.

Let E be as in the theorem, with $r_0 = 1$, set $a = \frac{1}{32}$, and choose a maximal set $X \subset E$ such that $|x - x'| \geq a$ for $x, x' \in X$ such that $x' \neq x$. Thus

$$\text{dist}(x, X) \leq a \ \text{ for } x \in E, \tag{2.1}$$

because otherwise we could add x to X. Next decompose X as a disjoint union

$$X = \bigcup_{1 \leq j \leq N} X_j, \tag{2.2}$$

where for each j,

$$|x - x'| \geq 16a = \frac{1}{2} \ \text{ when } x \text{ and } x' \text{ lie in } X_j \text{ and } x \neq x'. \tag{2.3}$$

Let us check that we can do this with an N that depends only on n. Define X_j by induction, to be a maximal set contained in $X \setminus \cup_{i<j} X_i$ and with the property (2.3); it is easy to see that we can stop as soon as j is larger than the maximal number of points in a ball of radius $16a$ that lie at mutual distances at least a. This number is in turn estimated by saying that the balls of radius $a/2$ centered at these points are all disjoint and contained in a ball of radius $17a$, which yields $N \leq 34^n$ by comparing Lebesgue measures.

For each $x \in X$, use (1.3) to find an affine d-plane P_x through x such that

$$d_{x,1}(E, P_x) \leq \varepsilon \tag{2.4}$$

(see the definitions (1.1) and (1.2)). Then denote by $\pi_x : \mathbb{R}^n \to P_x$ the orthogonal projection on P_x, and by $\pi_x^\perp$ the orthogonal projection on the vector space $P_x^\perp$ of dimension $n - d$ which is orthogonal to P_x. Let us check that

$$d_{x,1/4}(P_x, P_y) \leq 8\varepsilon \ \text{ for } x, y \in X \text{ such that } |x - y| \leq \frac{1}{2}. \tag{2.5}$$

If $z \in P_x \cap B(x, 1/4)$, we can use (2.4) to find $w \in E$ such that $|w - z| \leq \varepsilon$; then $w \in E \cap B(y, 1)$ and (2.4) for y says that we can find $z' \in P_y$ such that $|z' - w| \leq \varepsilon$ and hence $|z' - z| \leq 2\varepsilon$. By a similar argument, for each $z' \in P_y \cap B(x, 1/4)$ we can find $z \in P_x$ such that $|z - z'| \leq 2\varepsilon$, and (2.5) follows.

Thus in (2.5), P_x and P_y make a small angle; this will be useful because we want small Lipschitz graphs over P_x to be small Lipschitz graphs over P_y as well.

We shall now construct a nondecreasing sequence of sets S_j, $0 \leq j \leq N$. We start with $S_0 = X$, and our final set S_N will be a good choice of Σ_0, except for the fact that we shall not immediately take care of the many derivatives in (1.9). Notice that X gives the general position of Σ_0, so our problem will essentially consist in completing S_0 into a smooth surface that contains it; we shall only use the set E marginally, to prove that Σ_0 has no boundary.

We shall construct the S_j by induction, with the property that for each $x \in X$, there is an $A_j\varepsilon$-Lipschitz mapping $F_{j,x} : P_x \to P_x^\perp$ such that

$$S_j \cap B(x, 4a) \subset \mathcal{G}(F_{j,x}), \tag{2.6}$$

where $\mathcal{G}(F_{j,x})$ denotes the graph of $F_{j,x}$ over P_x, defined as in (1.7). The constants A_j will be chosen larger and larger, but since we can take ε as small as we want, the $A_j\varepsilon$ will stay small.

In order to prove (2.6), we shall first check that for each $x \in X$,

$$|\pi_x^\perp(y) - \pi_x^\perp(z)| \leq A_j'\varepsilon|\pi_x(y) - \pi_x(z)| \text{ for } y, z \in S_j \cap B(x, 4a) \tag{2.7}$$

for some constant A_j' (that will depend on A_{j-1} if $j \geq 1$). As soon as we have (2.7), we observe that π_x is injective on $S_j \cap B(x, 4a)$, so can define a function $F_{j,x}$ from $H = \pi_x(S_j \cap B(x, 4a))$ to $P_x^\perp$, by the relation $F_{j,x}(\pi_x(y)) = \pi_x^\perp(y)$ for $y \in S_j \cap B(x, 4a)$. In addition, (2.7) says that $F_{j,x}$ is $A_j'\varepsilon$-Lipschitz on H, and $S_j \cap B(x, 4a)$ is its graph.

We then use the Whitney extension theorem (see [St], page 170) to extend $F_{j,x}$ into an $A_j\varepsilon$-Lipschitz function defined on P_x, and we get (2.6). Since we allow ourselves to take A_j larger than A_j', we don't need to use Kirszbraun's theorem (see 2.10.43 in [F]) and the construction of the extension is simpler. Let us now check that (2.7) holds for $j = 0$. Recall that we took $S_0 = X$. Let $y, z \in S_0 \cap B(x, 4a) = X \cap B(x, 4a)$ be given. Then

$$|\pi_x^\perp(y) - \pi_x^\perp(x)| = \mathrm{dist}(y, P_x) \leq \varepsilon \tag{2.8}$$

because P_x goes through x, and then because $y \in X \subset E$ and by (2.4) (also see the definition (1.2)). Similarly, $|\pi_x^\perp(z) - \pi_x^\perp(x)| \leq \varepsilon$. If $y = z$, the inequality in (2.7) is obvious. Otherwise, $|y - z| \geq a$ by definition of X, and

$$|\pi_x^\perp(y) - \pi_x^\perp(z)| \leq 2\varepsilon \leq 2a^{-1}\varepsilon|y - z|; \tag{2.9}$$

then $|\pi_x(y) - \pi_x(z)| \geq |y - z| - |\pi_x^\perp(y) - \pi_x^\perp(z)| \geq |y - z|/2$ and (2.7) holds with $A_0' = 4a^{-1} = 128$. So there are Lipschitz functions $F_{0,x}$ such that (2.6) holds for $j = 0$.

Next assume that $0 \leq j < N$ and that we constructed S_j and the $F_{j,x}$ with the property (2.6). We take

$$S_{j+1} = S_j \cup \bigcup_{w \in X_{j+1}} \mathcal{G}(F_{j,w}) \cap B(w, 3a). \tag{2.10}$$

Let us now prove (2.7) for $j + 1$. Let $x \in X$ and $y, z \in S_{j+1} \cap B(x, 4a)$ be given; we need to prove that

$$|\pi_x^\perp(y) - \pi_x^\perp(z)| \leq A_{j+1}'\varepsilon|\pi_x(y) - \pi_x(z)|. \tag{2.11}$$

If y, z both lie on S_j, (2.11) simply follows from (2.7) for j. So we may assume that one of the two points (say y for definiteness) lies in $\mathcal{G}(F_{j,w}) \cap B(w, 3a)$ for some $w \in X_{j+1}$.

