Existence of periodic solutions for a damped vibration problem with $(q, p)-$Laplacian

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Abstract

In this paper, some existence theorems are obtained for periodic solutions of a damped vibration problem with $(q, p)-$Laplacian by using variational methods. Our results extend some results in some known literatures.

1. Introduction and Main results

In this paper, we consider the following dynamical system

$$\begin{cases} \frac{d}{dt}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)) + g(t)|\dot{u}_1(t)|^{q-2}\dot{u}_1(t) = \nabla_{u_1}F(t, u_1(t), u_2(t)), \ \text{a.e. } t \in [0, T] \\ \frac{d}{dt}(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)) + g(t)|\dot{u}_2(t)|^{p-2}\dot{u}_2(t) = \nabla_{u_2}F(t, u_1(t), u_2(t)), \ \text{a.e. } t \in [0, T] \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \end{cases} \tag{1.1}$$

where

$$u(t) = (u_1(t), u_2(t)) = (u_1^1(t), u_1^2(t), \cdots, u_1^N(t), u_2^1(t), u_2^2(t), \cdots, u_2^N(t))^\tau,$$

$1 < p < \infty$, $1 < q < \infty$, $T > 0$, $g \in L^\infty(0, T; \mathbb{R})$, $G(t) = \int_0^t g(s)ds$, $G(T) = 0$ and $F : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

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(A) *$F(t,x)$ is measurable in t for every $x = (x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and continuously differentiable in (x_1, x_2) for a.e. $t \in [0,T]$, and there exist $a_1, a_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that*

$$|F(t,x_1,x_2)|, \quad |\nabla_{x_1} F(t,x_1,x_2)|, \quad |\nabla_{x_2} F(t,x_1,x_2)| \leq [a_1(|x_1|) + a_2(|x_2|)]b(t)$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Moreover, we also consider the following $p-$Laplacian system

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + g(t)|\dot{u}(t)|^{p-2}\dot{u}(t) = \nabla F(t,u(t)), & \text{a.e. } t \in [0,T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.2}$$

where $u \in \mathbb{R}^N, 1 < p < \infty$, $T > 0$, $g \in L^\infty(0,T;\mathbb{R})$, $G(t) = \int_0^t g(s)ds$, $G(T) = 0$ and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

$(A)'$ *$F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that*

$$|F(t,x)|, \quad |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

When $p = q = 2$ and $F(t,x_1,x_2) = F_1(t,x_1)$, it has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods (see [1-8]). Many solvability conditions are given, such as the coercive condition (see [1]), the periodicity condition (see [8]); the convexity condition (see [2]); the subadditive condition (see [7]). For system (1.2), there are also some results (for example, [9-12]). For system (1.1), recently, in [13], by using the least action principle and the saddle point theorem, Pasca and Tang considered system (1.1) with $g(t) \equiv 0$ under the following assumptions: there exist $f_j, g_j \in L^1(0,T;\mathbb{R}^+), j = 1,2$ and $\alpha_1 \in [0, q-1), \alpha_2 \in [0, p-1)$ such that

$$|\nabla_{x_1} F(t,x_1,x_2)| \leq f_1(t)|x_1|^{\alpha_1} + g_1(t) \tag{1.3}$$

$$|\nabla_{x_2} F(t,x_1,x_2)| \leq f_2(t)|x_2|^{\alpha_2} + g_2(t) \tag{1.4}$$

for all $(x_1,x_2) \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. By using saddle point theorem and the least action principle, they obtained system (1.1) with $g(t) \equiv 0$ has at least one solution.

In [14], Wu and Chen considered the following damped vibration problem

$$\begin{cases} \ddot{u}(t) + g(t)\dot{u}(t) = \nabla_u F(t,u(t)), & \text{a.e. } t \in [0,T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.5}$$

By using the least action principle, Theorem 2 in [15] and the saddle point theorem, the authors obtained some existence results of solutions for system (1.5). Moreover, recently, in [18]-[23], the authors also considered the existence and multiplicity of solutions for damped vibration problem and p-Laplacian system. In this paper, we will establish some similar results for system (1.1) and system (1.2). Now we state our main results.

Theorem 1.1. *Assume the following condition holds:*
(F_1)

$$\liminf_{\sqrt{|x_1|^2+|x_2|^2}\to+\infty} \frac{F(t,x_1,x_2)}{|x_1|^q+|x_2|^p} > 0 \quad \text{uniformly for a.e. } t \in [0,T].$$

Then system (1.1) has at least one solution in $W_T^{1,q} \times W_T^{1,p}$. Let q' and p' be such that $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{p}+\frac{1}{p'}=1$. Furthermore, if the following condition also holds:
(F_2) there exist $\delta>0$, $a \in \left[0, \frac{G_0}{qG_1}\left(\frac{q'+1}{T}\right)^{1/q'}\right]$ and $b \in \left[0, \frac{G_0}{pG_1}\left(\frac{p'+1}{T}\right)^{1/p'}\right]$ such that

$$-a|x_1|^q - b|x_2|^p \le F(t,x_1,x_2) \le 0, \quad \forall\, |x_1| \le \delta, \quad |x_2| \le \delta,$$

where $G_0 = \min_{t\in[0,T]} e^{G(t)}$ and $G_1 = \max_{t\in[0,T]} e^{G(t)}$, then system (1.1) has at least two nonzero solutions in $W_T^{1,q} \times W_T^{1,p}$, where

$W_T^{1,p} = \{u : \mathbb{R} \to \mathbb{R}^N | u$ is absolutely continuous, $u(0)=u(T)$ and $\dot{u} \in L^p([0,T])\}$.
with the norm defined by

$$\|u\|_{[W_T^{1,p}]} = \left(\int_0^T e^{G(t)}|u(t)|^p dt + \int_0^T e^{G(t)}|\dot{u}(t)|^p dt\right)^{1/p}.$$

Remark 1.1. Note that $G(t)$ is continuous on $[0,T]$ and so is $e^{G(t)}$. Hence, $e^{G(t)}$ has the maximal and minimal value on $[0,T]$.

