Generalized Köthe p-dual spaces

O. Galdames Bravo

Abstract

Let us consider a Banach function space X. The Köthe dual space can be characterized as the space of multipliers from X to L^1. We extend this characterization to the space of multipliers from X to L^p in order to define the Köthe p-dual space of X. We analyze the properties of this space so as to use it as a tool for studying p-th power factorable operators. In particular, we compute q-concavity for these spaces and type and cotype when X is an AM-space. As main applications, we give a characterization for Hilbert Banach function spaces, as well as a factorization for p-th power factorable operators through an $L^{p,\infty}$-space.

1 Introduction and preliminaries

Generalized duality was investigated in 1989 in [15] as spaces of multipliers (see also [2]). A complete bibliography on this topic can be found in the recent papers [10, 11]. The point of view of these papers and the references therein is in general oriented to the study of structural aspects and representation of these spaces of multipliers. The purpose of the present paper is to investigate some geometric aspects of these spaces. To be precise, we study p-concavity, type and cotype of multipliers from Banach function spaces to the space of p-integrable real-valued functions with respect to a positive and finite scalar measure. The advantage of using these spaces lies in the fact that they inherit some geometry from the L^p-space, for example the p-convexity (see e.g. [2, Lem. 5.1]). We will also define the natural operator associated with an operator $T\colon X \to E$ and the Köthe

Received by the editors in February 2013 - In revised form in August 2013.
Communicated by F. Bastin.
2010 *Mathematics Subject Classification* : Primary 46A20, Secondary 32F10, 46B20.
Key words and phrases : Banach function space, Köthe dual space, space of multipliers, p-concavity, type, cotype.

p-dual X^p so as to define the Köthe p-adjoint of T. As applications we will use the properties of the so-called p-th power factorable operators and easy properties of spaces of multipliers in order to factorize the Köthe p-adjoint through an L^p-space. We will see that a p-th power factorable operator may factorize through L^p-spaces and $L^{p,\infty}$-spaces. Also, we will show that the Köthe r-dual space of a p-convex Banach function space is q-concave for every $q \leq 1$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Computing type and cotype for some Köthe p-dual of an AM-space and applying Kwapień's Theorem we will obtain a characterization of Hilbert spaces as Köthe 2-dual space and some examples of operators can be written as Köthe p-adjoint operators. We provide an example by using some results of [15] for Orlicz spaces, and results of [9] for the Riesz transform.

Let (Ω, Σ, μ) be a complete σ-finite measure space and L^0 the space of μ-measurable real-valued functions defined on Ω that are equal μ-a.e.. A *(quasi-) Banach function space* X is a linear subspace of L^0, with complete (quasi-)norm $\|\cdot\|_X$, so that for each $g \in X$ and $h \in L^0$ with $|h| \leq |g|$ μ-a.e., implies that $h \in X$ and $\|h\|_X \leq \|g\|_X$. Other authors use this space, with slight differences, see e.g. [2, 4, 14, 16]. An element $h \in X$ is a *weak unit* if it is such that $h > 0$ μ-a.e., and it is a *weak order unit* if it is such that $g \wedge nh \uparrow g$ for every $g \in X^+$. For instance if $\chi_\Omega \in X$, then it is a weak order unit and $L^\infty \subseteq X$ (see [16, Prop. 2.2(iv)]). Clearly, a weak order unit is a weak unit. Given $1 \leq p \leq \infty$, we denote the conjugate of p by $p' := \frac{p}{p-1}$. We write B_X for the unit ball of X.

Let $1 \leq p < \infty$. Recall that a Banach function space X is *p-convex* if there is a constant K_p such that for every finite set $f_1, \ldots, f_n \in X$, the inequality

$$\Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{1/p} \Big\|_X \leq K_p \Big(\sum_{i=1}^n \|f_i\|_X^p \Big)^{1/p}$$

holds. Let $1 \leq q < \infty$. It is said that X is *q-concave* if there is a constant K^q such that

$$\Big(\sum_{i=1}^n \|f_i\|_X^q \Big)^{1/q} \leq K^q \Big\| \Big(\sum_{i=1}^n |f_i|^q \Big)^{1/q} \Big\|_X$$

holds. The following definitions can be found in [14, Defn. 1.e.12] and also in [5, Sec. 7.7]. The Rademacher functions are in $L^2[0,1]$ and are defined by

$$r_k(t) := (-1)^j\,, \qquad t \in \Big[\frac{j}{2^k}, \frac{j+1}{2^k} \Big[\,,\ j = 0, \ldots, 2^k - 1\,,\ k = 1, 2, 3, \ldots .$$

Let E be a Banach space. We say that E has type $p \in [1,2]$ if for every $x_1, \ldots, x_n \in E$ exists $C \geq 0$

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|_E^2 dt \right)^{1/2} \leq C \left(\sum_{k=1}^n \|x_k\|_E^p \right)^{1/p} . \tag{1}$$

And has cotype $q \in [2, \infty]$ if for every $x_1, \ldots, x_n \in E$ exists $C \geq 0$

$$\left(\sum_{k=1}^n \|x_k\|_E^q \right)^{1/q} \leq C \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|_E^2 dt \right)^{1/2} . \tag{2}$$

Every Banach space has cotype ∞ and type 1 [14, p. 73]. Let us recall Kwapień's characterization of Hilbert spaces [13]: X is isomorphic to a Hilbert space if and only if has type 2 and cotype 2. See [19, Chap. 3] for more details and applications.

Let $1 \leq p < \infty$ and X a Banach function space, we call the p-th power space of X the space

$$X_{[p]} := \{f \in L^0 : |f|^{1/p} \in X\},$$

equipped with the quasi-norm $\|f\|_{X_{[p]}} := \||f|^{1/p}\|_X^p$. This space is a Banach function space if and only if X is p-convex with constant 1 (see e.g. [16, Prop. 2.23(iii)]).