Case 1. We first assume that $z \in \mathcal{G}(F_{j,w})$. Then

$$|\pi_w^{\perp}(y) - \pi_w^{\perp}(z)| \leq A_j\varepsilon|\pi_w(y) - \pi_w(z)|, \tag{2.12}$$

just because $F_{j,w}$ is $A_j\varepsilon$-Lipschitz. This is not exactly (2.11), because we project on different planes, but the difference will be small. Indeed, $|w - x| \leq |w - y| + |y - x| \leq 3a + 4a < 1/2$, so (2.5) says that

$$d_{x,1/4}(P_x, P_w) \leq 10\varepsilon. \tag{2.13}$$

The reader will not be surprised to learn that (2.11) follows from (2.12) and (2.13), but let us check this anyway. Denote by $\widetilde{P}_x$ and $\widetilde{P}_w$ the vector planes parallel to P_x and P_w, and by $\widetilde{\pi}_x$ (respectively $\widetilde{\pi}_w$) the orthogonal projection on $\widetilde{P}_x$ (respectively $\widetilde{P}_w$); we want to check that

$$||\widetilde{\pi}_x - \widetilde{\pi}_w|| \leq 100\varepsilon. \tag{2.14}$$

Let $v \in \mathbb{R}^n$ be given; we want to estimate $|\widetilde{\pi}_x(v) - \widetilde{\pi}_w(v)|$, and we may as well assume that v is a unit vector. We start with the case when $v \in \widetilde{P}_x$, and we first check that

$$\mathrm{dist}(v, \widetilde{P}_w) \leq 25\varepsilon. \tag{2.15}$$

Observe that both x and $x + v/5$ lie in $P_x \cap B(x, 1/4)$, so by (2.13) we can find x' and x'' in P_w such that $|x' - x| \leq 10\varepsilon/4$ and $|x'' - x - v/5| \leq 10\varepsilon/4$; then $v' = 5(x'' - x')$ lies in $\widetilde{P}_w$, and $|v' - v| = 5|(x'' - x - v/5) + (x - x')| \leq 25\varepsilon$, as needed for (2.15).

Because of (2.15), we get that $|\widetilde{\pi}_x(v) - \widetilde{\pi}_w(v)| = |v - \widetilde{\pi}_w(v)| = \mathrm{dist}(v, \widetilde{P}_w) \leq 25\varepsilon$ for every unit vector $v \in \widetilde{P}_x$.

Now suppose v is a unit vector in $P_x^{\perp}$. We set $\xi = \widetilde{\pi}_w(v)$, and check that

$$\mathrm{dist}(\xi, \widetilde{P}_x) \leq 25\varepsilon. \tag{2.16}$$

Pick $x' \in P_w$ such that $|x' - x| \leq 10\varepsilon/4$ (use (2.13) as before), and, since $x' + \xi/5 \in P_w \cap B(x, 1/4)$, use (2.13) to find $y \in P_x$ such that $|y - x' - \xi/5| \leq 10\varepsilon/4$. Recall that $x \in P_x$; then $5(y - x) \in \widetilde{P}_x$, and $|5(y - x) - \xi| \leq 5[|y - x' - \xi/5| + |x' - x|] \leq 25\varepsilon$, which proves (2.16). Thus

$$|\xi - \widetilde{\pi}_x(\xi)| = \mathrm{dist}(\xi, \widetilde{P}_x) \leq 25\varepsilon. \tag{2.17}$$

Also, for each unit vector $e \in \widetilde{P}_x$, use (2.15) to find $e' \in \widetilde{P}_w$, with $|e' - e| \leq 25\varepsilon$, and observe that

$$\begin{aligned} |\langle e, \xi\rangle| &\leq |\langle e', \xi\rangle| + |e' - e||\xi| \leq |\langle e', \xi\rangle| + 25\varepsilon = |\langle e', v\rangle| + 25\varepsilon \\ &\leq |\langle e, v\rangle| + |e' - e| + 25\varepsilon \leq |\langle e, v\rangle| + 50\varepsilon = 50\varepsilon \end{aligned} \tag{2.18}$$

because $\xi - v \perp e'$ by definition of $\xi = \widetilde{\pi}_w(v)$, and $v \in P_x^{\perp}$. By (2.18), $|\widetilde{\pi}_x(\xi)| \leq 50\varepsilon$ and, adding up with (2.17), $|\xi| \leq 75\varepsilon$. Then $|\widetilde{\pi}_x(v) - \widetilde{\pi}_w(v)| = |\widetilde{\pi}_w(v)| = |\xi| \leq 75\varepsilon$ for every unit vector $v \in P_x^{\perp}$. Since we also have $|\widetilde{\pi}_x(v) - \widetilde{\pi}_w(v)| \leq 25\varepsilon$

for every unit vector $v \in \widetilde{P}_x$, we get that $|\widetilde{\pi}_x(v) - \widetilde{\pi}_w(v)| \leq 100\varepsilon|v|$ for every $v \in \mathbb{R}^n$. That is, (2.14) holds.

Return to our y and $z \in S_{j+1} \cap B(x, 4a)$, recall that $\pi_x^\perp$ denotes the orthogonal projection on the vector space $P_x^\perp$, define $\pi_w^\perp$ similarly, and observe that

$$\begin{aligned} |(\pi_x^\perp(y) - \pi_x^\perp(z)) - \pi_w^\perp(y) - \pi_z^\perp(z))| &= |\pi_x^\perp(y-z) - \pi_w^\perp(y-z)| \\ &= |\widetilde{\pi}_x(y-z) - \widetilde{\pi}_w(y-z)| \\ &\leq ||\widetilde{\pi}_x - \widetilde{\pi}_w||\,|y-z| \leq 100\varepsilon|y-z| \end{aligned} \tag{2.19}$$

because $\pi_x^\perp + \widetilde{\pi}_x = I = \pi_w^\perp + \widetilde{\pi}_w$. Now (2.11) follows from (2.12) and (2.19), as soon as we take $A'_{j+1} \geq A_j + 100$. This completes our proof of (2.11) when $z \in \mathcal{G}(F_{j,w})$.

Case 2. We still need to check (2.11) when $z \notin \mathcal{G}(F_{j,w})$. Notice that $z \in \mathcal{G}(F_{j,w'}) \cap B(w', 3a)$ for some other $w' \in X_{j+1}$ is impossible, because we would get that

$$|w - w'| \leq |w-y| + |y-x| + |x-z| + |z-w'| \leq 3a + 4a + 4a + 3a < 16a, \tag{2.20}$$

which is forbidden by (2.3). By (2.10) and because $z \notin \mathcal{G}(F_{j,w})$, $z \in S_j$. In addition, by (2.6) and because $z \notin \mathcal{G}(F_{j,w})$, z lies out of $B(w, 4a)$, and so $|z-y| \geq |z-w| - |w-y| \geq 4a - |w-y| \geq a$ (recall from the line below (2.11) that $y \in \mathcal{G}(F_{j,w}) \cap B(w, 3a)$). If we prove that

$$|\pi_x^\perp(y) - \pi_x^\perp(z)| \leq C\varepsilon \tag{2.21}$$

for some C that depends on n and A'_j, (2.11) will follow as soon because (2.21) also implies that $|\pi_x(y) - \pi_x(z)| \geq a/2$. Observe that

$$|\pi_x^\perp(x) - \pi_x^\perp(z)| \leq 4aA'_j\varepsilon \tag{2.22}$$

by (2.7), and because $z \in B(x, 4a)$ by definition and we just checked that $z \in S_j$. We are left with

$$\begin{aligned} |\pi_x^\perp(y) - \pi_x^\perp(x)| &\leq |\pi_x^\perp(y) - \pi_x^\perp(w)| + |\pi_x^\perp(w) - \pi_x^\perp(x)| \\ &= |\pi_x^\perp(y) - \pi_x^\perp(w)| + \operatorname{dist}(w, P_x) \\ &\leq |\pi_x^\perp(y) - \pi_x^\perp(w)| + \varepsilon \end{aligned} \tag{2.23}$$

