Arithmetics on beta-expansions with Pisot bases over $F_q((x^{-1}))$

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Abstract

In this paper we consider finite β-expansions in the field of formal series with Pisot basis β. We are studying the arithmetic operations on β-expansions and provide bounds on the number of fractional digits arising in multiplication for arbitrary β-polynomials noted $L_{\odot}$. This value is given explicitly for families of Pisot basis. The last part of this paper is devoted to quadratic Pisot series where we will give the exact value for $L_{\odot}$.

1 Introduction

The β-expansions of real numbers were first introduced by A. Rényi [11]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors. Let $\beta > 1$ be a real number and let $[\beta]$ be the integer part of β. The β-expansion of a real number $x \in [0, 1)$ is defined as the sequence

$$d_\beta(x) = (x_i)_{i\geq 1} = x_1 x_2 x_3 \ldots$$

with values in $\{0, 1, \ldots, [\beta]\}$ produced by the β-transformation $T_\beta : x \mapsto \beta x$ (mod1) as follows:

$$x_i = [\beta T_\beta^{i-1}(x)].$$

It is easy to see that $x = \sum\limits_{i\geq 1} x_i \beta^{-i}$.

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For $x \geq 1$, we consider the integer n such that $\beta^n \leq x < \beta^{n+1}$. So, we have $\frac{x}{\beta^{n+1}} < 1$ and then we can represent x by shifting $d_\beta(\frac{x}{\beta^{n+1}})$ by $n+1$ digits to the left, that is, we can write $d_\beta(\frac{x}{\beta^{n+1}}) = (y_i)_{i\geq 1}$ and therefore we define the β-expansion of x by

$$d_\beta(x) = (x_i)_{i\geq -n} = x_{-n} \cdots x_0 \bullet x_1 x_2 \cdots$$

where $y_i = x_{i-n-1}$ and then we have

$$x = \sum_{i\geq -n} x_i \beta^{-i} = \sum_{i=-n}^{0} x_i \beta^{-i} + \sum_{i\geq 1} x_i \beta^{-i}.$$

The β-integer part of x is $[x]_\beta = \sum\limits_{i=-n}^{0} x_i \beta^{-i}$ and the β-fractional part is $\{x\}_\beta = \sum\limits_{i\geq 1} x_i \beta^{-i}$. This is a natural generalization for the expansion in integers basis. A β-expansion is finite if $(x_i)_i$ is eventually 0. It is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ for all $k \geq m$; if $x_k = x_{k+p}$ holds for all k, then it is purely periodic. We denote by

$$\mathrm{Fin}(\beta) = \{x \in \mathbb{R}_+ : d_\beta(x) \text{ is finite}\}.$$

It was proved in [1] that if $\mathbb{N} \subset \mathrm{Fin}(\beta)$ then β is a Pisot number, that is, a real algebraic integer greater than 1 with all conjugates strictly inside the unit circle. Note that it is not the case if we have only $d_\beta(1)$ finite. Let $\mathbb{Z}[\beta]$ be the smallest ring containing $\mathbb{Z}$ and β. We denote by $\mathbb{Z}[\beta]_{\geq 0}$ the non negative elements of $\mathbb{Z}[\beta]$. We say that the number x satisfies the finiteness property if:

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}.$$

This property was introduced by Frougny and Solomyak [6]. They showed that if β satisfies the finiteness property then β is a Pisot number. Note that there are Pisot numbers without the finiteness property, especially, all numbers β such that $d_\beta(1)$ is infinite.

The set of β-integers, denoted by $\mathbb{Z}_\beta$, is the set of real numbers x such that $\{|x|\}_\beta = 0$. The sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$ are not stable under usual operations like addition and multiplication. Although, it is sometimes useful in computer science to consider this operation in β-arithmetics. That's why, it is important to study what fractional parts may appear as a result of addition and multiplication of β-integers.

