Spacelike hypersurfaces with nonzero constant k-th mean curvature and two distinct principal curvatures in anti-de Sitter spaces

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Abstract

In this paper, we investigate the spacelike hypersurfaces in anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$ with nonzero constant k-th mean curvature H_k and two distinct principal curvatures one of which is simple, and characterize such hypersurfaces as hyperbolic cylinders.

1 Introduction

Let $M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian space form with constant sectional curvature c, we separately call it de Sitter space $\mathbb{S}_1^{n+1}(c)$, Lorentzian-Minkowski space $\mathbb{L}^{n+1}$ or anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$, with respect to $c > 0$, $c = 0$ or $c < 0$. Let M be an n-dimensional complete hypersurface in $M_1^{n+1}(c)$, we recall that M is said to be spacelike if its induced metric is positive definite.

The study of spacelike hypersurfaces in Lorentzian space forms has got increasing interest motivated by their importance in problems related to physics, more specifically in the theory of general relativity. Concerning to the mathematical viewpoint, such hypersurfaces appear in several uniqueness problems, for instance, constant mean curvature spacelike hypersurfaces exhibit nice Bernstein's type properties.

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In recent years, one of the principal research subjects already current in this theory is to characterize complete spacelike hypersurfaces with constant mean curvature (or constant scalar curvature) and two distinct principal curvatures one of which is simple. For example, let M be an n-dimensional $(n \geqslant 3)$ complete maximal spacelike hypersurface with two distinct principal curvatures λ (with multiplicity m) and μ in anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. If $\inf(\lambda-\mu)^2 > 0$, Cao-Wei [2] proved that M is isometric to hyperbolic cylinder $\mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$, $1 \leqslant m \leqslant n-1$. Wu [13] and Yang-Liu [16] extend that result to the case of constant mean curvature with the restriction that one of the principal curvatures is simple (indeed Wu's paper [13] includes the same result about the case of constant scalar curvature).

In [4], Chu-Zhai investigated the similar problem for spacelike hypersurface M in $\mathbb{H}_1^{n+1}(-1)$ with constant scalar curvature $n(n-1)R$ (instead of constant mean curvature) and two distinct principal curvatures one of which is simple. Although Wu [13] pointed out that the condition $\inf(\lambda-\mu)^2 > 0$, either for the case of constant mean curvature or constant scalar curvature, can not be dropped down, by replacing the condition $\inf(\lambda-\mu)^2 > 0$ with the squared norm of the second fundamental form $S \geqslant \frac{(n-1)(2-n-nR)}{n-2} + \frac{n-2}{2-n-nR}$ or $S \leqslant \frac{(n-1)(2-n-nR)}{n-2} + \frac{n-2}{2-n-nR}$, Chu-Zhai also proved that M is isometric to hyperbolic cylinder $\mathbb{H}^{n-1}(\frac{nR}{n-2}) \times \mathbb{H}^1(\frac{nR}{2-n-nR})$.

Since the k-th mean curvature H_k, $1 \leqslant k \leqslant n$, are generalizations of the mean curvature, scalar curvature and Gauss-Kronecker curvature, people may also expect that the similar results should hold for spacelike hypersurfaces of constant k-th mean curvature $(k \geqslant 2)$ and two distinct principal curvatures λ (with multiplicity $n-1$) and μ satisfying $\inf(\lambda-\mu)^2 > 0$. But this is not true, see also [13]. However, Suh-Wei [11] and Yang [15] considered complete $(k-1)$-maximal spacelike hypersurface with two distinct principal curvatures λ (with multiplicity $n-1$) and μ in $\mathbb{H}_1^{n+1}(-1)$, and proved that if $\inf(\lambda-\mu)^2 > 0$, then $S \geqslant \frac{n(n^2-2k+n)}{k(n-k)}$, equality holds if and only if $M = \mathbb{H}^1(c_1) \times \mathbb{H}^{n-1}(c_2)$, $1 \leqslant k \leqslant n-1$.