by (2.4) for x, and because $|w-x| \leq |w-y| + |y-x| \leq 7a < 1/2$. Let us again use (2.14) (its proof is valid also in case 2; in fact neither the statement nor the proof involves z); this yields

$$\begin{aligned} |\pi_x^\perp(y) - \pi_x^\perp(w)| = |\pi_x^\perp(y-w)| &\leq |\pi_w^\perp(y-w)| + |y-w|\,||\pi_x^\perp - \pi_w^\perp|| \\ &= |\pi_w^\perp(y-w)| + |y-w|\,||\pi_x - \pi_w|| \\ &\leq |\pi_w^\perp(y-w)| + 100\varepsilon|y-w| \end{aligned} \tag{2.24}$$

because $\pi_x^\perp$ is a linear projection and $\pi_x^\perp + \widetilde{\pi}_x = I = \pi_w^\perp + \widetilde{\pi}_w$. Now recall that $y \in \mathcal{G}(F_{j,w}) \cap B(w, 3a)$ (see below (2.11)), and so does w (because $w \in X = S_0 \subset S_j$, and by (2.6)); then

$$\begin{aligned} |\pi_w^\perp(y-w)| = |\pi_w^\perp(y) - \pi_w^\perp(w)| &= |F_{j,w}(\pi_w(y)) - F_{j,w}(\pi_w(w))| \\ &\leq A_j\varepsilon|\pi_w(y) - \pi_w(w)| \leq A_j\varepsilon|y-w| \leq A_j\varepsilon \end{aligned} \tag{2.25}$$

because $F_{j,w}$ is $A_j\varepsilon$-Lipschitz. Altogether, $|\pi_x^\perp(y) - \pi_x^\perp(z)| \leq 4aA'_j\varepsilon + \varepsilon + 300a\varepsilon + A_j\varepsilon$, by (2.22)-(2.25), which proves (2.21) and then (2.11) in our second case.

This completes our proof of (2.7) for $j+1$. Thus we have the local Lipschitz description of S_{j+1}, with (2.6), and this completes our construction of sets S_j by induction.

Let us now check that for $1 \leq j \leq N$ and $x \in X_j$,

$$S_N \cap B(x, 2a) = \mathcal{G}(F_{j-1,x}) \cap B(x, 2a). \tag{2.26}$$

Set $\mathcal{G} = \mathcal{G}(F_{j-1,x})$. By (2.10), $\mathcal{G} \cap B(x, 3a) \subset S_j \subset S_N$ so we get a first inclusion. Notice also that since the Lipschitz constant is small and $\mathcal{G}$ goes through x,

$$\pi_x(\mathcal{G} \cap B(x, 3a)) \text{ contains } P_x \cap B(x, 2a). \tag{2.27}$$

Now let $z \in S_N \cap B(x, 2a)$ be given. Then $\pi_x(z) \in P_x \cap B(x, 2a)$; by (2.27) we can find $w \in \mathcal{G} \cap B(x, 3a)$ such that $\pi_x(w) = \pi_x(z)$. Then $w \in S_N \cap B(x, 3a)$. But (2.6) says that $S_N \cap B(x, 4a)$ is contained in a Lipschitz graph, so π_x is injective on $S_N \cap B(x, 4a)$, hence $w = z$ and $z \in \mathcal{G}$. So $S_N \cap B(x, 2a) \subset \mathcal{G}$, as needed for (2.26).

Now we check the properties (1.4) and (1.5) for S_N. First let $y \in E$ be given. By (2.1), we can find $x \in X$ such that $|x - y| \leq a$. Let j be such that $x \in X_j$, and set $w = \pi_x(y)$ and $z = w + F_{j-1,x}(w) \in \mathcal{G}(F_{j-1,x})$. We know from (2.26) that $z \in S_N$, and so

$$\begin{aligned} \operatorname{dist}(y, S_N) &\leq |y - z| = |\pi_x^\perp(y) - \pi_x^\perp(z)| = |\pi_x^\perp(y) - F_{j-1,x}(w)| \\ &\leq |\pi_x^\perp(y) - \pi_x^\perp(x)| + |\pi_x^\perp(x) - F_{j-1,x}(w)| \\ &= \operatorname{dist}(y, P_x) + |\pi_x^\perp(x) - F_{j-1,x}(w)| \end{aligned} \tag{2.28}$$

because $\pi_x(y) = \pi_x(z) = w$ and then $x \in P_x$. But $x \in X \subset S_N$, so $x \in \mathcal{G}(F_{j-1,x})$ by (2.26) and hence $F_{j-1,x}(x) = \pi_x^\perp(x)$. Thus

$$|\pi_x^\perp(x) - F_{j-1,x}(w)| = |F_{j-1,x}(x) - F_{j-1,x}(w)| \leq A_{j-1}\varepsilon|x - w| \leq C\varepsilon \tag{2.29}$$

because $F_{j-1,x}$ is $A_{j-1}\varepsilon$-Lipschitz. Since $\operatorname{dist}(y, P_x) \leq \varepsilon$ by (2.4), we deduce from (2.28) that $\operatorname{dist}(y, S_N) \leq C\varepsilon$. So (1.4) holds.

For (1.5), let $y \in S_N$ be given, and let $j(y)$ be the first index such that $y \in S_j$; if $j(y) = 0$, y lies in $X \subset E$, and $\operatorname{dist}(y, E) = 0$. Otherwise, (2.10) says that $y \in \mathcal{G}(F_{j-1,x}) \cap B(x, 3a)$ for some $x \in X_j$. Set $w = \pi_x(y)$; then $w \in P_x \cap B(x, 3a)$ and hence $\operatorname{dist}(w, E) \leq \varepsilon$ by (2.4). Now

$$\begin{aligned} \operatorname{dist}(y, E) &\leq |y - w| + \operatorname{dist}(w, E) \leq |y - w| + \varepsilon = |\pi_x^\perp(y - w)| = |\pi_x^\perp(y - x)| \\ &= |F_{j-1,x}(w) - F_{j-1,x}(x)| + \varepsilon \leq A_{j-1}\varepsilon|x - w| + \varepsilon \leq C\varepsilon, \end{aligned} \tag{2.30}$$

where we used the facts that $w = \pi_x(y)$, that $\pi_x^\perp(w - x) = 0$ because x and w lie in P_x, that $y = w + F_{j-1,x}(w)$ and $x = x + F_{j-1,x}(w)$ (both points lie in $\mathcal{G}(F_{j-1,x})$; also see the definition (1.7)), and that $F_{j-1,x}$ is $A_{j-1}\varepsilon$-Lipschitz. This proves (1.5) for S_N.

The local Lipschitz description (1.7)-(1.8) for S_N (but with no extra smoothness yet) will easily follow from (2.26). Indeed, if $y \in S_N$, (1.5) and (2.1) imply that $\mathrm{dist}(y, X) \leq a + \varepsilon$; we choose $j \in [1, N]$ and $x \in X_j$ so that $|x - y| \leq a + \varepsilon$ and try to take $P_y = P_x$ and $F_y = F_{j-1,x}$; then (1.8) follows from (2.26) because $B(y, 10^{-2}) \subset B(x, 2a)$. (Recall that $a = \frac{1}{32}$.) We also get (1.9) for $k = 1$ (i.e., $F_y = F_{j-1,x}$ is $C\varepsilon$-Lipschitz). Now P_x does not necessarily pass through y, but we can just translate it, modify F_y accordingly, and get an acceptable pair (P_y, F_y) that works.