Theorem 1.2. *Assume the following condition holds:*
(F_3)

$$\liminf_{|x|\to+\infty} \frac{F(t,x)}{|x|^p} > 0 \quad \text{uniformly for a.e. } t \in [0,T].$$

Then system (1.2) has at least one solution in $W_T^{1,p}$. Furthermore, if the following condition also holds:
(F_4) there exist $\delta>0$ and $a \in \left[0, \frac{G_0}{pG_1}\left(\frac{p'+1}{T}\right)^{1/p'}\right]$ such that

$$-a|x|^p \le F(t,x) \le 0, \quad \forall\, |x| \le \delta,$$

then system (1.2) has at least two nonzero solutions in $W_T^{1,p}$.

Theorem 1.3. *If the following conditions hold:*
(F_5)

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^{p-1}} > -\infty \quad \text{uniformly for a.e. } t \in [0,T];$$

(F_6) *whenever $\{u_n\} \subset W_T^{1,p}$ is such that $\|u_n\|_{[W_T^{1,p}]} \to \infty$ and $\frac{|\bar{u}_n|}{\|u_n\|_{[W_T^{1,p}]}}\left(\int_0^T e^{G(t)}dt\right)^{1/p} \to 1$, as $n \to \infty$,*

$$\liminf_{n\to\infty} \int_0^T e^{G(t)} \left(\nabla F(t,u_n(t)), \frac{\bar{u}_n}{|\bar{u}_n|}\right) dt < 0,$$

then system (1.2) has at least one solution in $W_T^{1,p}$.

Remark 1.2. Theorem 1.1 generalizes Theorem 3.1 in [14]. In fact, it follows from Theorem 1.1 by letting $p = q = 2$ and $F(t, x_1, x_2) = F_1(t, x_1)$. Theorem 1.2 and Theorem 1.3 generalize Theorem 3.1 and Theorem 3.3 in [14] by letting $p = 2$. Moreover, we also obtain multiplicity results by adding some conditions like (F_2) and (F_4). Moreover, in [22], the authors investigated system (1.2) with $g(t) \equiv 0$ and they obtained some existence and multiplicity results of solutions. Our Theorem 1.3 generalizes Theorem 1 in [22].

2. Variational structure and some Preliminaries

The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \left[\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right]^{1/p}.$$

Set

$$\|u\|_p = \left(\int_0^T |u(t)|^p dt\right)^{1/p} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Lemma 2.1. (see [11] or [12]) *Each $u \in W_T^{1,p}$ and each $v \in W_T^{1,q}$ can be written as $u(t) = \bar{u} + \tilde{u}(t)$ and $v(t) = \bar{v} + \tilde{v}(t)$ with*

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \int_0^T \tilde{u}(t)dt = 0, \bar{v} = \frac{1}{T}\int_0^T v(t)dt, \int_0^T \tilde{v}(t)dt = 0.$$

Let q' and p' be such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\|\tilde{u}\|_\infty \le \left(\frac{T}{p'+1}\right)^{1/p'} \left(\int_0^T |\dot{u}(s)|^p ds\right)^{1/p},$$

$$\|\tilde{v}\|_\infty \le \left(\frac{T}{q'+1}\right)^{1/q'} \left(\int_0^T |\dot{v}(s)|^q ds\right)^{1/q}, \tag{2.1}$$

and

$$\int_0^T |\tilde{u}(s)|^p ds \le \frac{T^p\Theta(p,p')}{(p'+1)^{p/p'}} \int_0^T |\dot{u}(s)|^p ds,$$

$$\int_0^T |\tilde{v}(s)|^q ds \le \frac{T^q\Theta(q,q')}{(q'+1)^{q/q'}} \int_0^T |\dot{v}(s)|^q ds, \tag{2.2}$$

where

$$\Theta(p,p') = \int_0^1 \left[s^{p'+1} + (1-s)^{p'+1}\right]^{p/p'} ds,$$

$$\Theta(q,q') = \int_0^1 \left[s^{q'+1} + (1-s)^{q'+1}\right]^{q/q'} ds.$$

Obviously, $W_T^{1,p}$ is a reflexive Banach space and the norm $\|\cdot\|_{W_T^{1,p}}$ is equivalent to the norm defined by

$$\|u\|_{[W_T^{1,p}]} = \left(\int_0^T e^{G(t)}|u(t)|^p dt + \int_0^T e^{G(t)}|\dot{u}(t)|^p dt\right)^{1/p}$$

because of $g \in L^\infty(0,T;\mathbb{R})$.