The main idea proving such kinds of results is motivated by Otsuki's work [8], where, Otsuki studied the minimal hypersurfaces in a unit $(n+1)$-sphere $S^{n+1}(1)$ $(n \geqslant 3)$ with two distinct principal curvatures and proved that if the multiplicities of the two principal curvatures are both greater than 1, then they are Clifford minimal hypersurfaces. As for the case when the multiplicity of one of the two principal curvatures is $n-1$, it corresponds to an ordinary differential equation. Otsuki's method is generalized by many authors to study hypersurfaces with constant k-th mean curvature and two distinct principal curvatures in Riemannian space forms (see e.g., [6], [9], [12]) or spacelike hypersurfaces in de Sitter space (see e.g., [5], [7], [10], [14]).

In this paper, we are interested in characterizing complete spacelike hypersurfaces with non-zero constant k-th mean curvature H_k and two distinct principal curvatures λ and μ (which is assumed to be simple) immersed in anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$ by analysing the behavior of its k-th mean curvature. Roughly speaking, we characterize such hypersurfaces as hyperbolic cylinders, according to $H_k > 0$ for $1 \leqslant k \leqslant n$, or $H_k < 0$ for $3 \leqslant k \leqslant n$.

This paper consists of five sections. In Section 2 we first recall the structure equations and basic formulas as well as an key ordinary differential equation (see Eq.(7) in Section 2 and Eq.(11) in Section 5). Then some key lemmas are given in Section 3. Section 4 is devoted to state and prove our main theorems according to the constant k-th mean curvature $H_k > 0$ for $1 \leqslant k \leqslant n-1$, or $H_k < 0$ for $3 \leqslant k \leqslant n-1$. The case of $k = n$ is discussed in Section 5 individual.

2 Preliminaries

Let $\mathbb{R}_2^{n+2}$ be $\mathbb{R}^{n+2}$ equipped with the indefinite inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2},$$

for $x = (x_1, \cdots, x_{n+1}, x_{n+2})$, $y = (y_1, \cdots, y_{n+1}, y_{n+2}) \in \mathbb{R}^{n+2}$. Then, for $c < 0$, we have the $(n+1)$-dimensional anti-de Sitter space

$$\mathbb{H}_1^{n+1}(c) = \Big\{ x \in \mathbb{R}^{n+2} : \langle x, x \rangle = \frac{1}{c} \Big\}.$$

Suppose that M be an n-dimensional spacelike hypersurface of $\mathbb{H}_1^{n+1}(c)$. For any $p \in M$, we choose a local pseudo-Riemannian orthonormal frame field $\{e_1, \cdots, e_{n+1}\}$ in $\mathbb{H}_1^{n+1}(c)$ around p such that $e_1, \cdots, e_n$ are tangent to M and e_{n+1} is the unit timelike normal vector field. Take the corresponding dual coframe $\{\omega_1, \cdots, \omega_{n+1}\}$ with the matrix of connection one forms being ω_{ij}. A well-known argument [3] shows that the forms ω_{in+1} may be expressed as $\omega_{in+1} = \sum_j h_{ij} \omega_j$, $h_{ij} = h_{ji}$. The second fundamental form and the mean curvature of M are given by $B = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$ and $H = \frac{1}{n} \sum_i h_{ii}$ respectively. Then the structure equations of M are

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl} \omega_k \wedge \omega_l.$$

The Gauss equations are given by

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{1}$$

For $1 \leqslant k \leqslant n$, the k-th mean curvature H_k of M is defined by

$$\binom{n}{k} H_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{2}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $\lambda_i (1 \leqslant i \leqslant n)$ are the principal curvatures of M. In particular, when $k = 1$, $H_1 = H$ is nothing but the mean curvature of M; while $k = 2$, a simple calculation by using Gauss equations (1) of M yields $H_2 = c - R$, where R is the normalized scalar curvature of M. When $k = n$, H_n is the well-known Gauss-Kronecker curvature. So we know that the k-th mean curvature H_k generalizes the mean curvature, scalar curvature and Gauss-Kronecker curvature naturally.