Our proof is not complete yet, because we want our smooth surface Σ_0 to satisfy (1.9) for all k. If we were only interested in a finite number of derivatives, we could modify the argument above, use Whitney extensions with functions of class C^k, and conclude as before. But since we decided to require an infinite number of derivatives, the simplest way seems to start from S_N and smooth it out. Notice that if we do not move it by more than $C\varepsilon$, (1.4) and (1.5) stay true (with larger constants), so the reader should not expect trouble here.

Let us rapidly describe the argument. The details would be standard and boring, so we skip them. We shall use the same sets X_j, $1 \leq j \leq N$, as before, and define sets $T_0 = S_N, T_1, \dots$, and finally $\Sigma_0 = T_N$, by induction. For $j = 1, \dots N$, we obtain T_j from T_{j-1} by smoothing T_j in the balls $B(x, \frac{3a}{2})$, $x \in X_j$. Since these balls are far from each other, we can do this independently in all these balls. We start from a description of T_{j-1} as a Lipschitz graph near $B(x, \frac{3a}{2})$ (which we get from an induction assumption), use a convolution with a smooth function with a support of small diameter to smooth out T_{j-1} in $B(x, \frac{3a}{2})$, interpolate smoothly in a small ring near $\partial B(x, \frac{3a}{2})$, and get the next T_j. Each time we get a Lipschitz graph with a slightly worse constant, but this does not matter. The smoothness that has been produced in the previous stages is preserved too, and at the end, $\Sigma_0 = T_N$ is the desired smooth set. This completes our proof of Theorem 1.10. ■

Remark 2.31. In [DS], one also obtains parameterizations of sets that contain E, assuming that for $x \in E$ and $0 < r \leq 1$, we can find a d-dimensional affine plane $P(x, r)$ through x such that

$$\mathrm{dist}(y, P(x, r)) \leq \varepsilon r \quad \text{for } y \in E \cap B(x, r), \tag{2.32}$$

and also that the $P(x, r)$ vary sufficiently slowly. Here we can do something similar, suppose that for $x \in E$, we can find a d-plane P_x through x, such that

$$\mathrm{dist}(y, P_x) \leq \varepsilon \quad \text{for } y \in E \cap B(x, 1), \tag{2.33}$$

and such that (2.5) holds. Then our construction gives a set S_N, with a Lipschitz description near E, as in (2.26) included. We also get (1.4) as before, but of course not (1.5). But slightly more unpleasantly, S_N has a boundary; that is, we only get (2.26) for balls centered on $X \subset E$, but not necessarily for balls centered on points of S_N far from E. For instance, the parts near E may be far from each other and then S_N stops somewhere in between. It is possible that we can connect all the boundary pieces of S_N, and get a larger set Σ_0 which is smooth and without boundary, but this could be unpleasant to do. The author did not find any obvious topological obstruction either.

Remark 2.34. We can probably prove a variant of Theorem 1.10 where, when the set E is unbounded, we only require that $\gamma(x, r(x)) \leq \varepsilon$ for some $r(x)$ that depends sufficiently slowly on $x \in E$, instead of systematically demanding that $r(x) = r_0$ (a fixed number). We shall not give any detail.

3 Approximation from the inside of Reifenberg-flat domains

In this section we apply Theorem 1.10 to prove Theorem 1.18. Let E be as in the statement. For each $r \in (0, r_0]$, we can apply Theorem 1.10 (with r_0 replaced with r), and we get a smooth surface Σ_r that satisfies the (1.4)-(1.9) with the radius r. The general plan is easy to guess: we shall move Σ_r in the direction of the unit normal to get two new surfaces $\Sigma_{r,1}$ and $\Sigma_{r,2}$ (one on each side of Σ_r), which will bound the desired domains $W_{r,1}$ and $W_{r,2}$. The slightly unpleasant part of the argument will be to take care of the topology. In particular, we shall give a more direct proof of (1.20), that relies only on the simple argument of [S] (rather that the full Reifenberg machine), and which is a little reminiscent of arguments in Chapter II.4 of [DS].

Let us first check that for $0 < r \leq r_0$,

$$\Sigma_r \text{ is connected.} \tag{3.1}$$

When E is connected, we can use the proof of (1.25) to get this directly, but let us also prove (3.1) under the weaker assumption that E is $\frac{r_0}{20}$-connected (as in Remark 1.26). The proof of (1.25) shows that Σ_{r_0} is connected, and then (1.4) and (1.5) imply that E is $10C_0\varepsilon r_0$-connected. Then set $r_1 = r_0/10$, use the fact that (if $\varepsilon < \varepsilon_2$ is small enough) E is $\frac{r_1}{20}$-connected to prove that Σ_{r_1} is connected, and continue. We get that E is ρ-connected for every $\rho > 0$, and it is easy to deduce from this that it is connected. So the weaker $\frac{r_0}{20}$-connectedness implies that E is connected, and we get a full proof of (3.1).

By [S], Σ_r is orientable, which means that there is a continuous (in fact, smooth) choice of unit normal to Σ_r; denote by $n(y)$ this unit normal, computed at the point $y \in \Sigma_r$. The two sets that we want to consider are

$$\Sigma_{r,1} = \left\{ z \in \mathbb{R}^n \, ; \, z = y + 5C_0\varepsilon r n(y) \text{ for some } y \in \Sigma_r \right\} \tag{3.2}$$

and

$$\Sigma_{r,2} = \left\{ z \in \mathbb{R}^n \, ; \, z = y - 5C_0\varepsilon r n(y) \text{ for some } y \in \Sigma_r \right\}, \tag{3.3}$$

where C_0 is as in (1.4) and (1.5). These will eventually be our $\partial W_{r,j}$, but we shall take care of the topology later. Our first task is to show that the $\Sigma_{r,j}$ are nice surfaces, in the sense that they satisfy the constraints (1.22)-(1.24) that we require from the boundaries $\partial W_{r,j}$. For this we will use the good description of Σ_r that comes from (1.6)-(1.9) (think that Σ_r is sufficiently smooth for $n(y)$ to vary slowly at the scale r).

Let $y \in \Sigma_r$ be given; we want to study the $\Sigma_{r,j}$ near y. Recall from (1.6)-(1.9) that there is a hyperplane P_y through y and a smooth function $F_y : P_y \to P_y^{\perp}$, such that Σ_r coincides with the graph of F_y in $B(y, 10^{-2}r)$. Let $\mathcal{G}$ denote the graph of F_y

over P_y, defined as in (1.7). We want to compute a unit normal to $\mathcal{G}$, and this will be simpler after a change of basis. Without loss of generality, we can assume that $y = 0$, and that we chose an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ such that P_y is the plane of equation $x_n = 0$. Then, for $w \in P_y$, a unit normal to $\mathcal{G}$ at $w + F_y(w)$ is

$$(3.4) \qquad \nu(w) = a(w)\big(e_n - \sum_{j=1}^{n-1} e_j \frac{\partial F_y}{\partial w_j}(w)\big), \text{ with } a(w) = \big|e_n - \sum_{j=1}^{n-1} e_j \frac{\partial F_y}{\partial w_j}(w)\big|^{-1}.$$

Let us first study $\mathcal{G}$ and the analogues of the $\Sigma_{r,j}$ for $\mathcal{G}$, and then we will use (1.7) to return to the $\Sigma_{r,j}$. For $w \in P_y$, set

$$(3.5) \qquad z(w) = w + F_y(w) \in \mathcal{G},$$

$$(3.6) \qquad \xi_\pm(w) = z(w) \pm 5C_0\varepsilon r\nu(w) = w + F_y(w) \pm 5C_0\varepsilon r\nu(w),$$

and then

$$(3.7) \qquad \mathcal{G}_\pm = \big\{\xi_\pm(w)\,;\, w \in P_y\big\}.$$

We want to show that $\mathcal{G}_\pm$ is a smooth Lipschitz graph, and for this we will just need to estimate some derivatives and apply the implicit function theorem.