Moreover, in order to consider system (1.1), we need to use the space W defined by

$$W = W_T^{1,q} \times W_T^{1,p}$$

with the norm $\|(u_1,u_2)\|_{[W]} = \|u_1\|_{[W_T^{1,q}]} + \|u_2\|_{[W_T^{1,p}]}$. It is clear that W is a reflexive Banach space. Let $\varphi_{(q,p)} : W \to \mathbb{R}$ given by

$$\varphi_{(q,p)}(u_1,u_2) = \frac{1}{q}\int_0^T e^{G(t)}|\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T e^{G(t)}|\dot{u}_2(t)|^p dt + \int_0^T e^{G(t)} F(t,u_1(t),u_2(t))dt. \quad (2.3)$$

Lemma 2.2. *The functional $\varphi_{(q,p)}$ is continuously differentiable and weakly lower semicontinuous on W.*

Proof. Let

$$L(t,x_1,x_2,y_1,y_2) = e^{G(t)}\left[\frac{1}{q}|y_1|^q + \frac{1}{p}|y_2|^p + F(t,x_1,x_2)\right].$$

Then it follows from Lemma 4 in [13] that $\varphi_{(q,p)}$ is continuously differentiable on W and

$$\begin{aligned}
\langle \varphi'_{(q,p)}(u_1,u_2),(v_1,v_2)\rangle = \int_0^T [&(D_{x_1}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),v_1(t)) + \\
&(D_{y_1}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),\dot{v}_1(t)) + (D_{x_2}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),v_2(t)) + \\
&(D_{y_2}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),\dot{v}_2(t))]dt. \\
= \int_0^T e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t),\dot{v}_1(t))dt &+ \int_0^T e^{G(t)}(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t),\dot{v}_2(t))dt \\
+ \int_0^T e^{G(t)}(\nabla_{x_1}F(t,u_1(t),u_2(t)),v_1(t))dt &+ \int_0^T e^{G(t)}(\nabla_{x_2}F(t,u_1(t),u_2(t)),v_2(t))dt
\end{aligned}$$

Moreover, by Remark 3 in [13], we know that $\varphi_{(q,p)}$ is weakly lower semi-continuous on W.

Lemma 2.3. *If $u \in W$ is a solution of Euler equation $\varphi'_{(q,p)}(u_1, u_2) = 0$, then $u = (u_1, u_2)$ is a solution of system (1.1).*

Proof. Since $\varphi'_{(q,p)}(u_1, u_2) = 0$,

$$0 = \langle \varphi'_{(q,p)}(u_1, u_2), (v_1, v_2) \rangle = \int_0^T e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t), \dot{v}_1(t))dt + \int_0^T e^{G(t)}(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t), \dot{v}_2(t))dt + \int_0^T e^{G(t)}(\nabla_{x_1} F(t, u_1(t), u_2(t)), v_1(t))dt + \int_0^T e^{G(t)}(\nabla_{x_2} F(t, u_1(t), u_2(t)), v_2(t))dt$$

for all $v = (v_1, v_2) \in W$. Then

$$\int_0^T e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t), \dot{v}_1(t))dt + \int_0^T e^{G(t)}(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t), \dot{v}_2(t))dt = -\int_0^T e^{G(t)}(\nabla_{x_1} F(t, u_1(t), u_2(t)), v_1(t))dt - \int_0^T e^{G(t)}(\nabla_{x_2} F(t, u_1(t), u_2(t)), v_2(t))dt$$

for all $v = (v_1, v_2) \in W$. Let $v_2 = 0$. Then

$$\int_0^T e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t), \dot{v}_1(t))dt = -\int_0^T e^{G(t)}(\nabla_{x_1} F(t, u_1(t), u_2(t)), v_1(t))dt$$

for all $v_1 \in W_T^{1,q}$. Then by Fundamental Lemma and Remark 1 in [3, p. 6-7], we know that $e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t))$ has a weak derivative and

$$\left[e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t))\right]' = e^{G(t)}\nabla_{x_1} F(t, u_1(t), u_2(t)), \text{ a.e.} t \in [0, T], \tag{2.4}$$

$$e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)) = \int_0^t e^{G(s)}\nabla_{x_1} F(s, u_1(s), u_2(s))ds + c \text{ a.e. } t \in [0, T], \tag{2.5}$$

$$\int_0^T e^{G(t)}\nabla_{x_1} F(t, u_1(t), u_2(t))dt = 0, \tag{2.6}$$

where c is a constant. We identify the equivalence class $e^{G(t)}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t))$ and its continuous represent $\int_0^T e^{G(s)}\nabla_{x_1} F(s, u_1(s), u_2(s))ds + c$. Then by (2.5), (2.6), $G(T) = 0$ and the existence of $\dot{u}_1$, one has

$$\dot{u}_1(0) - \dot{u}_1(T) = u_1(0) - u_1(T) = 0.$$

Moreover, by (2.4), we know

$$\frac{d}{dt}(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)) + g(t)|\dot{u}_1(t)|^{q-2}\dot{u}_1(t) = \nabla_{x_1} F(t, u_1(t), u_2(t)), \text{ a.e. } t \in [0, T].$$

Similarly, if we let $v_1 = 0$, we can obtain that

$$\dot{u}_2(0) - \dot{u}_2(T) = u_2(0) - u_2(T) = 0$$

and

$$\frac{d}{dt}(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)) + g(t)|\dot{u}_2(t)|^{p-2}\dot{u}_2(t) = \nabla_{x_2}F(t, u_1(t), u_2(t)), \text{ a.e. } t \in [0, T].$$

Hence, (u_1, u_2) is a solution of system (1.1). We complete the proof.