Now, we assume that M be a spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with constant k-th mean curvature $H_k \neq 0$ and two distinct principal curvatures λ (with multiplicity $n-1$) and μ. By virtue of the definition of H_k, we have from (2) that $\binom{n}{k}H_k = \binom{n-1}{k}\lambda^k + \binom{n-1}{k-1}\lambda^{k-1}\mu$, equivalently,

$$\lambda^{k-1}((n-k)\lambda + k\mu) = nH_k. \tag{3}$$

Notice our assumption $H_k \neq 0$, (3) implies $\lambda \neq 0$. Choose a proper direction, such that $\lambda > 0$, then $\lambda^k > 0$ and

$$\mu = \frac{nH_k - (n-k)\lambda^k}{k\lambda^{k-1}}, \tag{4}$$

$$\lambda - \mu = n\frac{\lambda^k - H_k}{k\lambda^{k-1}} \neq 0. \tag{5}$$

On the other hand, we know from [17, Theorem 1.3] (see also [13]) that M is the locus of a family of moving submanifolds $M_1^{n-1}(s)$ (where s is the arc length parameter of the integral curve corresponding to μ), and λ^k, H_k satisfy the following differential equation of order 2:

$$\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} + \overline{w}\frac{ck\lambda^{k-2} + (n-k)\lambda^k - nH_k}{k\lambda^{k-2}} = 0, \tag{6}$$

where $\overline{w}(s) = |\lambda^k - H_k|^{-\frac{1}{n}}$, $s \in (-\infty, +\infty)$. Let $P_{H_k}(t) = ckt^{\frac{k-2}{k}} + (n-k)t - nH_k$, $t > 0$. Note that $\lambda^k > 0$, then (6) can be rewritten as

$$\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} + \overline{w}\frac{P_{H_k}(\lambda^k)}{k\lambda^{k-2}} = 0. \tag{7}$$

3 Key Lemmas

In order to prove our main theorems in Section 4, we need the following lemmas. In this section, we always assume that the k-th mean curvature H_k, for some $1 \leqslant k \leqslant n-1$, is a non-zero constant. The case of $k = n$ will be discussed in Section 5 individual.

Lemma 3.1. For $t > 0$, $c < 0$, $n \geqslant 3$, let $P_{H_k}(t) = ckt^{\frac{k-2}{k}} + (n-k)t - nH_k$.

Case 1 When $H_k > 0$ with $1 \leqslant k \leqslant n-1$, then $P_{H_k}(t)$ has a unique positive real root t_1. Furthermore, $P_{H_k}(t) \leqslant 0$ (resp. $\geqslant 0$) for $t \in (0, t_1]$ (resp. $t \in [t_1, +\infty)$).

Case 2 When $H_k < 0$ with $3 \leqslant k \leqslant n-1$.

(1) If $H_k < \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$, then $P_{H_k}(t) > 0$ for $t \in (0, +\infty)$.

(2) If $H_k = \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$, then $P_{H_k}(t) = 0$ has a unique positive real root, denoted by t_2, and $P_{H_k}(t)$ reaches its minimal value at t_2, $P_{H_k}(t) \geqslant 0$ for $t \in (0, +\infty)$.

(3) If $H_k > \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$, then $P_{H_k}(t) = 0$ has two distinct positive real roots t_3, t_4 (here and in the sequel we assume $t_3 < t_4$). At that time, $P_{H_k}(t)$ reaches its minimal value at $t_0 = (\frac{c(k-2)}{k-n})^{\frac{k}{2}} > 0$, and $t_3 < t_0 < t_4$. Furthermore, $P_{H_k}(t) \geqslant 0$ for $t \in (0, t_3] \cup [t_4, +\infty)$, $P_{H_k}(t) \leqslant 0$ for $t \in [t_3, t_4]$.