Recall that given two Banach function spaces X and Y over the same measure, the *space of multipliers* from X to Y is defined by

$$X^Y := \{g \in L^0 : gX \subseteq Y\}.$$

The seminorm $\|f\|_{X^Y} := \sup_{g \in B_X} \|fg\|_Y$, gives a Banach function space structure for X^Y when the μ-a.e. order is considered and X has weak unit (see [15, Prop. 2]). For more details we refer the reader to [2, Sec. 2]. Notice that the Köthe dual X' of X may be characterized by $X' := X^{L^1}$. We will use the following characterizations of order continuity and Fatou properties (see [14, p. 28–30] for more details). A Banach function space X is order continuous if and only if $X' = X^*$, where X^* denotes the topological dual space, and is Fatou if and only if $X'' = X$.

The Köthe p-dual space was introduced in [8, Defn. 2.1] and is defined by $X^p := X^{L^p}$. Observe that the classical Lebesgue space L^p is obtained as a particular case when $X = L^\infty$ (see Lemma 2.1(4)). By definition, when X has weak unit, the norm in X^p is

$$\|f\|_{X^p} := \sup_{g \in B_X} \|gf\|_p .$$

For example, let $1 \leq r \leq p \leq \infty$ and $q \geq 1$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $L^q = (L^p)^r$ where the spaces are based on a σ-finite measure (see [15, Prop. 3]). It is important pointing out that the expression X^p may be trivial, for example $(L^r)^s = \{0\}$ whenever $r < s$ and the measure μ is non-atomic. A sufficient condition can be found in [20, Thm. 1.8], in which is established for the case of non-atomic measure that $X^Y = \{0\}$ if

$$\inf\{p \geq 1 : X \text{ is } p\text{-concave}\} < \sup\{p \geq 1 : Y \text{ is } p\text{-convex}\}. \tag{3}$$

For instance, if X is a Banach function space such that the inclusion $L^\infty \subset X$ is proper, then $X^\infty = \{0\}$, since L^∞ is ∞-convex.

Let us define now a class of operators that will be relevant in the paper [16, Defn. 5.1].

Definition 1.1. Let $1 \leq p < \infty$, X an order continuous quasi-Banach function space with weak order unit χ_Ω, and E a Banach space. We say that an operator $T\colon X \to E$ is *p-th power factorable* if there exists an operator $T_{[p]}\colon X_{[p]} \to \mathbf{E}$, which equals T over $X \subseteq X_{[p]}$. In other words, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, hook, "i_{[p]}"'] & E \\ X_{[p]} \arrow[ur, "T_{[p]}"'] & \end{tikzcd}$$

where $i_{[p]}$ is the natural continuous inclusion.

The following relations will be useful throughout the paper. The main properties of $X_{[p]}$ were presented in [16, Sec. 2.2] for the case of finite measure, which implies in their definition of quasi-Banach function space, that χ_Ω is a weak order unit in X (see also [20, Sec. 3]). We have replaced the requirement of finite measure by the requirement of $\chi_\Omega \in X$. Let us recall some of them.

Proposition 1.2. *Let X be a quasi-Banach function space such that $\chi_\Omega \in X$.*

1. *Let $0 < p, q < \infty$. Then ${X_{[p]}}_{[q]} = X_{[pq]}$.*
2. *Let $p \in (0, \infty)$. Then, X is order continuous if and only if $X_{[p]}$ is order continuous.*
3. *If $0 < p \leq q < \infty$ we have that $X_{[p]} \subseteq X_{[q]}$, in particular $X = X_{[1]} \subseteq X_{[p]}$ for all $1 \leq p < \infty$ and $X_{[p]} \subseteq X$ for all $0 < p \leq 1$.*
4. *Let $0 < p < \infty$. $(X_{[p]})' = (X^{L^p})_{[p]}$.*

The main duality identification involving $X_{[p]}$ and X^p is the last property (4).

The following represents an example of a geometric property that is inherited by the multiplier space and will be useful in what follows. It was established without proof in [2, Lem. 5.1], see also [20, Prop. 3.1].

Lemma 1.3. *Let X, Y be Banach function spaces where X has a weak unit and let $1 \leq p \leq \infty$. If Y is p-convex, then X^Y is p-convex with the same constant.*

2 The Köthe p-dual space

In [15, Sec. 2, Cor. 1], [10, Sec. 2] and references therein we can find lists of simple properties involving general spaces of multipliers. Let us now compile some of these properties for the setting of Köthe p-dual and p-th power spaces of Banach function spaces and using our own notation.

Lemma 2.1. *Let X and Y be Banach function spaces with weak unit.*

1. *If $0 < p \le q \le \infty$, then $X^q \subseteq X^p$.*
2. *If $0 < p \le \infty$ and $X \subseteq Y$ then $Y^p \subseteq X^p$.*
3. *If $0 < p \le q \le \infty$, then $(X^p)^q \subseteq (X^q)^p$.*
4. *Let $0 < p \le \infty$, then $L^p = (L^\infty)^p$.*

The proof is straightforward.