Recall from (1.9) that F_y is $\Lambda_1\varepsilon$-Lipschitz, so $\big|\sum_{j=1}^{n-1} e_j \frac{\partial F_y}{\partial w_j}(w)\big| \leq \Lambda_1\varepsilon$; then the normalizing factor $a(w)$ in (3.4) stays close to 1 (if ε_2 is chosen small enough),

$$(3.8) \qquad |\nu(w) - e_n| \leq C\varepsilon,$$

and (1.9) and (3.4) yield

$$(3.9) \qquad |D^k\nu(w)| \leq C_k\varepsilon r^{-k}$$

for $w \in P_y$ and $k \geq 1$. Here and below, C_k denotes a constant that depends only on k and n. From (3.6), (3.9), and (1.9) we also deduce that

$$(3.10) \qquad |D^k[\xi_\pm(w) - w]| \leq C_k\varepsilon r^{1-k}$$

(where for $k = 0$, we also use the fact that $F_y(y) = 0$). Let π denote the orthogonal projection on P_y, and set $\psi_\pm = \pi \circ \xi_\pm$; then

$$(3.11) \qquad |D\psi_\pm(w) - I| \leq |D\pi \circ DF_y(w)| + 5C_0\varepsilon r|D\pi \circ D\nu(w)| \leq C\varepsilon$$

by (3.6), (1.9), and (3.9). By the implicit function theorem (or rather the contracting fixpoint theorem), $\psi_\pm$ is a bijection from P_y to P_y; denote by $\varphi_\pm : P_y \to P_y$ its inverse. Notice that $\varphi_\pm$ is differentiable and

$$(3.12) \qquad D\varphi_\pm(u) = D\psi_\pm^{-1}(\varphi_\pm(u));$$

then $\varphi_\pm$ is $(1 + C\varepsilon)$-Lipschitz, by (3.11). In addition, (3.10) also holds for $\psi_\pm$ (trivially), and then for its inverse $\varphi_\pm$ (because of (3.12)), and also the composition $f_\pm = \xi_\pm \circ \varphi_\pm$. That is,

$$(3.13) \qquad ||D^k[f_\pm - I]||_\infty \leq C_k\varepsilon r^{1-k}.$$

It is easy to see that

$$\mathcal{G}_\pm \text{ is the graph of } f_\pm - I : P_y \to P_y^\perp, \tag{3.14}$$

so we have a good Lipschitz graph description of $\mathcal{G}_\pm$, of the same type as in (1.6)-(1.9) and (1.23)-(1.24). Let us also check that

$$4C_0\varepsilon r \leq \operatorname{dist}(\xi, \mathcal{G}) \leq 5C_0\varepsilon r \ \text{ for } \xi \in \mathcal{G}_\pm. \tag{3.15}$$

Let $\xi \in \mathcal{G}_\pm$ be given, and write $\xi = \xi_\pm(w) = z(w) \pm 5C_0\varepsilon r\nu(w)$ for some $w \in P_y$. The second inequality is trivial because $z(w) \in \mathcal{G}$; for the first one we shall just use the last coordinates. We proceed by contradiction and suppose that $B(\xi, 4C_0\varepsilon r)$ contains a point $z_1 \in \mathcal{G}$. Write $z_1 = z(w_1)$ and observe that

$$\begin{aligned} |\xi - z_1| &\geq |\langle \xi - z_1, e_n\rangle| \geq |\langle \xi - z(w), e_n\rangle| - |\langle z(w) - z_1, e_n\rangle| \\ &= |\langle 5C_0\varepsilon r\nu(w), e_n\rangle| - |\langle z(w) - z(w_1), e_n\rangle| \\ &\geq 5C_0\varepsilon r - 5C_0\varepsilon r|\nu(w) - e_n| - |F_y(w) - F_y(w_1)| \\ &\geq 5C_0\varepsilon r - C\varepsilon^2 r - \Lambda_1\varepsilon|w - w_1| \geq 5C_0\varepsilon r - C\varepsilon^2 r \geq 4C_0\varepsilon r \end{aligned} \tag{3.16}$$

by (3.6), (3.8), because F_y is $\Lambda_1\varepsilon$-Lipschitz and $|w - w_1| \leq |z(w) - z(w_1)| \leq |z(w) - \xi| + |\xi - z_1| \leq 10C_0\varepsilon r$, and if ε_2 is small enough. This contradiction completes our proof of (3.15).

Now we compare the $\mathcal{G}_\pm$ and the $\Sigma_{r,j}$. Since by (1.7), $\Sigma_r = \mathcal{G}$ inside of $B(y, 10^{-2}r)$, $z(w) = w + F_y(w)$ is also a parameterization of Σ_r near y, and $\nu(w) = \pm n(z(w))$ in a neighborhood of y. Possibly at the price of replacing e_n with $-e_n$ and F_y by $-F_y$, we may assume that the sign is $+$. Then (1.7) and a short connectedness argument show that

$$\nu(w) = n(z(w)) \ \text{ for } w \in P_y \cap B(y, 10^{-2}r/2). \tag{3.17}$$

Let us check that

$$\Sigma_{r,1} \cap B(y, 10^{-2}r/3) = \mathcal{G}_+ \cap B(y, 10^{-2}r/3). \tag{3.18}$$

If $\xi \in \mathcal{G}_+ \cap B(y, 10^{-2}r/3)$, then by (3.7) $\xi = \xi_+(w) = z(w) + 5C_0\varepsilon r\nu(w)$ for some $w \in P_y$. Notice that $w = \pi(z(w)) \in B(y, 10^{-2}r/2)$ because $|\xi - z(w)| = 5C_0\varepsilon r$, so $\nu(w) = n(z(w))$ by (3.17). Also, $z(w) \in \mathcal{G}_+ \cap B(y, 10^{-2}r/2) \subset \Sigma_r$, by (1.8), and now $\xi = z(w) + 5C_0\varepsilon rn(z(w))$ lies in $\Sigma_{r,1}$, by (3.2).

Conversely, if $\xi \in \Sigma_{r,1} \cap B(y, 10^{-2}r/3)$, then we can write $\xi = z + 5C_0\varepsilon rn(z)$ for some $z \in \Sigma_r$. Obviously $z \in B(y, 10^{-2}r/2)$, so $\Sigma_r = \mathcal{G}$ near z (by (1.8)); then $z = z(w)$ for some $w \in P_y$. In addition, $w = \pi(z) \in B(y, 10^{-2}r/2)$, so $n(z) = \nu(w)$ by (3.7). Thus $\xi = z(w) + 5C_0\varepsilon r\nu(w) \in \mathcal{G}_+$, as needed for (3.18). The proof of (3.18) also yields

$$\Sigma_{r,2} \cap B(y, 10^{-2}r/3) = \mathcal{G}_- \cap B(y, 10^{-2}r/3). \tag{3.19}$$

Let us now deduce from this that

$$4C_0\varepsilon r \leq \operatorname{dist}(\xi, \Sigma_r) \leq 5C_0\varepsilon r \ \text{ for } \xi \in \Sigma_{r,1} \cup \Sigma_{r,2}. \tag{3.20}$$

Let ξ lie in some $\Sigma_{r,j}$, and use the definition (3.2) or (3.3) to write $\xi = y \pm 5C_0\varepsilon rn(y)$ for some $y \in \Sigma_r$; then $\xi \in B(y, 10^{-2}r/3)$ and, with the notation above, $\xi \in \mathcal{G}_+ \cup \mathcal{G}_-$ (by (3.18) or (3.19)); now (3.20) follows from (3.15) and the fact that $\Sigma_r = \mathcal{G}$ in $B(y, 10^{-2}r)$.