Proof. A straightforward computation gives $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} = c(k-2)t^{-\frac{2}{k}} + (n-k)$, and $\frac{\mathrm{d}^2P_{H_k}(t)}{\mathrm{d}t^2} = -\frac{2(k-2)}{k}ct^{-\frac{2+k}{k}}$.

Case 1 When $H_k > 0$, the conclusions for $k = 1, 2$ are obvious. For $3 \leqslant k \leqslant n-1$, it is clear that $\frac{\mathrm{d}^2P_{H_k}(t)}{\mathrm{d}t^2} > 0$, which implies that $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t}$ is a strictly monotone increasing function of t. Let $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} = 0$, we get $t_0 = (\frac{c(k-2)}{k-n})^{\frac{k}{2}} > 0$. Thus, if $0 < t < t_0$ (resp. $t > t_0$), then $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} < 0$ (resp. > 0), and $P_{H_k}(t)$ is a strictly monotone decreasing function (resp. increasing function). Since $\lim_{t\to 0^+} P_{H_k}(t) = -nH_k < 0$, $\lim_{t\to+\infty} P_{H_k}(t) = +\infty$, from the monotonic property of $P_{H_k}(t)$, we infer that $P_{H_k}(t) = 0$ has a unique positive real root and the result follows.

Case 2 When $H_k < 0$ with $3 \leqslant k \leqslant n-1$, we also know that $\frac{\mathrm{d}^2P_{H_k}(t)}{\mathrm{d}t^2} > 0$, and $P_{H_k}(t)$ has the same monotonicity as Case 1. So $P_{H_k}(t)$ attains its minimum at t_0. It is easy to check that

$$P_{H_k}(t_0) = 2c\Big(\frac{c(k-2)}{k-n}\Big)^{\frac{k-2}{2}} - nH_k.$$

On the other hand, $\lim_{t\to 0^+} P_{H_k}(t) = -nH_k > 0$, $\lim_{t\to+\infty} P_{H_k}(t) = +\infty$, the conclusions follow immediately from the continuous property and monotone property of $P_{H_k}(t)$. ∎

Lemma 3.2. For $t > 0$, $n \geqslant 3$, let $f(t) = \frac{1}{k^2t^{\frac{2k-2}{k}}}\Big\{(n-1)k^2t^2 + ((n-k)t - nH_k)^2\Big\}$, then

$$f(t_i) = (n-1)t_i^{\frac{2}{k}} + c^2t_i^{-\frac{2}{k}}, \ \ i = 1, 2, 3, 4,$$

where t_i is the positive real roots of the equation $P_{H_k}(t) = 0$ obtained in Lemma 3.1. Furthermore,

(1) If $H_k > 0$ for $1 \leqslant k \leqslant n-1$, then $f(t)$ is a monotone increasing (resp. decreasing) function for $t \geqslant H_k$ (resp. $0 < t \leqslant H_k$).

(2) If $H_k < 0$ for $3 \leqslant k \leqslant n-1$, denote $t'_0 = \frac{n(1-k)H_k}{n-2k+k^2}$. Then $t'_0 > 0$ and $f(t)$ is monotone increasing (resp. decreasing) of t for $t \geqslant t'_0$ (resp. $0 < t \leqslant t'_0$).