Lemma 2.2. *Let X be a Banach function space such that $\chi_\Omega \in X$.*

1. *If $1 \le p < \infty$, then $X^p \subseteq (X^p)_{[p]} \subseteq X'$.*
2. *If $0 \le p \le q < \infty$, then $(X_{[p]})^q = (X^{pq})_{[p]}$ and $(X^q)_{[p]} = (X_{[p]})^{q/p}$.*
3. *If $1 \le p \le q \le \infty$, then $(X_{[p]})^q \subseteq X^p$.*
4. *If $s \le r \le t$ and $1 \le p \le q \le \infty$, then*
$$(X_{[q]})^t \subseteq (X_{[q]})^r \subseteq (X_{[p]})^r \subseteq (X_{[p]})^s .$$
5. *If $rq \le tp$ and $1 \le p \le q \le \infty$, then $(X_{[p]})^t \subseteq (X_{[q]})^r$.*
6. *Let $q \le r, p \le \infty$ be such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. If $X \subseteq L^r$, then $L^p \subseteq X^q$.*
7. *Let $0 < p, q \le \infty$, then $X^p = \big((X_{[1/q]})^{pq}\big)_{[q]}$. In particular $X' = \big((X_{[1/q]})^q\big)_{[q]}$.*
8. *Let $1 \le p \le \infty$, then $X^p \subseteq (X_{[p]})' \subseteq X'$.*
9. *Let $1 \le p \le \infty$ and let Y be a Banach function space such that $\chi_\Omega \in Y$, then $Y \subseteq X^p$ if and only if $X \subseteq Y^p$.*
10. *$X^p \subseteq L^p$.*
11. *$X^p = L^p$ if and only if $X = L^\infty$.*

Proof. From (1) to (5) are easy using Proposition 1.2 and Lemma 2.1. Let us prove the rest.

(6) We use Hölder's inequality
$$\|f\|_{X^q} = \sup_{g \in B_X} \|gf\|_q \le \sup_{g \in B_X} \|g\|_r \|f\|_p = \sup_{g \in B_X} \|g\chi_\Omega\|_r \|f\|_p = \|\chi_\Omega\|_{X^r} \|f\|_p ,$$
hence $L^p \subseteq X^q$, which completes the proof.

(7) We proceed directly using the definition of Köthe dual space. $L^p = (L^1)_{[1/p]}$, then apply properties (1) and (4) of Proposition 1.2
$$X^p = ((X^p)_{[1/q]})_{[q]} = ((X_{[1/q]})^{(L^p)_{[1/q]}})_{[q]} = ((X_{[1/q]})^{((L^1)_{[1/p]})_{[1/q]}})_{[q]} = \big((X_{[1/q]})^{pq}\big)_{[q]} .$$

(8) Apply Proposition 1.2(3) and Lemma 2.1(2).

(9) By hypothesis and Proposition 1.2(4) we have that $Y_{[p]} \subseteq (X^p)_{[p]} = (X_{[p]})'$, hence $X_{[p]} \subseteq (X_{[p]})'' \subseteq (Y_{[p]})'$. Let $g \in Y^p$. Then, for some $C > 0$, it holds

$$\|g\|_{Y^p} = \||g|^p\|_{(Y^p)_{[p]}}^{1/p} = \||g|^p\|_{(Y_{[p]})'}^{1/p} \leq C \, \||g|^p\|_{X_{[p]}}^{1/p} = C \, \|g\|_X \, .$$

The converse is analogous.

(10) Let be $f \in X^p$, since $\chi_\Omega \in X$ then $f = f\chi_\Omega \in L^p$. In order to prove the continuity we consider the following inequalities

$$\begin{aligned} \|f\|_p = \|f\chi_\Omega\|_p = \|\chi_\Omega\|_X \left\| f \frac{\chi_\Omega}{\|\chi_\Omega\|_X} \right\|_p &\leq \|\chi_\Omega\|_X \sup_{g \in X \setminus \{0\}} \left\| f \frac{g}{\|g\|_X} \right\|_p \\ &= \|\chi_\Omega\|_X \sup_{g \in B_X} \|fg\|_p = \|\chi_\Omega\|_X \|f\|_{X^p} \, . \end{aligned}$$

So $X^p \subseteq L^p$.

(11) We only need to show that $X^p = L^p$ implies that $X = L^\infty$. It follows from the statement (9), since $L^p \subseteq X^p$ implies that $X \subseteq (L^p)^p = L^\infty$. Then $X = L^\infty$. ∎

Example 2.3. Let $L^p(m)$ ($L^p_w(m)$) denote the space of (weakly) p-integrable real-valued functions with respect to the vector measure m. Standard works on this topic are [7, 16], see [6] for the case $p = 1$. By Lemma 2.2(2) we have

$$L^p_w(m) = (L^1_w(m))_{[1/p]} = \big(L^1_w(m)''\big)_{[1/p]} = ((L^1_w(m)')_{[1/p]})^p = \big((L^p_w(m))^p\big)^p$$

Applying [2, Prop. 5.3] we obtain that $L^p_w(m)$ is Fatou and p-convex with constant one.

Let us now study operators defined on Köthe p-dual spaces. They provide a natural p-th power factorization, for $1 \leq p < \infty$, as the next diagram shows

$$\begin{tikzcd} X^p \arrow[r, hook, "\kappa_p"] \arrow[d, hook, "k_{[p]}"'] & X' \arrow[r, hook, "\kappa"] & X^* \arrow[r, "T"] & E^* \\ (X^p)_{[p]} \arrow[r, equal] & (X_{[p]})' \arrow[u, hook] \arrow[urr, dotted] & & \end{tikzcd} \tag{4}$$

The next definition and lemma were introduced in [8].

Definition 2.4. Let $1 \leq p < \infty$. Let X be a Banach function space, E a Banach space and $T \colon E \to X$ an operator. Then we define an operator $T^p \colon X^p \to E^*$ as $T^p := T'|_{X^p}$, that we will call Köthe *p-adjoint* operator.

The following lemma gives an easy example of p-th power factorable operators. Now, we provide an expression for the extension map, which will be useful in the sequel.