Let us start to deal with the topology. Recall from the discussion below (3.1) that Σ_r is connected, hence orientable. In addition,

$$\mathbb{R}^n \setminus \Sigma_r \text{ has exactly two components;} \tag{3.21}$$

see for instance [S]. For each $y \in \Sigma_r$, we know from (1.8) that in $B(y, 10^{-2}r)$, Σ_r coincides with a Lipschitz graph $\mathcal{G} = \mathcal{G}(y)$. Denote by $U_1(y)$ the connected component of $\mathbb{R}^n \setminus \Sigma_r$ that contains the points of $B(y, 10^{-3}r) \setminus \Sigma_r$ that lie in the direction of $n(y)$; in the coordinates that we used to describe $\Sigma_r = \mathcal{G}$ near y, this is the component that contains the part of $B(y, 10^{-3}r) \setminus \mathcal{G}$ above $\mathcal{G}$. It is clear that $U_1(y)$ is locally constant on Σ_r, i.e., that $U_1(y) = U_1(y')$ when $|y' - y| \leq 10^{-3}r$. Since Σ_r is connected, $U_1(y)$ is constant; we shall denote by $U_{r,1}$ its constant value on Σ_r. Similarly, we can define $U_{r,2}$ to be the connected component of $\mathbb{R}^n \setminus \Sigma_r$ that contains the points of $B(y, 10^{-3}r) \setminus \Sigma_r$ that lie in the direction of $-n(y)$, because this component does not depend on y.

Each of the two components of $\mathbb{R}^n \setminus \Sigma_r$ gets arbitrarily close to Σ_r, hence it meets some $B(y, 10^{-3}r)$ and it is equal to $U_{r,1}$ or $U_{r,2}$. This proves that $U_{r,1} \neq U_{r,2}$; and these are the two components of $\mathbb{R}^n \setminus \Sigma_r$.

We have a local Lipschitz description of the two $\Sigma_{r,j}$, that comes from (3.18) and (3.19) (also see (3.13)-(3.14)), and from the connectedness of Σ_r we easily deduce that $\Sigma_{r,j}$ is connected: given $z_1, z_2 \in \Sigma_{r,j}$, pick $y_i \in \Sigma_r \cap B(z_i, 10^{-3}r)$, connect y_1 to y_2 by a $(10^{-3}r)$-chain of points $y \in \Sigma_r$, use (3.18) or (3.19) to connect z_1 to successive points $z \in \Sigma_{r,j} \cap B(z, 10^{-3}r)$ by paths in $\Sigma_{r,j}$, and finally connect the last z to z_2.

By [S], $\mathbb{R}^n \setminus \Sigma_{r,j}$ has exactly two connected components, which we denote by $W_{r,j}$ and $W'_{r,j}$. Since Σ_r is connected and does not meet $\Sigma_{r,j}$ (by 3.20)), it is contained in one of these components, and we choose their names so that

$$\Sigma_r \subset W'_{r,j} \text{ and } \Sigma_r \cap W_{r,j} = \emptyset. \tag{3.22}$$

For $y \in \Sigma_r$, we can use (3.18) and (3.19) to localize these components near y. We have three Lipschitz graphs, $\mathcal{G}_-$, $\mathcal{G}$, and $\mathcal{G}_+$, which do not meet (by (3.15)), and $\mathcal{G}_-$ and $\mathcal{G}_+$ cut $\mathbb{R}^n$ into three open region: the region R_- below $\mathcal{G}_-$, the region R_0 between $\mathcal{G}_-$ and $\mathcal{G}_+$ (and which contains $\mathcal{G}$), and the region R_+ above $\mathcal{G}_+$. Let us check that

$$\Sigma_{r,j} \cap B(y, 10^{-2}r/4) \subset U_{r,j} \text{ for } j = 1, 2, \tag{3.23}$$

$$R_0 \cap B(y, 10^{-2}r/4) \subset W'_{r,1} \cup W'_{r,2}, \tag{3.24}$$

$$R_+ \cap B(y, 10^{-2}r/4) \subset W_{r,1}, \tag{3.25}$$

and

$$R_- \cap B(y, 10^{-2}r/4) \subset W_{r,2}. \tag{3.26}$$

Let us first prove (3.23) for $j = 1$. Recall from the definitions below (3.21) that $U_{r,1}$ is the common value, for $y \in \Sigma_r$, of $U_1(y)$, which is the component of $\mathbb{R}^n \setminus \Sigma_r$ that contains the points of $B(y, 10^{-3}r) \setminus \Sigma_r$ that lie in the direction of $n(y)$. Clearly this component also contains $\mathcal{G}_+ \cap B(y, 10^{-2}r/4)$, which by (3.18) is the same as $\Sigma_{r,1} \cap B(y, 10^{-2}r/4)$. This proves (3.23) for $j = 1$; the proof for $j = 2$ is the same.

Next, if $x \in R_0 \cap B(y, 10^{-2}r/4)$, the vertical line segment from x to the point of $\mathcal{G}$ with the same projection is contained in $B(y, 10^{-2}r/3) \setminus [\mathcal{G}_+ \cup \mathcal{G}_-]$, hence (by (3.18) and (3.19)) in $B(y, 10^{-2}r/3) \setminus [\Sigma_{r,1} \cup \Sigma_{r,2}]$. Since $y \in \Sigma_r \subset W'_{r,j}$ for $j = 1, 2$, we get that $x \in W'_{r,j}$, which proves (3.24).

We now check (3.25), starting with a specific y. Since $W_{r,1}$ is a component of $\mathbb{R}^n \setminus \Sigma_{r,1}$, we can find $\xi \in \Sigma_{r,1}$ such that $\mathrm{dist}(\xi, W_{r,1}) \leq C_0 \varepsilon r$. Then (by (3.20)) we can pick $y \in \Sigma_r$ such that $|\xi - y| \leq 5C_0 \varepsilon r$. For this specific y, $\xi \in \mathcal{G}_+ \cap B(y, 10^{-2}r/4)$ (by (3.18)), so

$$\mathrm{dist}(\mathcal{G}_+ \cap B(y, 10^{-2}r/4), W_{r,1}) \leq C_0 \varepsilon r. \tag{3.27}$$

Let $a \in W_{r,1}$ be such that $\mathrm{dist}(a, \mathcal{G}_+ \cap B(y, 10^{-2}r/4) \leq 2C_0 \varepsilon r$. Observe that $a \notin \overline{R}_-$, because $\mathrm{dist}(\overline{R}_-, \mathcal{G}_+) \geq \mathrm{dist}(\mathcal{G}_-, \mathcal{G}_+) \geq \mathrm{dist}(\mathcal{G}_-, \mathcal{G}) \geq 4C_0 \varepsilon r$ by (3.15). But a does not lie in R_0 (by (3.24)), nor in $\Sigma_{r,1}$ (by definition of $W_{r,1}$), so we get that $a \in R_+$. It is then easy to connect a to any point of $R_+ \cap B(y, 10^{-2}r/4)$, by a path contained in $B(y, 10^{-2}r/3) \setminus \mathcal{G}_+ = B(y, 10^{-2}r/3) \setminus \Sigma_{r,1}$ (by (3.18)). So (3.25) holds for this specific y.