The notations $L_\oplus$ and $L_\odot$ are introduced in [6]. They represent the maximal possible finite length of the β-fractional parts which may appear when one adds or multiplies two β-integers. Consider the sets:

$$S = \{n \in \mathbb{N} : \forall x, y \in \mathbb{Z}_\beta, x + y \in \mathrm{Fin}(\beta) \Longrightarrow \beta^n(x+y) \in \mathbb{Z}_\beta\}$$

and

$$P = \{n \in \mathbb{N} : \forall x, y \in \mathbb{Z}_\beta, xy \in Fin(\beta) \Longrightarrow \beta^n(xy) \in \mathbb{Z}_\beta\}$$

and define

$$L_{\oplus} = \begin{cases} \min S & \text{if} \quad S \neq \emptyset \\ +\infty & \text{if} \quad S = \emptyset \end{cases} \quad \text{and} \quad L_{\odot} = \begin{cases} \min P & \text{if} \quad P \neq \emptyset \\ +\infty & \text{if} \quad P = \emptyset \end{cases}$$

Many authors are interested in the case where $L_{\oplus}$ and $L_{\odot}$ are finite. Indeed, if the sum or the product of two β-integers belongs to $\mathrm{Fin}(\beta)$, then the length of the β-fractional part of this sum or product is bounded by a constant which only depends on β. In this case, one can decide whether a given improper expansion can be renormalized into a finite or an ultimately periodic expansion, in the sense that, if during the renormalisation process one gets a β-fractional part of the length greater than $L_{\oplus}$, then, the improper expansion corresponds to a real number that does not belong to $\mathrm{Fin}(\beta)$. Conversely, if the set of the length sums of two β-integers is unbounded, then performing arithmetics in $\mathbb{Z}_{\beta}$ will be very difficult if not impossible, since one can not compute in a finite time any operation on β-integers.

C. Frougny and B. Solomyak in [6] showed that $L_{\oplus}$ is finite when β is a Pisot number. The case of Pisot quadratic unit numbers has been studied in [5] where the authors gave exact values for $L_{\oplus}$ and $L_{\odot}$, when $\beta > 1$ is a solution either of the equation $x^2 = mx - 1$, $m \in \mathbb{N}$, $m \geq 3$ or of the equation $x^2 = mx + 1$, $m \in \mathbb{N}$. They showed that in the first case $L_{\oplus} = L_{\odot} = 1$ and in the second case $L_{\oplus} = L_{\odot} = 2$. In [7], the authors have generalized the last results to other quadratic Pisot numbers. However, when β is of algebraic higher degree, it is a difficult problem to compute the exact value of $L_{\oplus}$ or $L_{\odot}$ and even to compute upper and lower bounds for these two quantities. Several examples are studied in [2], where a method is described in order to compute upper bounds for $L_{\oplus}$ and $L_{\odot}$ for Pisot numbers satisfying additional algebraic properties. For example, in the Tribonacci case, that is, when β is the positive root, of the polynomial $x^3 - x^2 - x - 1$, we have $L_{\oplus} = 5$ and $L_{\odot}$ is still unknown until now. However, it is only proven in [2] that $4 \leq L_{\odot} \leq 5$.

In [4], J. Bernat determinate the exact value of $L_{\oplus}$ for several cases of cubic Pisot unit numbers.

In this paper, we study a similar concepts for the field of formal series over a finite field, i.e the arithmetic operations on β-expansions and the bounds on the number of fractional digits arising in multiplication for arbitrary β-polynomials noted $L_{\odot}$. It is well known that the field of formal power series over finite fields has a lot of properties in common to number fields (the finite extension of $\mathbb{Q}$). The paper is in this direction and showed many properties which seem difficult or impossible to show in the number field case. It is organized as follows: In Section 2, we define the field of formal power series over finite field as well as the analog to Pisot and Salem numbers. We will also define the β-expansion algorithm for formal power series. In Section 3, we study the arithmetic operations on β-expansions and provide bounds on the number of fractional digits arising in multiplication for arbitrary β-polynomials noted $L_{\odot}$ with Pisot basis. Section 4 is devoted to give explicit values of $L_{\odot}$ in quadratic Pisot basis.