Lemma 2.5. *Let $1 \leq p < \infty$ and $T\colon E \to X$ be an operator, where X is a quasi-Banach function space that contains χ_Ω and X^p is order continuous. If $p \geq 1$, then the Köthe p-adjoint operator T^p is p-th power factorable, and the extension operator is $(T^p)_{[p]} = (i_{[p]} \circ T)^* = T^* \circ i^*_{[p]}$.*

Proof. The first assertion is immediate from (4) (see [8, Prop. 2.2]). For the second assertion, by definition of Köthe p-adjoint we have $T^p = (T^p)_{[p]} \circ k_{[p]}$, where $k_{[p]}\colon X^p \hookrightarrow (X^p)_{[p]}$ is the canonic inclusion and $(T^p)_{[p]}$ is the unique extension. On the other hand $i_{[p]}\colon X \hookrightarrow X_{[p]}$ is canonic and so is $(i_{[p]})^*\colon (X_{[p]})^* = (X_{[p]})' = (X^p)_{[p]} \hookrightarrow X^*$, since X is order continuous by Proposition 1.2(2), so is $X_{[p]}$. Then, by uniqueness of the extension, we have that $(T^p)_{[p]}\colon (X_{[p]})' \overset{(i_{[p]})'}{\hookrightarrow} X^* \overset{T^*}{\to} E^*$. ■

Remark 2.6. Let us comment the factorization of $T^2\colon X^2 \to E^*$. Under the requirements of Lemma 2.5. On one hand, the previous Lemma 2.5 asserts that $T^2 = (T^2)_{[2]} \circ i_{[2]}$. On the other hand by [20, Prop. 3.1] X^2 is 2-convex. If X is 4-convex, hence $X_{[2]}$ is 2-convex, and so $(X^2)_{[2]} = (X_{[2]})'$ is 2-concave (see e.g. [4, Lem. 2]). Therefore $i_{[2]}\colon X^2 \to L^2 \to (X^2)_{[2]}$ (see [14, Cor. 1.f.15(iii)]). In consequence T^2 factors through a Hilbert space. In general, when X is (pp')-convex, and T is a positive operator, i.e $Tx \geq 0$ for every $x \geq 0$, by means of [4, Cor. 5] we can conclude that T^p factors through an L^p-space (see [5, Sect. 18.6]).

Example 2.7. Let $(\mathbb{R}, \Sigma, \mu)$ be a finite measure space. Let φ be an Orlicz function, i.e. convex, continuous, increasing and unbounded, defined on $[0, \infty)$, so that $\varphi(0) = 0$. The Orlicz space is defined by

$$L^\varphi := \{f \in L^0 : \inf\{\lambda > 0 : \int_{\mathbb{R}} \varphi(|f(\omega)|/\lambda)\, d\mu \leq 1\} < \infty\}\,.$$

The one-dimensional Riesz transform $R\colon L^\varphi \to L^\varphi$, defined by

$$(Rf)(x) := c \int_{\mathbb{R}} \frac{x-y}{|x-y|} f(y)\, d\mu(y)\,,$$

where $c := \frac{\Gamma(1/2)}{\pi^{1/2}}$, is continuous (see [9, Thm. 3.11]). Let $1 \leq p < \infty$ and let $L^\varphi := (L^{\varphi_0})^{2p}$. For a suitable φ_0, it is an Orlicz space again over a non-atomic, finite and positive measure. To be precise, thanks to [15, Thm. 4], φ_0 can be chosen as an Orlicz function such that (A): $(uv)^{2p} \leq \varphi_0(u) + \varphi(v)$ for every $u, v \geq 0$, and (B): $u^{1/(2p)} \leq \varphi^{-1}(u)\varphi^{-1}(u)$ for every $u \geq 0$. L^φ is Fatou and $2p$-convex, as we have seen. Let us choose $p = 1$. Then by [2, Prop. 5.3]

$$L^\varphi = (L^{\varphi_0})^2 \subseteq (L^{\varphi_0})' = L^{\varphi_1}$$

where φ_1 satisfies (A) and (B) for φ_0 and $2p = 1$. The Riesz transform $R\colon L^{\varphi_1} \to L^{\varphi_1}$ has Köthe 2-adjoint $R^2\colon L^\varphi \to L^{\varphi_0}$, and R^2 is again a Riesz transform. In virtue of the remark above, it factors through a Hilbert space.

Remark 2.8. In order to summarize the relations and spaces that we considering in this paper we present the following diagram, for $1 \leq q \leq p < \infty$ and $u, v > 1$. X is a Banach function space over a finite measure and E is a Banach space and $T\colon E \to X$ is an operator.

$$
\begin{tikzcd}
 & & & X' \arrow[dlll, "\kappa"'] \arrow[dll, "T'"] \arrow[drr, "T'"] \arrow[drrr, "\kappa"] & & & \\
X^* \arrow[r, "T^*"'] & E^* & X^p \arrow[l, "T^p"'] \arrow[ur, hook] \arrow[rr, hook] \arrow[d, hook, "k_{[p]}"] & & X^q \arrow[ul, hook'] \arrow[r, "T^q"] \arrow[d, hook', "k_{[q]}"'] & E^* & X^* \arrow[l, "T^*"] \\
 & & (X^p)_{[p]} \arrow[ul, dotted, "(T^p)_{[p]}"] \arrow[rr, hook] \arrow[d, equal] & & (X^q)_{[q]} \arrow[ur, dotted, "(T^q)_{[q]}"'] \arrow[d, equal] & & \\
 & (X_{[p]})^* \arrow[uul, hook, "i^*_{[p]}"] & (X_{[p]})' \arrow[l, hook', "\kappa"'] \arrow[uul, "(T_p)'"] \arrow[rr, hook] & & (X_{[q]})' \arrow[r, hook, "\kappa"] \arrow[uur, "(T_q)'"'] & (X_{[q]})^* \arrow[uur, hook', "i^*_{[q]}"'] & \\
 & & (X_{[p]})^u \arrow[ul, hook, "\kappa_p"] \arrow[u, hook] \arrow[rr, hook, shift left] & & (X_{[q]})^v \arrow[ll, hook, shift left] \arrow[u, hook'] \arrow[ur, hook', "\kappa_q"'] & &
\end{tikzcd}
$$

3 Some geometrical aspects

The geometric structure of the space of multipliers is strongly connected with the notions of concavity and convexity (see e.g. [14, Sect. 1.c, 1.d and 1.e]). Let us now study the p-convexity, q-concavity, type and cotype of the spaces X^r. Let us state a first corollary.