Let $\varphi_p : W_T^{1,p} \to \mathbb{R}$ given by

$$\varphi_p(u) = \frac{1}{p}\int_0^T e^{G(t)}|\dot{u}(t)|^p dt + \int_0^T e^{G(t)}F(t, u(t))dt. \tag{2.7}$$

Lemma 2.4. *The functional φ_p is continuously differentiable and weakly lower semi-continuous on $W_T^{1,p}$.*

Proof. It follows from Theorem 1.4 in [3] that φ_p is continuously differentiable on $W_T^{1,p}$ and

$$\begin{aligned}\langle \varphi_p'(u), v\rangle = &\int_0^T e^{G(t)}(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))dt \\ &+ \int_0^T e^{G(t)}(\nabla F(t, u(t)), v(t))dt, \quad u, v \in W_T^{1,p}.\end{aligned}$$

Moreover, by Remark 3 in [13], we know that φ_p is weakly lower semi-continuous on $W_T^{1,p}$.

Lemma 2.5. *If $u \in W_T^{1,p}$ is a solution of Euler equation $\varphi_p'(u) = 0$, then u is a solution of system (1.2).*

Proof. Similar to the proof of Lemma 2.3, the proof is easy to be completed.

We will use the following lemmas to seek the critical points of $\varphi_{(q,p)}$ and φ_p.

Lemma 2.6. (see [3], Theorem 1.1) *If φ is weakly lower semi-continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 2.7. (see [16]) *Let φ be a C^1 function on $X = X_1 \oplus X_2$ with $\varphi(0) = 0$, satisfying (PS) condition and assume that, for some $\rho > 0$,*

$$\begin{aligned}&\varphi(u) \geq 0, \ \textit{for } u \in X_1, \|u\| \leq \rho, \\ &\varphi(u) \leq 0, \ \textit{for } u \in X_2, \|u\| \leq \rho.\end{aligned}$$

Assume also that φ is bounded below and $\inf_X \varphi < 0$, then φ has at least two nonzero critical points.

Lemma 2.8. (see [17], Theorem 4.6) *Let $X = X_1 \oplus X_2$, where X is a real Banach space and $X_1 \neq \{0\}$ and is finite dimensional. Suppose $\varphi \in C^1(X, \mathbb{R})$, satisfies (PS) condition, and*

(I1) there is a constant α and a bounded neighborhood D of 0 in X_1 such that $\varphi|_{\partial D} \leq \alpha$ and

(I2) there is a constant $\beta > \alpha$ such that $\varphi|_{X_2} \geq \beta$.

Then φ possesses a critical value $c \geq \beta$. Moreover c can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in\bar{D}} \varphi(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{D}, X) | h = id \text{ on } \partial D\}.$$

3. Proofs of Theorems

Proof of Theorem 1.1.
By (F_1), there is $0 < \varepsilon < \min\left\{1, \liminf_{\sqrt{|x_1|^2+|x_2|^2}\to+\infty} \frac{F(t,x_1,x_2)}{|x_1|^q+|x_2|^p}\right\}$ and $M > 0$ such that

$$F(t, x_1, x_2) > \frac{\varepsilon}{q+p}|x_1|^q + \frac{\varepsilon}{q+p}|x_2|^p \tag{3.1}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\sqrt{|x_1|^2 + |x_2|^2} > M$ and a.e. $t \in [0, T]$.

Set $a_M = \max_{|x_1|\leq M,|x_2|\leq M} [a(|x_1|) + a_2(|x_2|)]$. Then by (3.1) and assumption (A), we have

$$F(t, x_1, x_2) > \frac{\varepsilon}{q+p}|x_1|^q + \frac{\varepsilon}{q+p}|x_2|^p - \frac{\varepsilon}{q+p}M^q - \frac{\varepsilon}{q+p}M^p - a_M b(t) \tag{3.2}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$. Then

$$\begin{aligned}
&\varphi_{(q,p)}(u_1, u_2)\\
&= \frac{1}{q}\int_0^T e^{G(t)}|\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T e^{G(t)}|\dot{u}_2(t)|^p dt + \int_0^T e^{G(t)}F(t, u_1(t), u_2(t))dt\\
&\geq \frac{1}{q}\int_0^T e^{G(t)}|\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T e^{G(t)}|\dot{u}_2(t)|^p dt + \frac{\varepsilon}{q+p}\int_0^T e^{G(t)}|\dot{u}_1(t)|^q dt\\
&+ \frac{\varepsilon}{q+p}\int_0^T e^{G(t)}|\dot{u}_2(t)|^p dt - \frac{\varepsilon}{q+p}(M^q + M^p)\int_0^T e^{G(t)}dt - a_M\int_0^T e^{G(t)}b(t)dt\\
&\geq \frac{\varepsilon}{q+p}\|u_1\|^q_{[W_T^{1,q}]} + \frac{\varepsilon}{q+p}\|u_2\|^p_{[W_T^{1,p}]} -\\
&\qquad \frac{\varepsilon}{q+p}(M^q + M^p)\int_0^T e^{G(t)}dt - a_M\int_0^T e^{G(t)}b(t)dt.
\end{aligned} \tag{3.3}$$

for all $(u_1, u_2) \in W$. Obviously, $\varphi_{(q,p)} \to +\infty$ as $\|(u_1, u_2)\|_{[W]} \to \infty$. Hence, $\varphi_{(q,p)}$ has a bounded minimizing sequence. Thus, by Lemma 2.2 and Lemma 2.6, we know that $\varphi_{(q,p)}$ has a minimum on W. So system (1.1) has at least one solution in W.