2 β-expansions in $\mathbb{F}_q((x^{-1}))$

Let $\mathbb{F}_q$ be a finite field with q elements, $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$, $\mathbb{F}_q(x)$ the field of rational functions and $\mathbb{F}_q((x^{-1}))$ the field of formal power series of the form:

$$f = \sum_{k=-\infty}^{\ell} f_k x^k, \quad f_k \in \mathbb{F}_q$$

where

$$\ell = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{if} \quad f \neq 0, \\ -\infty & \text{if} \quad f = 0. \end{cases}$$

We define the absolute value $|f| = q^{\deg f}$. Thus, $\mathbb{F}_q((x^{-1}))$, equipped with this absolute value, is a complete metric space, it is the completion of $\mathbb{F}_q(x)$. Since the above absolute value is not archimedean, then it fulfills the strict triangle inequality:

$$|f+g| \leq \max(|f|,|g|) \quad \text{and} \quad |f+g| = \max(|f|,|g|) \quad \text{if} \quad |f| \neq |g|.$$

Consider $f \in \mathbb{F}_q((x^{-1}))$ and define the polynomial part $[f] = \sum_{k=0}^{\ell} f_k x^k$ where the empty sum is defined to be zero. Thus, $[f] \in \mathbb{F}_q[x]$ and $f - [f] \in M_0$ where $M_0 = \{f \in \mathbb{F}_q((x^{-1})) : \ |f| < 1\}$.

Since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, then any algebraic element over $\mathbb{F}_q[x]$ can be valuated (see [10]). However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed, such an element is not necessarily a formal power series.

An element $\beta \in \mathbb{F}_q((x^{-1}))$ is called a Pisot (resp a Salem) element if it is an algebraic integer over $\mathbb{F}_q[x]$ with $|\beta| > 1$ and $|\beta_j| < 1$ for all conjugates β_j (resp $|\beta_j| \leq 1$ and there exist at least one conjugate β_k with $|\beta_k| = 1$). P. Bateman and A. L. Duquette [3] characterized the Pisot and Salem elements in $\mathbb{F}_q((x^{-1}))$:

Theorem 2.1. *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and*

$$P(y) = y^n - A_{n-1}y^{n-1} - \cdots - A_0, \ \ A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then

(*i*) *β is a Pisot element if and only if $|A_{n-1}| > \max_{2 \leq j \leq n} |A_{n-j}|$.*

(*ii*) *β is a Salem element if and only if $|A_{n-1}| = \max_{2 \leq j \leq n} |A_{n-j}|$.*

Let $\beta, f \in \mathbb{F}_q((x^{-1}))$ where $|\beta| > 1$ and $f \in M_0$. A representation in base β (or β-representation) of f is a sequence $(d_i)_{i \geq 1}$, $d_i \in \mathbb{F}_q[x]$, such that

$$f \ = \ \sum_{i \geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f and noted $d_\beta(f)$. It is obtained by using the β-transformation T_β in M_0 which is given by the mapping:

$$\begin{array}{rcl} T_\beta : M_0 & \longrightarrow & M_0 \\ f & \longmapsto & \beta f - [\beta f]. \end{array}$$

The β-expansion of f is $d_\beta(f) = (d_i)_{i\geq 1}$ where $d_i = [\beta T_\beta^{i-1}(f)]$. We note that $d_\beta(f)$ is finite if and only if there is a $k \geq 0$ with $T_\beta^k(f) = 0$, $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) for which $T_\beta^{p+s}(f) = T_\beta^p(f)$. If $f \in M_0$ and $d_\beta(f) = (d_i)_{i\geq 1}$, we often write $f = 0 \bullet d_1 d_2 d_3 \cdots$.

As in the real case, we can define the β-expansion $d_\beta(f) = (d_i)_{i\geq -n}$ for any $f \in \mathbb{F}_q((x^{-1}))$:

$$f = \sum_{i\geq -n} d_i \beta^{-i}, \quad d_{-n} \neq 0. \tag{2.1}$$

In this case have $\deg_\beta(f) = n$ and if $d_\beta(f)$ is finite we define $\mathrm{ord}_\beta(f)$ as $\mathrm{ord}_\beta(f) = -\ell$ where ℓ is the bigger integer such that $d_\ell \neq 0$, i.e the smallest exponent in β appearing in the β-expansion of f.