Corollary 3.1. *Let $1 \leq p < \infty$. Suppose that E is a p'-convex Banach lattice, X is a quasi-Banach function space that contains χ_Ω and X^p is order continuous. If $T^p\colon X^p \to E^*$ is positive, then it factors through the space L^p.*

Proof. On one hand, from [14, Prop. 1.d.4(iii)], E^* is p-concave then by [14, Prop. 1.d.9], T^p is p-concave too, since T^p is positive. On the other hand thanks to Lemma 1.3 we have that X^p is p-convex, thus Maurey-Rosenthal's Theorem (see e.g. [4, Cor. 5]) ensure us that T^p factors by L^p. ■

Hence in case that T' is positive and E is ∞-convex, e.g. if $E = L^\infty$, T' will be p-factorable for every $p \geq 1$, i.e. it factors through L^p, whenever $T'(X') \subseteq L^1$. It is clear, since in this case $T\colon L^\infty \to X$, and so $T'\colon X' \to L^1$.

Let $p \in [1, \infty]$, it is well-known that X^p is p-convex with constant 1. (see e.g. [20, Prop 3.1]). However, this result does not hold for the p-concavity. For instance $(L^p)^p = L^\infty$, which is not p-concave for $p < \infty$. This theorem sheds some new light on the p-concave case. In fact it is a generalization of [14, Prop. 1.d.4(i)], see also [12]. The proof is adapted from [18, Lem. 2.2], which is deduced directly from the definitions. Recall that if X has not weak unit, then we only can define a seminorm for X^Y ([15, Prop. 2]).

Theorem 3.2. *Let $1 \leq r, p, q \leq \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Given X and Y, two Banach function spaces, such that Y is r-concave and X is p-convex with weak unit, we have that X^Y is q-concave.*

Proof. Since X has weak unit, X^Y is a Banach function space and the definition of q-concavity can be applied. We assume without loss of generality that the involved concavity and convexity constants are equal to 1. On one hand, let us take $n \in \mathbb{N}$, $f_1, \ldots, f_n \in X^Y$ and $g_1, \ldots, g_n \in B_X$. Thanks to [15, Prop. 3] it is clear that $\ell^q = (\ell^p)^r$, so for an element $(\tau_i)_i \in \ell^q$

$$\|(\tau_i)_i\|_q = \Big(\sum_i |\tau_i|^q\Big)^{1/q} = \sup_{(\lambda_i)_i \in B_{\ell^p}} \Big(\sum_i |\lambda_i \tau_i|^r\Big)^{1/r}. \tag{5}$$

On the other hand, since X is p-convex with constant 1, if $(\lambda_i)_i \in B_{\ell^p}$, (note that $\lambda_i g_i \in X$) we have

$$\begin{aligned} \Big\|\Big(\sum_{i=1}^n |\lambda_i g_i|^p\Big)^{1/p}\Big\|_X &\leq \Big(\sum_{i=1}^n \|\lambda_i g_i\|_X^p\Big)^{1/p} = \Big(\sum_{i=1}^n |\lambda_i|^p \|g_i\|_X^p\Big)^{1/p} \\ &\leq \Big(\sum_{i=1}^n |\lambda_i|^p\Big)^{1/p} \leq 1, \end{aligned} \tag{6}$$

hence $\big(\sum_{i=1}^n |\lambda_i g_i|^p\big)^{1/p} \in B_X$. So, applying (5), that Y is r-concave and (6), we obtain

$$\begin{aligned} \Big(\sum_{i=1}^n \|f_i g_i\|_Y^q\Big)^{1/q} &= \sup_{(\lambda_i)_i \in B_{\ell^p}} \Big(\sum_{i=1}^n |\lambda_i|^r \|f_i g_i\|_Y^r\Big)^{1/r} = \sup_{(\lambda_i)_i \in B_{\ell^p}} \Big(\sum_{i=1}^n \|\lambda_i g_i f_i\|_Y^r\Big)^{1/r} \\ &\leq \sup_{(\lambda_i)_i \in B_{\ell^p}} \Big\|\Big(\sum_{i=1}^n |\lambda_i g_i f_i|^r\Big)^{1/r}\Big\|_Y \leq \sup_{(\lambda_i)_i \in B_{\ell^p}} \Big\|\Big(\sum_{i=1}^n |\lambda_i g_i|^p\Big)^{1/p}\Big(\sum_{i=1}^n |f_i|^q\Big)^{1/q}\Big\|_Y \\ &\leq \sup_{g \in B_X} \Big\| g\,\Big(\sum_{i=1}^n |f_i|^q\Big)^{1/q}\Big\|_Y = \Big\|\Big(\sum_{i=1}^n |f_i|^q\Big)^{1/q}\Big\|_{X^Y}. \end{aligned} \tag{7}$$