Furthermore, if (F_2) also holds, we will use Lemma 2.7 to obtain more critical points of $\varphi_{(q,p)}$. Let $X = W$, $X_2 = \mathbb{R}^N \times \mathbb{R}^N$ and $X_1 = \tilde{W} = \tilde{W}_T^{1,q} \times \tilde{W}_T^{1,p}$ which is the subspace of W given by

$$\tilde{W} = \{(u_1, u_2) \in W |\ (\bar{u}_1, \bar{u}_2) = (0, 0)\}.$$

By (3.3), we know that $\varphi_{(q,p)} \to +\infty$ as $\|(u_1,u_2)\|_{[W]} \to \infty$. So $\varphi_{(q,p)}$ satisfies (PS) condition and is bounded below. Take $\rho = \frac{\delta}{c_1}$, where c_1 is a positive constant such that $\|u_1\|_\infty \le c_1\|u_1\|_{W_T^{1,q}} \le c_1\|u\|_{[W]}$ and $\|u_2\|_\infty \le c_1\|u_2\|_{W_T^{1,p}} \le c_1\|u\|_{[W]}$ for all $(u_1,u_2) \in W$. It follows from (F_2) and Lemma 2.1 that for all $(u_1,u_2) \in X_1$ with $\|u\|_{[W]} \le \rho$,

$$
\begin{aligned}
&\varphi_{(q,p)}(u_1,u_2) \\
&= \frac{1}{q}\int_0^T e^{G(t)}|\dot u_1(t)|^q dt + \frac{1}{p}\int_0^T e^{G(t)}|\dot u_2(t)|^p dt + \int_0^T e^{G(t)}F(t,u_1(t),u_2(t))dt \\
&\qquad \ge \frac{1}{q}\int_0^T e^{G(t)}|\dot u_1(t)|^q dt + \frac{1}{p}\int_0^T e^{G(t)}|\dot u_2(t)|^p dt - \\
&\qquad\quad a\int_0^T e^{G(t)}|u_1(t)|^q dt - b\int_0^T e^{G(t)}|u_2(t)|^p dt \\
&\ge \frac{1}{q}G_0\int_0^T |\dot u_1(t)|^q dt + \frac{1}{p}G_0\int_0^T |\dot u_2(t)|^p dt - aG_1\left(\frac{T}{q'+1}\right)^{1/q'}\int_0^T |\dot u_1(t)|^q dt \\
&\qquad - bG_1\left(\frac{T}{p'+1}\right)^{1/p'}\int_0^T |\dot u_2(t)|^p dt.
\end{aligned} \tag{3.4}
$$

Since $a \le \frac{G_0}{qG_1}\left(\frac{q'+1}{T}\right)^{1/q'}$ and $b \le \frac{G_0}{pG_1}\left(\frac{p'+1}{T}\right)^{1/p'}$, (3.4) implies that $\varphi_{(q,p)}(u_1,u_2) \ge 0$ for all $(u_1,u_2) \in X_1$ with $\|u\|_{[W]} \le \rho$. By (F_2), it is easy to see that $\varphi_{(q,p)}(u_1,u_2) \le 0$ for all $(u_1,u_2) \in X_2$ for all $\|u\|_{[W]} \le \rho$.

If $\inf\{\varphi_{(q,p)}(u_1,u_2) : (u_1,u_2) \in W\} = 0$, then from above, we have $\varphi_{(q,p)}(u_1,u_2) = 0$ all $(u_1,u_2) \in X_2$ with $\|(u_1,u_2)\|_W \le \rho$. Hence, all $(u_1,u_2) \in X_2$ with $\|(u_1,u_2)\|_W \le \rho$ are minimal points of $\varphi_{(q,p)}$, which implies that $\varphi_{(q,p)}$ has infinitely many critical points. If $\inf\{\varphi_{(q,p)}(u_1,u_2) : (u_1,u_2) \in W\} < 0$, then by Lemma 2.7, $\varphi_{(q,p)}$ has at least two nonzero critical points. Hence, system (1.1) has at least two nontrivial solutions in W. We complete our proof.

Proof of Theorem 1.2. The proof is as essentially same as Theorem 1.1. So we omit it.

Proof of Theorem 1.3. We will use Lemma 2.8 to seek the critical point of φ_p. It is clear that $W_T^{1,p} = \tilde W_T^{1,p} \oplus \mathbb{R}^N$. Let $X_1 = \mathbb{R}^N$ and $X_2 = \tilde W_T^{1,p}$. First, we prove that φ satisfies the (PS) condition. Suppose that $\{u_n\} \subset W_T^{1,p}$ is a sequence such that

$$\varphi_p'(u_n) \to 0 \tag{3.5}$$

and there exists a constant $c_2 > 0$ such that $\varphi_p(u_n) \le c_2$, $n \in \mathbb{N}$. Then we can claim that $\{u_n\}$ is bounded in $W_T^{1,p}$. Otherwise, passing to a subsequence if necessary, we assume that $\|u_n\|_{[W_T^{1,p}]} \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{[W_T^{1,p}]}}$. Since $W_T^{1,p}$ is a reflexive Banach space, there is a point $v_0 \in H_T^1$ and a subsequence of $\{v_n\}$, still noted by $\{v_n\}$, such that