But then every point of $\Sigma_{r,1} \cap B(y, 10^{-2}r/4)$ lies very close to $W_{r,1}$, and (3.27) also holds for every $y' \in \Sigma_r$ such that $|y' - y| \leq 10^{-3}r$. By the argument above, we also get (3.25) for every such y' and eventually, because Σ_r is connected, for every $y' \in \Sigma_r$. This proves (3.25) for all $y \in \Sigma_r$. The proof of (3.26) is the same.

We now have a definition of $W_{r,1}$ and $W_{r,2}$, and we can start checking the conclusions of Theorem 1.18. First observe that

$$\partial W_{r,j} = \Sigma_{r,j}\,; \tag{3.28}$$

the first inclusion is obvious, and conversely every point of $\Sigma_{r,j}$ lies in $\partial W_{r,j}$: we choose $y \in \Sigma_r$ not too far and use (3.25) or (3.26). From (3.20), (1.4), and (1.5), we easily deduce that

$$3C_0 \varepsilon r \leq \mathrm{dist}(\xi, E) \leq 6C_0 \varepsilon r \ \text{ for } \xi \in \Sigma_{r,1} \cup \Sigma_{r,2}, \tag{3.29}$$

which proves (1.22) with $C_2 = 3C_0$. For (1.21), we just need to check that

$$\big\{ x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, E) > 6C_0 \varepsilon r \big\} \subset W_{r,1} \cup W_{r,2}. \tag{3.30}$$

Suppose $\mathrm{dist}(x, E) > 6C_0 \varepsilon r$, pick any point $a \in E$, and run along the segment $[x, a]$ until the first point z such that $\mathrm{dist}(z, E) = 6C_0 \varepsilon r$. Thus

$$\mathrm{dist}(\xi, E) > 6C_0 \varepsilon r \ \text{ for } \xi \in [x, z) \tag{3.31}$$

and, by (3.29), $[x, z)$ does not meet $\Sigma_{r,1} \cup \Sigma_{r,2}$. Pick $x' \in [x, z)$ close to z, so that $\mathrm{dist}(x', E) \leq 7C_0 \varepsilon r$ and hence $\mathrm{dist}(x', \Sigma_r) \leq 8C_0 \varepsilon r$ (by (1.4)). Pick $y \in \Sigma_r$ such that $|y - x'| \leq 8C_0 \varepsilon r$, and use y to localize. If x does not lie in $W_{r,1} \cup W_{r,2}$, then neither does x', and since $x' \in B(y, 10^{-3}r)$, (3.25) and (3.26) imply that $x' \in \overline{R}_0$. But (3.15) then implies that $\mathrm{dist}(x', \mathcal{G}) \leq 5C_0\varepsilon r$. In addition, $\mathcal{G} = \Sigma_r$ in $B(y, 10^{-2}r)$ (by (1.8)), so $\mathrm{dist}(x', \Sigma_r) \leq 5C_0\varepsilon r$, and (by (1.5)) $\mathrm{dist}(x', E) \leq 6C_0\varepsilon r$. This contradicts (3.31); (3.30) and (1.21) follow.

Let us also check that

$$\{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, E) < 3C_0\varepsilon r\} \subset W'_{r,1} \cap W'_{r,2} \subset \mathbb{R}^n \setminus [W_{r,1} \cup W_{r,2}]. \tag{3.32}$$

Suppose $\mathrm{dist}(x, E) < 3C_0\varepsilon r$, and let $a \in E$ be such that $|x - a| < 3C_0\varepsilon r$. Observe that $[x, a]$ does not meet $\Sigma_1 \cup \Sigma_2$, by (3.29). Then use (1.4) to find $b \in \Sigma_r$ such that $|b - a| \leq C_0\varepsilon r$; then $[a, b]$ does not meet $\Sigma_1 \cup \Sigma_2$ either, again by (3.29). But $b \in W'_{r,1} \cap W'_{r,2}$, by (3.22), so $x \in W'_{r,1} \cap W'_{r,2}$ too. The second part of (3.32) is immediate.

Now we establish the Lipschitz description of the $\partial W_{r,j}$ in (1.23) and (1.24). By (3.28), $\partial W_{r,j} = \Sigma_{r,j}$. Since $\Sigma_{r,j}$ stays close to Σ_r, by (3.20), a good Lipschitz description of $\Sigma_{r,j}$ in each $B(y, 10^{-2}r/3)$ will give a good description in each $B(z, 10^{-3}r)$, $z \in \Sigma_{r,j}$. But in $B(y, 10^{-2}r/3)$, (3.18) or (3.19) says that $\Sigma_{r,j}$ coincides with a $\mathcal{G}_\pm$, and now the desired Lipschitz description follows from (3.13) and (3.14).

The fact that for $z \in \Sigma_{r,j}$, $W_{r,j} \cap B(z, 10^{-3}r)$ lies on only one side of $\Sigma_{r,j}$ can also be seen locally, in some $B(y, 10^{-2}r/3)$, $y \in \Sigma_r$, where it follows from (3.24)-(3.26); the proof is the same as for (3.28).

Our last tasks will be to show that $\mathbb{R}^n \setminus E$ has exactly two connected components (as in (1.20)), and that for each r,

$$\text{the two } W_{r,j} \text{ are contained in different components of } \mathbb{R}^n \setminus E. \tag{3.33}$$

Once we do this, Theorem 1.18 will follow, just after renaming the $W_{r,j}$ so that $W_{r,1}$ is always contained in the same component, and $W_{r,2}$ in the other one.

Let us first check that for $j = 1, 2$,

$$\Sigma_{r,j} \subset U_{r,j} \tag{3.34}$$

where the $U_{r,j}$ are still the two connected components of $\mathbb{R}^n \setminus \Sigma_r$, defined below (3.21). By (3.20), each point of $\Sigma_{r,j}$ is contained in some $B(y, 10^{-3}r)$, and then (3.34) follows from (3.23). Let us deduce from this that

$$W_{r,j} \subset U_{r,j} \ \text{ for } j = 1, 2. \tag{3.35}$$

Let $z \in W_{r,j}$ be given, pick any $\xi \in \Sigma_{r,j}$, and let w be the first point of $\Sigma_{r,j}$ when we run from z to ξ along $[z, \xi]$. Then $[z, w) \subset W_{r,j}$ (because $W_{r,j}$ is a component of $\mathbb{R}^n \setminus \Sigma_{r,j}$) and hence $\mathrm{dist}([z, w), E) \geq 3C_0\varepsilon$, by (3.32). By (1.5), $\mathrm{dist}([z, w], E) \geq 2C_0\varepsilon$, so $[z, w]$ does not meet Σ_r, and $[z, w]$ is contained in some $U_{r,i}$. But $w \in U_{r,j}$ by (3.34), so $i = j$ and (3.35) follows.