Let $\varepsilon > 0$. Choose functions $\{g_1, \ldots, g_n\} \in B_X$ such that $\|f_i\|_{X^Y} \leq \|f_i g_i\|_Y + \varepsilon/(n^{1/q})$ for each $i = 1, \ldots, n$. Then applying (7)

$$\begin{aligned} \Big(\sum_{i=1}^n \|f_i\|_{X^Y}^q\Big)^{1/q} &\leq \Big(\sum_{i=1}^n (\|f_i g_i\|_Y + \frac{\varepsilon}{n^{1/q}})^q\Big)^{1/q} \\ &\leq \Big(\sum_{i=1}^n (\|f_i g_i\|_Y)^q\Big)^{1/q} + \Big(\sum_{i=1}^n (\frac{\varepsilon}{n^{1/q}})^q\Big)^{1/q} \\ &= \Big(\sum_{i=1}^n (\|f_i g_i\|_Y)^q\Big)^{1/q} + \varepsilon \leq \Big\|\Big(\sum_{i=1}^n |f_i|^q\Big)^{1/q}\Big\|_{X^Y} + \varepsilon, \end{aligned}$$

for every $\varepsilon > 0$, which yields us that X^Y is q-concave. ∎

Remark 3.3. The referee suggested the following alternative proof of theorem above under assumption that Y has Fatou property. Let $1 \leq p, q, r \leq \infty$ with $1/r = 1/p + 1/q$. If X is p-convex and Y is r-concave, both with constant 1, then Y is r'-convex and so by [11, Thm. 3] we get that $X \odot Y'$ is q'-convex because $1/q' = 1/r' + 1/p$, where $X \odot Y'$ denotes the pointwise product of Banach function spaces in the sense of the cited paper. Then by [11, Cor. 3] and Fatou property of Y we have with equality of the norms

$$(X \odot Y')' = X^{Y''} = X^Y,$$

which gives that X^Y is q-concave.

This theorem sometimes fails without the assumption of p-convexity for X.

Example 3.4. Let $1 \leq r, p, q < \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $X := L^s$ for $1 \leq r \leq s < p$, which is not p-convex, then $(L^s)^r = L^t$, where $\frac{1}{t} = \frac{1}{r} - \frac{1}{s}$ (Y is L^r in the previous theorem). Then $\frac{1}{t} = \frac{1}{r} - \frac{1}{s} = \frac{1}{q} + \frac{1}{p} - \frac{1}{s}$, and so $\frac{1}{t} - \frac{1}{q} = \frac{1}{p} - \frac{1}{s} < 0$ since $s < p$. Therefore, $q < t$, and so $(L^s)^r = L^t$ cannot be q-concave.

Remark 3.5. Let $1 \leq q < p \leq \infty$. For the case of Lebesgue spaces we have $(L^q)^p = \{0\}$, but this is not in general true. Take $r > q \geq 1$ and choose $s \geq 1$ so that $\frac{1}{s} = \frac{1}{q} - \frac{1}{r}$. Then, for $p \in (q, s)$ and $X := L^r$, we conclude that $X^q = L^s$ and so $(X^q)^p = L^t \neq \{0\}$, where $\frac{1}{t} = \frac{1}{p} - \frac{1}{s}$.

The following corollary provides conditions to obtain $(X^q)^p = \{0\}$ when $q < p$.

Corollary 3.6. *Let $1 \leq q < p \leq \infty$ and let $r > 1$ such that $\frac{1}{p} < \frac{1}{q} - \frac{1}{r}$. Let X be a r-convex Banach function with weak unit and based over a non-atomic measure. Then $(X^q)^p = \{0\}$.*

Proof. By the previous theorem X^q is s-concave for some $s \geq 1$ such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. Since $\frac{1}{p} < \frac{1}{q} - \frac{1}{r} = \frac{1}{s}$, we obtain that $s < p$ and the measure is non-atomic. Then the requirement (3) is satisfied, since L^p is p-convex, and $(X^q)^p = \{0\}$. ∎

To finish we will use the 2-Köthe dual space in order to find a simple characterization for Hilbert spaces. The proof of the following theorem uses the type and cotype inequalities for L^p spaces, which has type $\min\{2, p\}$ and cotype $\max\{2, p\}$ (see [5, Prop. 8.6] or [14, p. 73]). Recall that a Banach function AM-space such that satisfies $\|f \vee g\| = \max\{\|f\|, \|g\|\}$ for $f, g > 0$, where $(f \vee g)(\omega) := \max\{f(\omega), g(\omega)\}$. We show that we can provide a direct proof by using this abstract axiomatic definition, instead of writing the direct result for the case $X = L^\infty$. We will see in Corollary 3.8 that in fact, the AM-space involved is L^∞.

Theorem 3.7. *Let $1 \leq p \leq \infty$. Let X be a Banach function space with weak unit, which is an AM-space. Then X^p has type* $\min\{2, p\}$ *and cotype* $\max\{2, p\}$.

Proof. Since X has weak unit, X^Y is a Banach function space and the definition of type or cotype can be applied. The proof is divided in 2 parts.
Step 1: Type. Recall that the norm in X^p is $\|f\|_{X^p} = \sup\{\|fg\|_p : g \in B_X\}$. If $f_1, \dots, f_n \in X^p$ and $\varepsilon > 0$ we claim that there exists $g \in B_X$ such that

$$\left\| \sum_{k=1}^{n} r_k(t) f_k \right\|_{X^p} \leq \left\| g \sum_{k=1}^{n} r_k(t) f_k \right\|_p + \varepsilon^2, \qquad t \in [0,1], \tag{8}$$

where the election of $g \in B_X$ does not depends on the election of $t \in [0,1]$. Let us define $\psi_n \colon [0,1] \to X^p$ by

$$\psi_n(t) := \sum_{k=1}^{n} r_k(t) f_k .$$

By definition of the Rademacher functions, ψ_n has at most 2^n values, since r_k is defined on 2^k subdivisions of the same length of $[0,1]$ for $k = 1, \dots, n$. So, it is not hard to realize that r_n define the number of possible values of ψ_n. Let us select $t_j \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right[$ for $j = 0, \dots, 2^n - 1$, (e.g. $t_j := \frac{j}{2^n}$). Then $\psi_n(t) = \psi_n(t_j)$ for $t \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right[$. By definition, for each $j \in \{0, \dots, 2^n - 1\}$ there exists $g_j \in B_X$ such that $\|\psi_n(t)\|_{X^p} \leq \|g_j \psi_n(t)\|_p + \varepsilon^2$ for every $t \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right[$. Let us define $g := |g_0| \vee \cdots \vee |g_{2^n - 1}|$. Since, X is an AM-space we have that