$$v_n \rightharpoonup v_0, \ \text{in } W_T^{1,p}.$$

By Proposition 1.2 in [3], we know that $\{v_n\}$ converges uniformly to v_0 on $[0,T]$. Hence, there is a $M_2 > 0$ such that

$$\max_{0\le t\le T} |v_n(t)| \le M_2, \;\; n = 1, 2, \cdots . \tag{3.6}$$

By (F_5) and assumption $(A)'$, we know that there exist $\lambda < 0$ and $M_3 > 0$ such that

$$F(t,x) \ge \lambda |x|^{p-1} - a_{M_3} b(t), \tag{3.7}$$

where $a_{M_3} = \max_{|x|\le M_3} a(|x|)$. It follows from (3.5), (3.6) and (3.7) that

$$\begin{aligned}
\frac{c_2}{\|u_n\|^p_{[W_T^{1,p}]}} &\ge \frac{\varphi_p(u_n)}{\|u_n\|^p_{[W_T^{1,p}]}} \\
&= \frac{1}{p}\int_0^T e^{G(t)}|\dot v_n(t)|^p dt + \frac{1}{\|u_n\|^p_{[W_T^{1,p}]}}\int_0^T e^{G(t)} F(t,u_n(t))dt \\
&\ge \frac{1}{p}\int_0^T e^{G(t)}|\dot v_n(t)|^p dt + \frac{1}{\|u_n\|^p_{[W_T^{1,p}]}}\int_0^T e^{G(t)} [\lambda|u_n(t)|^{p-1} \\
&\qquad - a_{M_3} b(t)]dt \\
&= \frac{1}{p}\int_0^T e^{G(t)}|\dot v_n(t)|^p dt + \frac{\lambda}{\|u_n\|_{[W_T^{1,p}]}}\int_0^T e^{G(t)} |v_n(t)|^{p-1}dt \\
&\qquad - \frac{a_{M_3}\int_0^T b(t)dt}{\|u_n\|^p_{[W_T^{1,p}]}} \\
&\ge \frac{1}{p} - \frac{1}{p}\int_0^T e^{G(t)}|v_n(t)|^p dt - \frac{c_3}{\|u_n\|_{[W_T^{1,p}]}} - \frac{c_4}{\|u_n\|^p_{[W_T^{1,p}]}}
\end{aligned}$$

for some constants $c_3 > 0$ and $c_4 > 0$. It implies that $\int_0^T e^{G(t)}|v_0(t)|^p dt \ge 1$. On the other hand, by weak lower semi-continuity of the norm, we have

$$\|v_0\|_{[W_T^{1,p}]} \le \liminf \|v_n\|_{[W_T^{1,p}]} = 1.$$

Hence, $|\dot v_0(t)| = 0$ for a.e. $t \in [0,T]$, which implies that $|v_0(t)|$ is a constant for a.e. $t \in [0,T]$. Then $|v_0|^p = \frac{1}{\int_0^T e^{G(t)}dt}$. Therefore,

$$\begin{aligned}
\frac{|\bar u_n|}{\|u_n\|_{[W_T^{1,p}]}}\left(\int_0^T e^{G(t)}dt\right)^{1/p} &= \left|\frac{1}{T}\int_0^T \frac{u_n(t)}{\|u_n\|_{[W_T^{1,p}]}}dt\right| \left(\int_0^T e^{G(t)}dt\right)^{1/p} \\
&= \left|\frac{1}{T}\int_0^T v_n(t)dt\right| \left(\int_0^T e^{G(t)}dt\right)^{1/p} \\
&\to \left|\frac{1}{T}\int_0^T v_0 dt\right| \left(\int_0^T e^{G(t)}dt\right)^{1/p} = 1.
\end{aligned}$$

as $n \to \infty$. Hence, by (F_6), we have

$$\liminf_{n\to\infty} \int_0^T e^{G(t)} \left(\nabla F(t, u_n(t)), \frac{\bar{u}_n}{|\bar{u}_n|} \right) dt < 0.$$

However,

$$\int_0^T e^{G(t)} \left(\nabla F(t, u_n(t)), \frac{\bar{u}_n}{|\bar{u}_n|} \right) dt = \left\langle \varphi_p'(u_n), \frac{\bar{u}_n}{|\bar{u}_n|} \right\rangle \to 0, \text{ as } n \to \infty,$$

which is a contradiction. Hence $\{u_n\}$ is bounded in $W_T^{1,p}$. The following arguments are motivated by [9], [11] and [12]. Since $W_T^{1,p}$ is a reflexive Banach space, passing to a subsequence if necessary, we suppose that

$$u_n \rightharpoonup u \text{ in } W_T^{1,p}, \tag{3.8}$$

for some $u \in W_T^{1,p}$ and then

$$u_n \to u \text{ strongly in } C([0,T]; \mathbb{R}^N). \tag{3.9}$$

Note that

$$\begin{aligned} \langle \varphi_p'(u_n), u_n - u \rangle &= \int_0^T e^{G(t)} (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) dt \\ &\quad - \int_0^T e^{G(t)} (\nabla F(t, u_n(t)), u_n(t) - u(t)) dt. \end{aligned} \tag{3.10}$$