Proof. Notice that $P_{H_k}(t_i) = 0$, $i = 1, \ldots, 4$, so

$$\begin{aligned} f(t_i) &= \frac{1}{k^2t_i^{\frac{2k-2}{k}}}\Big\{(n-1)k^2t_i^2 + \big((ckt_i^{\frac{k-2}{k}} + (n-k)t_i - nH_k) - ckt_i^{\frac{k-2}{k}}\big)^2\Big\} \\ &= \frac{1}{k^2t_i^{\frac{2k-2}{k}}}\Big\{(n-1)k^2t_i^2 + (-ckt_i^{\frac{k-2}{k}})^2\Big\} = (n-1)t_i^{\frac{2}{k}} + c^2t_i^{-\frac{2}{k}}. \end{aligned}$$

Furthermore, it is easy to check that

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{2t^{\frac{2-3k}{k}}}{k^3} n(t-H_k)\Big((n-2k+k^2)t - n(1-k)H_k\Big).$$

Put

$$g(t) = (t-H_k)\big((n-2k+k^2)t - n(1-k)H_k\big), \ \ t>0.$$

(1) If $H_k > 0$, the conclusions are clear as $k=1$. While $k \geqslant 2$, $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 2n(k^2-2k+n)t + n(k-2)(n-k)H_k > 0$, which implies that $g(t)$ is a strictly monotone increasing function. Notice that $g(H_k) = 0$, so H_k is the only zero point of $g(t)$. Henceforth, if $0 < t \leqslant H_k$, then $g(t) \leqslant 0$ and $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leqslant 0$, it follows that $f(t)$ is a decreasing function; if $t \geqslant H_k$, then $g(t) \geqslant 0$ and $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \geqslant 0$, this means $f(t)$ is an increasing function.

(2) If $H_k < 0$, then $t - H_k \neq 0$ because of $t > 0$. Hence $g(t) = 0$ if and only if $t = \frac{n(1-k)H_k}{n-2k+k^2}$. Denote $t_0' = \frac{n(1-k)H_k}{n-2k+k^2}$, it is clear that $t_0' > 0$ because of $H_k < 0$ and $k \geqslant 3$. By monotone property and concavity of $g(t)$, we complete the proof of Lemma 3.2. ∎

The following two lemmas deal with two cases: for $3 \leqslant k \leqslant n-1$ and the constant k-th mean curvature $\frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}} < H_k < 0$ or $H_k = \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$, separately. The conclusions are used to prove our main Theorems 4.3 and 4.4 in Section 4.

Lemma 3.3. With $t_0' = \frac{n(1-k)H_k}{n-2k+k^2} > 0$ given in Lemma 3.2. By t_3, t_4 ($t_3 < t_4$) denote the positive real roots of $P_{H_k}(t) = 0$ for $H_k > \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$ proved in Lemma 3.1.

(1) If $t_0' \leqslant t_3$, then $\lambda^k \geqslant t_0'$.

(2) If $t_0' \geqslant t_4$, then $\lambda^k \leqslant t_0'$.

Proof. If $t_0' \leqslant t_3$, we assume on the contrary that $\lambda^k < t_0'$, then $\lambda^k < t_3$ and $P_{H_k}(\lambda^k) > 0$ from Case 2(3) of Lemma 3.1, thus (7) yields $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} < 0$, i.e. $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a strictly monotone decreasing function of s, and it has at most one zero point for $s \in (-\infty, +\infty)$. If $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ has no zero point, then $\overline{w}(s)$ is a monotone function of $s \in (-\infty, +\infty)$; if $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ has one zero point $s_0 \in (-\infty, +\infty)$, then $\overline{w}(s)$ is a monotone function on both $(-\infty, s_0]$ and $[s_0, +\infty)$. Since $\overline{w}(s)$ is bounded (cf. [12] or [13]), we know that both $\lim_{s\to-\infty} \overline{w}(s)$ and $\lim_{s\to+\infty} \overline{w}(s)$ exist and

$$\lim_{s\to-\infty} \frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s} = \lim_{s\to+\infty} \frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s} = 0.$$

This is impossible because $\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}$ is a strictly monotone decreasing function. We aim at $\lambda^k \geqslant t_0'$. Taking similar arguments for the case $t_0' \geqslant t_4$, we complete the proof. ∎

Applying the same method as in the proof of Lemma 3.3, together with Case 2(2) of Lemma 3.1, we have

Lemma 3.4. With t_2 the positive real root of $P_{H_k}(t) = 0$ for $H_k = \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$ proved in Lemma 3.1. Then $\lambda^k \geqslant t_0'$ (resp. $\leqslant$) as $t_0' \leqslant t_2$ (resp. $\geqslant$).