$$\|g\|_X = \max\{\|g_1\|_X, \dots, \|g_{2^n-1}\|_X\} \leq 1 .$$

On the other hand $|g_j| \leq |g|$, thus $|g_j \psi_n(t_j)| \leq |g \psi_n(t_j)|$ for all $j = 0, \dots, 2^n - 1$. Then $|g_j \psi_n(t)| \leq |g \psi_n(t)|$ and $\|g_j \psi_n(t)\|_p \leq \|g \psi_n(t)\|_p$ for every $j = 0, \dots, 2^n - 1$, $t \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right[$. Let $t \in [0,1]$, then there exists j such that $\psi_n(t) = \psi_n(t_j)$. Therefore (8) holds.

Let us now compute taking into account that (1) is satisfied in L^p.

$$\begin{aligned}
\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) f_k \right\|_{X^p}^2 dt \right)^{1/2} &\leq \left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) g f_k \right\|_p^2 + \varepsilon^2 \, dt \right)^{1/2} \\
&\leq \left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) g f_k \right\|_p^2 dt \right)^{1/2} + \left(\int_0^1 \varepsilon^2 \, dt \right)^{1/2} \\
&\leq C \left(\sum_{k=1}^{n} \|g f_k\|_p^p \right)^{1/p} + \varepsilon \leq C \left(\sum_{k=1}^{n} \sup_{g \in B_X} \|g f_k\|_p^p \right)^{1/p} + \varepsilon \\
&\leq C \left(\sum_{k=1}^{n} \|f_k\|_{X^p}^p \right)^{1/p} + \varepsilon .
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can assert that X^p has the same type as L^p, i.e. $\min\{2, p\}$.

Step 2: Cotype. Let $f_1, \ldots, f_n \in X^p$. For each $k \in \{1, \ldots, n\}$, choose $0 \geq g_k \in B_X$ such that $\|f_k\|_{X^p} \leq \|g_k f_k\|_p + \frac{\varepsilon}{n^{1/p}}$. Let us define $g_0 := g_1 \vee \cdots \vee g_n$. Since X is an AM-space, $g_0 \in B_X$, $g_i \leq g_0$ and $g_i|f_i| \leq g_0|f_i|$, hence $\|g_i f_i\|_p \leq \|g_0 f_i\|_p$ for $i = 1, \ldots, n$. Therefore, by (2) for L^p we have that

$$\begin{aligned}\left(\sum_{k=1}^n \|f_k\|_{X^p}^p\right)^{1/p} &\leq \left(\sum_{k=1}^n \left(\|g_0 f_k\|_p + \frac{\varepsilon}{n^{1/p}}\right)^p\right)^{1/p} \leq \left(\sum_{k=1}^n \|g_0 f_k\|_p^p\right)^{1/p} + \varepsilon \\ &\leq C\left(\int_0^1 \left\|g_0 \sum_{k=1}^n r_k(t) f_k\right\|_p^2 dt\right)^{1/2} + \varepsilon \\ &\leq C\left(\int_0^1 \sup_{g \in B_X} \left\|g \sum_{k=1}^n r_k(t) f_k\right\|_p^2 dt\right)^{1/2} + \varepsilon \\ &\leq C\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) f_k\right\|_{X^p}^2 dt\right)^{1/2} + \varepsilon .\end{aligned}$$

Which proves that X^p has cotype $\max\{2, p\}$. ■

Corollary 3.8. *Let H be a Banach function space such that $\chi_\Omega \in H$. Then H is isomorphic to a Hilbert space if and only if there exists a Banach function AM-space X, such that $H = X^2$.*

Proof. Assume that H is isomorphic to a Hilbert space, hence $H = H^*$. On one hand, let us prove that H^2 is an AM-space. By Lemma 2.2(9) taking $X := H^2$ and $Y := H$, we have trivially that $H \subseteq (H^2)^2$. Thus, by Lemma 2.2(10) we have that

$$H \subseteq (H^2)^2 \subseteq L^2 , \tag{9}$$

hence Lemma 2.1(6) (taking $X = H$, $q = 1$ and $p = r = 2$), implies that

$$L^2 \subseteq H' \subseteq H^* = H . \tag{10}$$

Then, $H = L^2$, thus $H^2 = (L^2)^2 = L^\infty$, which is an AM-space. On the other hand, inclusions (9) and (10) state that $H = (H^2)^2$. So choose $X := H^2$.

For the converse, by the previous Theorem 3.7, we have that X^2 has type 2 and cotype 2. Applying Kwapień's Theorem we obtain the result. ■

The Nikishin's Theorem provides the last application of the paper. We recall that the vector measure associated to an operator between Banach function spaces $T\colon X \to Y$, where X is order continuous, is denoted by $m_T\colon \Sigma \to E$ and defined by $m_T(A) := T(\chi_A)$, for every $A \in \Sigma$. We refer the reader to [16, Chap. 4] for more information and main properties about this concept. Recall also that a Banach function AL-space is such that satisfies $\|f + g\| = \|f\| + \|g\|$ for $f, g > 0$ and $f \wedge g = \min\{f, g\} = 0$.