Using this last notion, we define the set of β-polynomials as follow:

$$(\mathbb{F}_q[x])_\beta = \{f \in \mathbb{F}_q((x^{-1})) : \mathrm{ord}_\beta(f) \leq 0\}.$$

In the sequel, we will use the following notation:

$$\mathrm{Fin}(\beta) = \{f \in \mathbb{F}_q((x^{-1})) : d_\beta(f) \text{ is finite}\}.$$

Remark 2.1. *In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if $z, w \in \mathbb{F}_q((x^{-1}))$, we have $d_\beta(z + w) = d_\beta(z) + d_\beta(w)$ digitwise. We have also $d_\beta(cf) = c d_\beta(f)$ for every $c \in \mathbb{F}_q$.*

3 Study of $L_\odot$

First, we need to recall some results given in [8, 9, 12]. Let β be an algebraic formal power series with algebraic degree d such that $|\beta| > 1$. Then we have

Theorem 3.1. *[9] β is a Pisot series if and only if $d_\beta(1)$ is finite.*

Theorem 3.2. *[9] An infinite sequence $(d_j)_{j\geq 1}$ is the β-expansion of $f \in M_0$ if and only if it is a β-representation of f and $|d_j| < |\beta|$ for $j \geq 1$.*

Theorem 3.3. *[8] β is a Pisot series of algebraic degree d if and only if the β-expansion of $(x^{\deg(\beta)})$ is finite and $\mathrm{ord}_\beta(x^{\deg(\beta)}) = 1 - d$.*

Theorem 3.4. *[12] β is a Pisot series if and only if $\mathrm{Fin}(\beta) = \mathbb{F}_q[x, \beta^{-1}]$.*

Lemma 3.1. *[9] Let $P(Y) = A_n Y^n - A_{n-1} Y^{n-1} - \cdots - A_0$ where $A_i \in \mathbb{F}_q[x]$, for $i = 1, \ldots, n$. Then P admits a unique root in $\mathbb{F}_q((x^{-1}))$ with absolute value > 1 and all other roots are with absolute value < 1 if and only if $|A_{n-1}| > |A_i|$ for $i \neq n - 1$.*

Now we can define, by analogy with the real case, the quantity $L_\odot$ as follows:

$$L_\odot = \begin{cases} \min\ P & \text{if } P \neq \emptyset, \\ \infty & \text{if } P = \emptyset. \end{cases}$$

where

$$P = \{n \in \mathbb{N} : \forall p_1, p_2 \in (\mathbb{F}_q[x])_\beta,\ p_1.p_2 \in Fin(\beta) \Longrightarrow \beta^n(p_1.p_2) \in (\mathbb{F}_q[x])_\beta\}.$$

Note that $L_\odot$ designates the maximal finite shift after the comma for the product of two β-polynomials.

Remark 3.5.

1. *In the case of formal series the quantity $L_\oplus$ is not interesting, because we know that the sum of two β-polynomials is also a β-polynomial.*
2. *The case:* $\deg(\beta) = 1$ *is trivial because the product of two β-polynomials is a β-polynomial, that is, $L_\odot = 0$.*

To compute $L_\odot$ for the families of basis β, we propose the following quantitative study over the order in basis β of polynomials.

Lemma 3.6. *Let β be a Pisot series with minimal polynomial*

$$P(Y) = Y^d - A_{d-1}Y^{d-1} - A_{d-2}Y^{d-2} - \cdots - A_0.$$

If $d_\beta(x^k) = d^k_{-n_k} \ldots d^k_0 \bullet d^k_1 \ldots d^k_{\ell_k}$ where $d^k_{\ell_k} \neq 0$, then

$$\deg(d^k_{\ell_k}) \geq \deg(A_0)\ \textit{for all}\ k \geq m = \deg(\beta).$$

Proof. We proceed by induction on $k \geq m$. For $k = m$, let c be the dominant coefficient of A_{d-1}, so we have

$$x^m = c^{-1}\beta - (c^{-1}A_{d-1} - x^m) - \beta^{-1}c^{-1}A_{d-2} - \cdots - \beta^{1-d}c^{-1}A_0. \tag{3.1}$$

According to Theorem 3.2 and Lemma 3.1, the equality (3.1) is the β-expansion of x^m. Therefore, we have $d^m_{\ell_m} = -c^{-1}A_0$, hence, $\deg(d^m_{\ell_m}) = \deg(A_0)$.