Since $\{\|u_n\|_{[W_T^{1,p}]}\}$ is bounded and $\varphi'(u_n) \to 0$, we have

$$|\langle \varphi_p'(u_n), u_n - u \rangle| \le \|\varphi_p'(u_n)\| \, \|u_n - u\|_{[W_T^{1,p}]} \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

By assumption (A) and (3.9), one has

$$\int_0^T e^{G(t)} \big(\nabla F(t, u_n(t)), u_n(t) - u(t) \big) dt \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Hence, it follows from (3.10), (3.11) and (3.12) that

$$\int_0^T e^{G(t)} (|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t)) dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13}$$

On the other hand, it is easy to derive from (3.9) and the boundedness of $\{u_n\}$ that

$$\int_0^T e^{G(t)} (|u_n(t)|^{p-2} u_n(t), u_n(t) - u(t)) dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Set

$$\psi(u) = \frac{1}{p} \left(\int_0^T e^{G(t)} |u(t)|^p dt + \int_0^T e^{G(t)} |\dot{u}(t)|^p dt \right).$$

Then we have

$$\begin{aligned}\langle \psi'(u_n), u_n - u\rangle &= \int_0^T e^{G(t)}(|u_n(t)|^{p-2}u_n(t), u_n(t) - u(t))dt \\ &\quad + \int_0^T e^{G(t)}(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t))dt\end{aligned}$$

and

$$\begin{aligned}\langle \psi'(u), u_n - u\rangle &= \int_0^T e^{G(t)}(|u(t)|^{p-2}u(t), u_n(t) - u(t))dt \\ &\quad + \int_0^T e^{G(t)}(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t))dt.\end{aligned}$$

From (3.13) and (3.14), we obtain

$$\langle \psi'(u_n), u_n - u\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

On the other hand, it follows from (3.8) that

$$\langle \psi'(u), u_n - u\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.16}$$

By (3.15), (3.16) and by using the Hölder's inequality, we get

$$\begin{aligned}&\langle \psi'(u_n) - \psi'(u), u_n - u\rangle \\ = &\int_0^T e^{G(t)}(|u_n(t)|^{p-2}u_n(t), u_n(t) - u(t))dt \\ &\qquad + \int_0^T e^{G(t)}(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t))dt \\ &- \int_0^T e^{G(t)}(|u(t)|^{p-2}u(t), u_n(t) - u(t))dt \\ &\qquad - \int_0^T e^{G(t)}(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t))dt \\ = &\|u_n\|^p_{[W_T^{1,p}]} + \|u\|^p_{[W_T^{1,p}]} - \int_0^T e^{G(t)}(|u_n(t)|^{p-2}u_n(t), u(t))dt \\ &- \int_0^T e^{G(t)}(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}(t))dt \\ &- \int_0^T e^{G(t)}(|u(t)|^{p-2}u(t), u_n(t))dt - \int_0^T e^{G(t)}(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{u}_n(t))dt \\ = &\|u_n\|^p_{[W_T^{1,p}]} + \|u\|^p_{[W_T^{1,p}]} - \int_0^T \left(\left(e^{G(t)}\right)^{\frac{p-1}{p}} |u_n(t)|^{p-2}u_n(t), \left(e^{G(t)}\right)^{\frac{1}{p}} u(t)\right) dt \\ &- \int_0^T \left(\left(e^{G(t)}\right)^{\frac{p-1}{p}} |\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \left(e^{G(t)}\right)^{\frac{1}{p}} \dot{u}(t)\right) dt \\ &- \int_0^T \left(\left(e^{G(t)}\right)^{\frac{p-1}{p}} |u(t)|^{p-2}u(t), \left(e^{G(t)}\right)^{\frac{1}{p}} u_n(t)\right) dt \\ &- \int_0^T \left(\left(e^{G(t)}\right)^{\frac{p-1}{p}} |\dot{u}(t)|^{p-2}\dot{u}(t), \left(e^{G(t)}\right)^{\frac{1}{p}} \dot{u}_n(t)\right) dt\end{aligned}$$