Corollary 3.9. *Let X and Y be two Banach function spaces over a positive measure, such that X is order continuous and $\chi_\Omega \in X$. Let $T\colon X \to Y$ be a p-th power factorable operator. If $L^1(m_T)$ is Fatou and is either q-concave for some $q < \infty$ or an AL-space, then the (range) extension $X \overset{T}{\to} Y \overset{i}{\hookrightarrow} L^0$ factorize through an $L^{\min\{2,p\},\infty}$-space.*

Proof. Assume first that $L^1(m_T)$ is q-concave for some $q < \infty$. If $p > 2$, T is also r-th power factorable for any $r < p$, then let us choose $1 \le p \le 2$. Thanks to Theorem 5.7 in [16], T is extended to the space $L^p(m_T)$, via the integration map $I^{(p)}_{m_T}$. We have that $L^p(m_T)$ is p-convex and $L^1(m_T)$ is q-concave, then $L^p(m_T)$ is also r-concave for an $r < \infty$, and then it has type $min\{2,p\}$ ([14, Prop. 1.f.3(ii)]). Then we can assert that $L^p(m_T)$ has type $\min\{2,p\}$. Finally, we apply the Nikishin's Theorem (see e.g. [21, Thm. III.H.6]), and obtain that $i \circ I^{(p)}_{m_T}$ factorize through a Lorentz space $L^{\max\{2,p\},\infty}$, hence so is $i \circ T$.

Suppose now that $L^1(m_T)$ is an AL-space. Then $L^1(m_T)'$ is an AM-space ([1, Thm. 10.15]). Then, since $L^1(m_T)'$ is Fatou by Lemma 2.2(7), we have

$$L^p(m_T) = (L^1(m_T)'')_{[1/p]} = \left((L^1(m_T)')_{[1/p]}\right)^p.$$

Then, applying Theorem 3.7 above we obtain that $L^p(m_T)$ has type $\min\{2,p\}$. Again Nikishin's Theorem gives the result. ∎

Notice that compactness of the integration map I_{m_T} implies that $L^1(m_T)$ is an L^1-space and so an AL-space ([17, Thm. 1 and 4]). Therefore, this gives an example of the corollary above. Other conditions to obtain that $L^1(m)$ is an AL-space can be found in [3].

Acknowledgment This paper is part of the Ph.D. thesis which I recently have presented in the Polytechnic University of Valencia, which has been supervised by the Professors F. Mayoral and E.A. Sánchez-Pérez, whom I am greatly grateful. I would like to thank the referee for carefully reading this manuscript and for the alternative proof of Theorem 3.2 (see Remark 3.3).

References

[1] Aliprantis, C.D. and Burkinshaw, O. *Locally Solid Riesz Spaces*. Academic Press, New York, 1978.

[2] Calabuig, J.M., Delgado, O., and Sánchez-Pérez, E.A. Generalized perfect spaces. *Indagationes Mathematicae*, 19(3):359–378, 2008.

[3] Curbera, G.P. When L^1 of a vector measure is an AL-space. *Pacific Journal of Mathematics*, 162(2):287–303, 1994.

[4] Defant, A. Variants of the Maurey-Rosenthal theorem for quasi Köthe function spaces. *Positivity*, 5:153–175, 2001.

[5] Defant, A. and Floret, K. *Tensor Norms and Operator functions*. North Holland, Mathematics studies 176, 1993.

[6] Diestel, J. and Uhl, Jr., J.J. *Vector Measures*. American Mathematical Society, Providence, 1977.

[7] Fernández, A., Mayoral, F., Naranjo, F., Sáez, C., and Sánchez-Pérez, E.A. Spaces of p-integrable functions with respect to a vector measure. *Positivity*, 10:1–16, 2006.

[8] Galdames Bravo, O. and Sánchez-Pérez, E.A. Optimal range theorems for operators with p-th power factorable adjoints. *Banach Journal in Mathematical Analysis*, 6(1):61–73, 2012.

[9] Kokilashvili, V.M. and Krbec, M. *Weighted Inequalities in Lorentz And Orlicz spaces*. World Scientific Publishing Co. Pte. Ltd., 1991.

[10] Kolwicz, P., Leśnik, K., and Maligranda, L. Pointwise multipliers of Calderón-Lozanowskiĭ. *Mathematische Nachrichten*, 286(8–9):876–907, 2013.

[11] Kolwicz, P., Leśnik, K., and Maligranda, L. Pointwise products of some Banach function spaces and factorization. arXiv:1211.3135v1 [math.FA] 13 Nov 2012. `http://arxiv.org/pdf/1211.3135.pdf`

[12] Krivine, J.L. Théorèmes de factorisation dans les espaces réticulés. Séminaire Maurey-Schwartz 1973-74, Exposés 22-23, École Polytechnique, Paris.

[13] Kwapień, S. Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Mathematicae*, 44:583–595, 1972.

[14] Lindenstrauss, J. and Tzafriri, L. *Classical Banach Spaces II*. Springer-Verlag, Berlin, 1979.

[15] Maligranda, L. and Persson, L.E. Generalized duality of some Banach function spaces. *Indagationes Mathematicae*, 51(3):323–338, 1989.

[16] Okada, S., Ricker, W.J., and Sánchez-Pérez, E.A. *Optimal Domain And Integral Extensions of Operators, Acting in Function Spaces*. Birkhäuser-Verlag AG, Basel-Boston-Berlin, 2008.

[17] Okada, S., Ricker, W.J., and Rodríguez-Piazza, L. Compactness of the integration operator associated with a vector measure. *Studia Mathematica*, 150:133–149, 2002.

[18] Palazuelos, C., Sánchez-Pérez, E.A., and Tradacete, P. Maurey-Rosenthal factorization of p-summing operators and Dodds-Fremlin domination. *Journal of Operator Theory*, 68(1):205–222, 2012.

[19] Pisier, G. *Factorization of Linear Operators and Geometry of Banach Spaces*. American Mathematical Society, Providence, 1987.

[20] Schep, A.R. Products and factors of Banach function spaces. *Positivity*, 14:301–319, 2010.

[21] Wojtaszczyk, P. *Banach spaces for analysts*. Cambridge University Press, Cambridge, 1991.

Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia
Camino de Vera s/n,
46022 Valencia. Spain.
email : orlando.galdames@gmail.com