Now we need to relate the descriptions that we get at different scales. Pick an origin $x_0 \in E$ and, for $0 < r \leq r_0$, use (1.4) to choose $y(r) \in \Sigma_r$ such that $|y(r) - x_0| \leq C_0\varepsilon r$. Then denote by $P(r)$ and $\mathcal{G}(r)$ the plane and graph that we get from (1.6)-(1.9), applied to Σ_r and the point $y(r)$. We claim that

$$d_{x_0, 10^{-3}r}(P(r), P(s)) \leq C\varepsilon \text{ when } 0 < s \leq r \leq 9s. \tag{3.36}$$

Here d is still the normalized Hausdorff distance defined in (1.2), and the proof will be routine. First notice that $\mathcal{G}(r)$ and $P(r)$ both go through $y(r)$ (by definitions). If $a \in P(r) \cap B(x_0, 10^{-3}r)$, we can use (1.9) to find $z \in \mathcal{G}(r)$ such that $|z - a| \leq \Lambda_1 \varepsilon 10^{-3} r$; this point lies in Σ_r by (1.8), and by (1.4) and (1.5) we can find $w \in \Sigma_s$ such that $|w - z| \leq 2C_0 \varepsilon r$. Since $w \in B(y(s), 10^{-2}s)$, we can apply (1.8) again to see that $w \in \mathcal{G}(s)$, and then its projection b on $P(s)$ is such that $|b - w| \leq \Lambda_1 \varepsilon 10^{-3} s$. Thus we found $b \in P(s)$ such that $|b - a| \leq C\varepsilon r$. The same argument shows that for $b \in P(s) \cap B(x_0, 10^{-3}r)$, we can find $a \in P(r)$ such that $|b - a| \leq C\varepsilon r$, and (3.36) follows.

Denote by $e_n(r)$ the last vector of the basis that we chose to describe Σ_r near $y(r)$; thus $e_n(r)$ is orthogonal to $P(r)$, and (3.36) shows that $|e_n(r) \pm e_n(s)| \leq C\varepsilon$ when $0 < s \leq r \leq 9s$; the sign $\pm$ depends on r and s through the orientations that we chose on Σ_r and Σ_s.

Set $z_\pm(r) = x_0 + 10^{-4} r e_n(r)$ for $0 < r \leq r_0$; if follows (3.25) and (3.26) that $z_+(r) \in W_{r,1}$ and that $z_-(r) \in W_{r,2}$. But we can also look in the coordinates associated to s, use the fact that $|e_n(r) \pm e_n(s)| \leq C\varepsilon$, and get that $z_\mp(r) \in W_{s,1}$ and $z_\pm(r) \in W_{s,2}$. The signs are not important, but we want to know that $z_+(r)$ and $z_-(r)$ lie in different $W_{s,j}$.

We also need to know that when $r \geq 3s$, each $W_{r,j}$ is contained in a $W_{s,i}$. For each $z \in W_{r,j}$, (3.32) says that $\mathrm{dist}(z, E) \geq 3C_0 \varepsilon r \geq 9C_0 s$. Then by (3.30), $x \in W_{s,1} \cup W_{s,2}$. That is, $W_{r,j} \subset W_{s,1} \cup W_{s,2}$. By (3.35), $W_{s,1}$ and $W_{s,2}$ are contained in different connected components of $\mathbb{R}^n \setminus \Sigma_s$, so $W_{r,j}$, which is itself connected, is contained in a single $W_{s,i}$, as needed. Of course i is unique, since $W_{s,1} \cap W_{s,2} = \emptyset$.

Let us now use all this to check that when $0 < 3s \leq r \leq r_0$,

$$W_{r,1} \subset W_{s,1} \text{ and } W_{r,2} \subset W_{s,2}, \text{ or else } W_{r,1} \subset W_{s,2} \text{ and } W_{r,2} \subset W_{s,1}. \tag{3.37}$$

Obviously it is enough to check this when $3s \leq r \leq 9s$, because after this we can use intermediate radii and iterate. Suppose first that $W_{r,1} \subset W_{s,1}$. Then $z_+(r) \in W_{r,1} \cap W_{s,1}$, and we observed earlier that then $z_-(r) \in W_{r,2} \cap W_{s,2}$. In this case $W_{r,2} \subset W_{s,2}$ (the other option, $W_{r,2} \subset W_{s,1}$, is impossible), and we are happy. If $W_{r,1}$ is not contained in $W_{s,1}$, then it is contained in $W_{s,2}$, so $z_+(r) \in W_{r,1} \cap W_{s,2}$, then $z_-(r) \in W_{r,2} \cap W_{s,1}$, and the only possibility is that $W_{r,2} \subset W_{s,1}$; (3.37) follows.

We are now ready to show that for $0 < r \leq r_0$, $W_{r,1}$ and $W_{r,2}$ lie in different components of $\mathbb{R}^n \setminus E$. First observe that by (3.32), $\mathrm{dist}(x, E) \geq 3C_0 \varepsilon r$ for $x \in W_{r,j}$, so $W_{r,j}$ does not meet E and, by connectedness, is contained in a component of $\mathbb{R}^n \setminus E$. If $W_{r,1}$ and $W_{r,2}$ lie in the same component, we can find a (compact) curve in $\mathbb{R}^n \setminus E$, that meets $W_{r,1}$ and $W_{r,2}$. Since $\mathrm{dist}(\gamma, E) > 0$, (3.30) says that for s small enough, $\gamma \subset W_{s,1} \cup W_{s,2}$. Since the γ is connected and the $W_{s,i}$ are contained in different components of $\mathbb{R}^n \setminus \Sigma_s$ (by (3.35)), we see that γ is contained in a single $W_{s,i}$. But (3.37) precisely says that $W_{r,1}$ or $W_{r,2}$ is contained in the other $W_{s,j}$, hence does not meet $W_{s,i}$. This contradiction proves (3.33).

We finally need to check that $\mathbb{R}^n \setminus E$ has exactly two components. Suppose instead that it has at least three components (we just excluded a single one). Let z_1, z_2, z_3 lie in different components, and choose r so small that for all j, $\mathrm{dist}(z_j, E) > 6C_0 \varepsilon r$; then (3.30) says that all z_j lie in $W_{r,1} \cup W_{r,2}$, at least two of them lie in a same $W_{r,i}$, and this is a contradiction because $W_{r,i}$ is connected and does not meet E. This concludes our proof of Theorem 1.18. ■

References

[AH] P. Alexandroff and H. Hopf, Topologie. Springer, Berlin, 1935.

[DS] Guy David and Stephen Semmes, Analysis of and on uniformly rectifiable sets, Mathematical Surveys and Monographs 38, American Mathematical Society, Providence, RI, 1993. xii+356 pp.

[DT] G. David and T. Toro, Reifenberg parameterizations for sets with holes. Memoirs of the American Mathematical Society, Vol. 215, 2012, Number 1012.

[F] H. Federer, Geometric measure theory, Grundlehren der Mathematishen Wissenschaften 153, Springer Verlag 1969.

[R] E. R. Reifenberg, Solution of the Plateau problem for m-dimensional surfaces of varying topological type, Acta Mathematica **104** (1960), 1–92.

[S] H. Samelson, Orientability of hypersurfaces in $\mathbb{R}^n$. Proc. Amer. Math. Soc. 22, 1969, 301–302.

[St] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton university press 1970.

Univ Paris-Sud,
Laboratoire de mathématiques UMR-8628,
Orsay F-91405, France
and Institut Universitaire de France