Now, assume that $\deg(d^k_{\ell_k}) \geq \deg(A_0)$ and $d_\beta(x^k) = (d^k_i)_{-n_k \leq i \leq \ell_k}$ with $d^k_{\ell_k} \neq 0$. We have

$$x^k = d^k_{-n_k}\beta^{n_k} + \cdots + d^k_0 + d^k_1\beta^{-1} + \cdots + d^k_{\ell_k}\beta^{-\ell_k}.$$

Therefore, the equality

$$x^{k+1} = xd^k_{n_k}\beta^{n_k} + \cdots + xd^k_0 + xd^k_1\beta^{-1} + \cdots + xa^k_{\ell_k}\beta^{-\ell_k} \tag{3.2}$$

is a β-representation of x^{k+1} which is not necessary the β-expansion of x^{k+1}. Thus, we consider the set $H = \{0 \leq i \leq \ell_k : \deg(xd^k_i) = m\}$ and we have to distinguish two cases:

Case 1: $H = \emptyset$. According to Theorem 3.2, we have $d_\beta\left(x^{k+1}\right) = \left(xd^k_i\right)_{-n_k \leq i \leq \ell_k}$ and therefore we have $d^{k+1}_{\ell_{k+1}} = xd^k_{\ell_k}$. Hence $\deg\left(d^{k+1}_{\ell_{k+1}}\right) = \deg\left(xd^k_{\ell_k}\right) \geq \deg(A_0)$.

Case 2: $H \neq \emptyset$. Let $h = \sup H$ and $\alpha = -c^{-1}\gamma$ where γ is the dominant coefficient of d^k_h, for that we should make this discussion.

(a) If $h+d-1 > \ell_k$, then $\ell_{k+1} = h+d-1$ and $d^{k+1}_{\ell_{k+1}} = \alpha A_0$. So, $\deg(d^{k+1}_{\ell_{k+1}}) = \deg(A_0)$.

(b) If $h+d-1 < \ell_k$, then $h \neq \ell_k$, $\ell_{k+1} = \ell_k$ and $d^{k+1}_{\ell_{k+1}} = xd^k_{\ell_k}$. Thus, $\deg(d^{k+1}_{\ell_{k+1}}) \geq \deg(A_0)$.

(c) If $h+d-1 = \ell_k$, then $h \neq \ell_k$, $\ell_{k+1} = \ell_k$ and $d^{k+1}_{\ell_{k+1}} = xd^k_{\ell_k} + \alpha A_0$. Therefore, we obtain $\deg(d^{k+1}_{\ell_{k+1}}) \geq \deg(A_0)$. ■

Theorem 3.7. *Let β be a Pisot series. Then we have for every $k \geq 0$*

$$(1-d) + \mathrm{ord}_\beta(x^k) \leq \mathrm{ord}_\beta(x^{k+1}) \leq \mathrm{ord}_\beta(x^k).$$

Proof. Let β be a Pisot series and assume that $d_\beta(x^k) = d^k_{-n_k} \ldots d^k_0 \bullet a^k_1 \ldots d^k_{\ell_k}$ ($d^k_{\ell_k} \neq 0$). Obviously we have

$$x^k = d^k_{-n_k}\beta^{n_k} + \cdots + d^k_0 + \frac{d^k_1}{\beta} + \cdots + \frac{d^k_{\ell_k}}{\beta^{\ell_k}}.$$

Thus

$$x^{k+1} = xd^k_{-n_k}\beta^{n_k} + \cdots + xd^k_0 + \frac{xd^k_1}{\beta} + \cdots + \frac{xd^k_{\ell_k}}{\beta^{\ell_k}}.$$

Consider $H = \{0 \leq i \leq \ell_k : \deg(xd^k_i) = m\}$. As before we have to distinguish two cases:

Case 1: $H = \emptyset$. So, $d^{k+1}_{\ell_{k+1}} = xd^k_{\ell_k}$ and therefore $\mathrm{ord}_\beta(x^{k+1}) = \mathrm{ord}_\beta(x^k) = -\ell_k$.

Case 2: $H \neq \emptyset$. As in proof of Lemma 3.6, we consider h, α and the following subcases:

(a) If $h+d-1 > \ell_k$, then $\ell_{k+1} = h+d-1$ and $\mathrm{ord}_\beta(x^{k+1}) = -(h+d-1) \leq -\ell_k = \mathrm{ord}_\beta(x^k)$. On the other hand, since $h \leq \ell_k$, we have $\mathrm{ord}_\beta(x^{k+1}) = -h + (1-d) \geq -\ell_k + (1-d) = \mathrm{ord}_\beta(x^k) + (1-d)$.