$$
\begin{aligned}
&\geq \|u_n\|^p_{[W^{1,p}_T]} + \|u\|^p_{[W^{1,p}_T]} - \int_0^T \left(e^{G(t)}\right)^{\frac{p-1}{p}} |u_n(t)|^{p-1} \left|\left(e^{G(t)}\right)^{\frac{1}{p}} u(t)\right| dt \\
&\quad - \int_0^T \left(e^{G(t)}\right)^{\frac{p-1}{p}} |\dot{u}_n(t)|^{p-1} \left|\left(e^{G(t)}\right)^{\frac{1}{p}} \dot{u}(t)\right| dt \\
&\quad - \int_0^T \left(e^{G(t)}\right)^{\frac{p-1}{p}} |u(t)|^{p-1} \left|\left(e^{G(t)}\right)^{\frac{1}{p}} u_n(t)\right| dt \\
&\quad - \int_0^T \left(e^{G(t)}\right)^{\frac{p-1}{p}} |\dot{u}(t)|^{p-1} \left|\left(e^{G(t)}\right)^{\frac{1}{p}} \dot{u}_n(t)\right| dt \\
&\geq \|u_n\|^p_{[W^{1,p}_T]} + \|u\|^p_{[W^{1,p}_T]} - \left[\left(\int_0^T e^{G(t)}|u_n(t)|^p dt\right)^{\frac{p-1}{p}} \left(\int_0^T e^{G(t)}|u(t)|^p dt\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(\int_0^T e^{G(t)}|\dot{u}_n(t)|^p dt\right)^{\frac{p-1}{p}} \left(\int_0^T e^{G(t)}|\dot{u}(t)|^p dt\right)^{\frac{1}{p}}\right] \\
&\quad - \left[\left(\int_0^T e^{G(t)}|u(t)|^p dt\right)^{\frac{p-1}{p}} \left(\int_0^T e^{G(t)}|u_n(t)|^p dt\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(\int_0^T e^{G(t)}|\dot{u}(t)|^p dt\right)^{\frac{p-1}{p}} \left(\int_0^T e^{G(t)}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p}}\right] \\
&\geq \|u_n\|^p_{[W^{1,p}_T]} + \|u\|^p_{[W^{1,p}_T]} \\
&\quad - \left(\int_0^T e^{G(t)}|u(t)|^p dt + \int_0^T e^{G(t)}|\dot{u}(t)|^p dt\right)^{\frac{1}{p}} \\
&\qquad\qquad \left(\int_0^T e^{G(t)}|u_n(t)|^p dt + \int_0^T e^{G(t)}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p'}} \\
&\quad - \left(\int_0^T e^{G(t)}|u_n(t)|^p dt + \int_0^T e^{G(t)}|\dot{u}_n(t)|^p dt\right)^{\frac{1}{p}} \\
&\qquad\qquad \left(\int_0^T e^{G(t)}|u(t)|^p dt + \int_0^T e^{G(t)}|\dot{u}(t)|^p dt\right)^{\frac{1}{p'}} \\
&= \|u_n\|^p_{[W^{1,p}_T]} + \|u\|^p_{[W^{1,p}_T]} - \|u\|_{[W^{1,p}_T]} \|u_n\|^{p-1}_{[W^{1,p}_T]} - \|u_n\|_{[W^{1,p}_T]} \|u\|^{p-1}_{[W^{1,p}_T]} \\
&= \left(\|u_n\|^{p-1}_{[W^{1,p}_T]} - \|u\|^{p-1}_{[W^{1,p}_T]}\right)\left(\|u_n\|_{[W^{1,p}_T]} - \|u\|_{[W^{1,p}_T]}\right). \qquad (3.17)
\end{aligned}
$$

It follows that

$$
0 \leq \left(\|u_n\|^{p-1}_{[W^{1,p}_T]} - \|u\|^{p-1}_{[W^{1,p}_T]}\right)\left(\|u_n\|_{[W^{1,p}_T]} - \|u\|_{[W^{1,p}_T]}\right) \leq \langle \psi'(u_n) - \psi'(u), u_n - u\rangle, \qquad (3.18)
$$

which, together with (3.15)-(3.18) yields $\|u_n\|_{[W^{1,p}_T]} \to \|u\|_{[W^{1,p}_T]}$. By the uniform convexity of $W^{1,p}_T$ and (3.8), it follows from the Kadec-Klee property that $u_n \to u$ strongly in $W^{1,p}_T$. Thus we have verified that φ_p satisfies (PS) condition.

Next, we prove that φ_p satisfies (I_1) and (I_2). First, we claim that $\varphi_p(x) \to -\infty$, as $|x| \to \infty$ for all $x \in \mathbb{R}^N = X_1$. By using (F_6), the proof is the same as Lemma 3.3 in [14]. So we omit it. For $u \in X_2 = \tilde{W}_T^{1,p}$, it follows from (3.7), Hölder's inequality and (2.2) that

$$\begin{aligned}
\varphi_p(u) &= \frac{1}{p}\int_0^T e^{G(t)}|\dot{u}(t)|^p dt + \int_0^T e^{G(t)}F(t,u(t))dt \\
&\geq \frac{1}{p}\int_0^T e^{G(t)}|\dot{u}(t)|^p dt + \int_0^T e^{G(t)}[\lambda|u(t)|^{p-1} - a_{M_3}b(t)]dt
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{e^{G_0}}{p}\int_0^T |\dot{u}(t)|^p dt - |\lambda| e^{G_1}\int_0^T |u(t)|^{p-1}dt - a_{M_3}\int_0^T e^{G(t)}b(t)dt \\
&\geq \frac{e^{G_0}}{p}\int_0^T |\dot{u}(t)|^p dt - |\lambda| e^{G_1}T^{1/p}\left(\int_0^T |u(t)|^p dt\right)^{1/p'} - a_{M_3}\int_0^T e^{G(t)}b(t)dt \\
&\geq \frac{e^{G_0}}{p}\int_0^T |\dot{u}(t)|^p dt - c_5\left(\int_0^T |\dot{u}(t)|^p dt\right)^{1/p'} - a_{M_3}\int_0^T e^{G(t)}b(t)dt,
\end{aligned}$$

where $c_5 = |\lambda| e^{G_1}T^{1/p}\left(\frac{T^p\Theta(p,p')}{(p'+1)^{p/p'}}\right)^{1/p'}$. Note that the norm $\|\dot{u}\|_{L^p}$ is equivalent to the norm $\|u\|_{[W_T^{1,p}]}$ in $\tilde{W}_T^{1,p}$. Hence, $\varphi_p(u) \to +\infty$ as $\|u\|_{[W_T^{1,p}]} \to \infty$ for all $u \in X_2$. Thus we complete the proof.

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