(b) If $h+d-1 < \ell_k$, then $\ell_{k+1} = \ell_k$ and $\mathrm{ord}_\beta(x^{k+1}) = \mathrm{ord}_\beta(x^k) = -\ell_k \geq \mathrm{ord}_\beta(x^k) + (1-d)$.

(c) If $h+d-1 = \ell_k$, then $\ell_{k+1} = \ell_k$ and $a^{k+1}_{-\ell_{k+1}} = xa^k_{-\ell_k} + \alpha A_0 \neq 0$ because by Lemma 3.6, we have $\deg(d^k_{\ell_k}) \geq \deg(A_0)$. Thus, $\mathrm{ord}_\beta(x^{k+1}) = \mathrm{ord}_\beta(x^k) = -\ell_k \geq \mathrm{ord}_\beta(x^k) + (1-d)$. ■

Combining Theorem 3.7 (the sequence $(\mathrm{ord}_\beta(x^k)_{k\geq 0}$ is decreasing) with Remark 2.1, we deduce:

Corollary 3.8. *Let β be a Pisot series such that $m = \deg(\beta) \geq 1$. Then $L_\odot$ is finite and*

$$L_\odot = \mathrm{ord}_\beta(x^{2m-2}).$$

Now, we continue our study by giving bounds on $L_\odot$ when β is a Pisot series.

Theorem 3.9. *Let β be a Pisot series of algebraical degree d and let $m = \deg(\beta) > 1$. Then $L_\odot$ is finite and*

$$(d-1) \leq L_\odot \leq (d-1)(m-1).$$

Proof. By induction on $k \geq m$, it follows that

$$(k-m+1)(d-1) \leq \mathrm{ord}_\beta(x^k) \leq (d-1).$$

In fact, the case $k = m$ is contained in Theorem 3.3 and the induction step is contained in Theorem 3.7. ∎

To show that the two bounds of Theorem 3.9 are reached, we propose the two following propositions that ensue from Theorem 3.3.

Proposition 3.10. *Let β be a Pisot series of minimal polynomial*

$$P(Y) = Y^d + A_{d-1}Y^{d-1} + \cdots + A_0,$$

with $\deg(A_0) = \deg(\beta) - 1 = m - 1$. *Then*

$$L_\odot = (d-1)(m-1).$$

Indeed, in this case for all positive integer s, we have $\mathrm{ord}_\beta(x^{m+s}) = (s+1)(1-d)$.

Proposition 3.11. *Let β be a Pisot series of minimal polynomial*

$$P(Y) = Y^d + A_{d-1}Y^{d-1} + \cdots + A_0,$$

with $A_i \in \mathbb{F}_q$ for $i \neq d-1$. Then

$$L_\odot = (d-1).$$

Indeed, we know that

$$x^m = c^{-1}\beta - (c^{-1}A_{d-1} - x^m) - \frac{c^{-1}A_{d-2}}{\beta} - \cdots - \frac{c^{-1}A_0}{\beta^{d-1}}.$$

So, for all $0 < h < m-1$, we have

$$x^{m+h} = x^h c^{-1}\beta - x^h(c^{-1}A_{d-1} - x^m) - \frac{c^{-1}A_{d-2}x^h}{\beta} - \cdots - \frac{c^{-1}A_0 x^h}{\beta^{d-1}}.$$

Hence $\mathrm{ord}_\beta(x^{m+h}) \geq 1-d$. On the other hand, according to Theorem 3.7, we obtain $\mathrm{ord}_\beta(x^{m+h}) \leq \mathrm{ord}_\beta(x^m) = 1-d$.

4 $L_\odot$ for quadratic Pisot series

In this section, we give the exact value for all quadratic Pisot basis. Therefore, in the special case of quadratic Pisot unit series, it is easy to deduce the following corollary from Proposition 3.11.

Corollary 4.1. *Let β be a quadratic Pisot unit series. Then $L_\odot = 1$.*

So far, we are interested in results for $L_\odot$ for general algebraic series β. Hence, we shall focus on quadratic Pisot series. For this case, we give the exact value for $L_\odot$. Before giving this value, we propose the following quantitative study over the order in base β of polynomials.

Theorem 4.2. *Let β be a quadratic Pisot series of minimal polynomial $P(Y) = Y^2 + AY + D$ with $\deg(\beta) = m \geq 1$ and $\deg(D) = s$. Then for any $k \geq m$, we have*

$$d_\beta(x^k) = d^k_{-n_k} \dots d^k_0 \bullet d^k_1 \dots d^k_{\ell_k}, \quad (d^k_{\ell_k} \neq 0)$$

where

$$\mathrm{ord}_\beta(x^k) = \ell_k = \left[\frac{k-s}{m-s}\right] \text{ and } \deg(d^k_{\ell_k}) = s + \overline{(k-s)}$$

with $\overline{(k-s)}$ is the rest of the Euclidean division of $(k-s)$ by $(m-s)$.

Proof. Let c be the dominant coefficient of A, so from Theorem 3.3, we have

$$cx^m = \beta - (A - cx^m) - \frac{D}{\beta}.$$

Now, let $D_1 = -c^{-1}D$, clearly we have $\deg(D_1) = \deg(D)$. We will proceed by induction on $k \geq m$. The result holds for $k = m$, in fact, we have $\mathrm{ord}_\beta(x^m) = -\left[\frac{m-s}{m-s}\right] = -1$ and $d^m_1 = D_1$. Assume that $\mathrm{ord}_\beta(x^k) = -\left[\frac{k-s}{m-s}\right] = -\ell_k$ and $\deg(d^k_{\ell_k}) = s + \overline{(k-s)}$. We have

$$x^k = d^k_{-n_k}\beta^{n_k} + \dots + d^k_0 + \frac{d^k_1}{\beta} + \dots + \frac{d^k_{\ell_k}}{\beta^{\ell_k}}.$$

Therefore,

$$x^{k+1} = xd^k_{-n_k}\beta^{n_k} + \dots + xd^k_0 + \frac{xd^k_1}{\beta} + \dots + \frac{xd^k_{\ell_k}}{\beta^{\ell_k}}$$

is a β-representation of x^{k+1}, then, we distinguish two cases:

Case 1: $\deg(d^k_{\ell_k}) = m - 1$.

On one hand, in this case, we have $\mathrm{ord}_\beta(x^{k+1}) = \mathrm{ord}_\beta(x^k) - 1$ and $d^{k+1}_{\ell_{k+1}} = \alpha D_1$ where α is the dominant coefficient of $d^k_{\ell_k}$. On the other hand, we have $\deg(d^k_{\ell_k}) = m - 1$ and that means $\overline{k-s}$ is equal to $(m - s - 1)$. Thus,

$$\left[\frac{k+1-s}{m-s}\right] = \left[\frac{k-s}{m-s}\right] + 1 \text{ and } \deg(d^{k+1}_{\ell_{k+1}}) = s + \overline{k+1-s} = s.$$

Case 2: $\deg(d^k_{\ell_k}) < m-1$.

In this case we have $\mathrm{ord}_\beta(x^{k+1}) = \mathrm{ord}_\beta(x^k)$ and exactly

$$d^{k+1}_{\ell_{k+1}} = \begin{cases} xd^k_{\ell_k} & \text{if } \deg(d^k_{\ell_k-1}) < m-1, \\ \gamma D_1 + xd^k_{\ell_k} & \text{if } \deg(d^k_{\ell_k-1}) = m-1, \end{cases}$$

where γ is the dominant coefficient of $d^k_{\ell_k-1}$. Hence $\deg(d^{k+1}_{\ell_{k+1}}) = 1+\deg(d^k_{\ell_k})$. Since we have $\deg(d^k_{\ell_k}) < m-1$, so $\overline{k-s} < m-s-1$. Thus,

$$\left[\frac{k-s}{m-s}\right] = \left[\frac{k+1-s}{m-s}\right]$$

and

$$s + (\overline{k+1-s}) = 1 + (s + (\overline{k-s})). \qquad \blacksquare$$

Theorem 4.3. *Let β be a quadratic Pisot series of minimal polynomial $P(Y) = Y^2 + AY + D$ with $\deg(\beta) = m \geq 1$ and $\deg(D) = s$. Then*

$$L_\odot = 1 + \left[\frac{m-2}{m-s}\right].$$

Proof. We know from Corollary 3.8, that $L_\odot = \mathrm{ord}_\beta(x^{2m-2})$. So, by Theorem 4.2, we get

$$L_\odot = \left[\frac{2m-2-s}{m-s}\right] = 1 + \left[\frac{m-2}{m-s}\right]. \qquad \blacksquare$$

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