4 Main theorems

In this section, we will characterize complete spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with non-zero constant k-th mean curvature H_k and two distinct principal curvatures, one of which is simple. Since the lemmas used in the proof processing are slightly different between two situations $H_k > 0$ and $H_k < 0$, so we state and prove our results separately.

4.1 The case of $H_k > 0$

Theorem 4.1. Let M be an n-dimensional ($n \geqslant 3$) complete spacelike hypersurface in anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$ with constant k-th $(1 \leqslant k \leqslant n-1)$ mean curvature $H_k > 0$ and two distinct principal curvatures, one of which is simple. If the squared norm S of the second fundamental form of M satisfies

$$S \geqslant (n-1)t_1^{\frac{2}{k}} + c^2 t_1^{-\frac{2}{k}} \tag{8}$$

or

$$S \leqslant (n-1)t_1^{\frac{2}{k}} + c^2 t_1^{-\frac{2}{k}}, \tag{9}$$

then M is isometric to the Riemannian product $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$ for some constants $c_1 < 0$, $c_2 < 0$ with $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and t_1 is the positive real root of $P_{H_k}(t) = 0$ given in Lemma 3.1.

Proof. Since $P_{H_k}(H_k) = ckH_k^{\frac{k-2}{k}} - kH_k < 0$, it follows from Case 1 of Lemma 3.1 that $H_k < t_1$. We also assert that $\lambda^k > H_k$. In fact, if on the contrary $\lambda^k < H_k$ (because of $\lambda^k \neq H_k$ from (5)), then $\lambda^k < t_1$. Review the process of the proof of Lemma 3.1(1), it is not difficult to find that $P_{H_k}(\lambda^k) < 0$. Now (7) implies $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} > 0$, i.e. $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a strictly monotone increasing function of s. Similar to the proof of Lemma 3.3 we get a contradiction which implies the assertion $\lambda^k > H_k$ holds.

Put $t = \lambda^k$ into $f(t)$ defined in Lemma 3.2, together with (4), it is easy to verify that $f(\lambda^k) = S$. In the following, keep in mind that $H_k < t_1$ as we have proved at the beginning of the proof of Theorem 4.1.

Case 1 If the assumption (8) in Theorem 4.1 holds, i.e., $S = f(\lambda^k) \geqslant f(t_1)$, we know from Lemma 3.2(1) that $\lambda^k \geqslant t_1$, thus Lemma 3.1(Case 1) tells us $P_{H_k}(\lambda^k) \geqslant 0$. So we have $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} \leqslant 0$ from (7), this means that $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a monotone decreasing function of s. Therefore, $\overline{w}(s)$ must be monotonic when s tends to infinity. On the other hand, since $\overline{w}(s)$ is bounded (cf. [12]), we find that both $\lim_{s\to-\infty} \overline{w}(s)$ and $\lim_{s\to+\infty} \overline{w}(s)$ exist and we have

$$\lim_{s\to-\infty} \frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s} = \lim_{s\to+\infty} \frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s} = 0.$$

By the monotonicity of $\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}$, we see that $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s} \equiv 0$, thus $\overline{w}(s)$ is a constant. Then, according to $\overline{w}(s) = |\lambda^k - H_k|^{-\frac{1}{n}}$ and (4), we infer that λ and μ are constants on M. Therefore, the results due to Abe-Koike-Yamaguchi [1] lead to M be an isoparametric hypersurface and isometric to the hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $c_1 < 0, c_2 < 0, \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.

Case 2 If the assumption (9), instead of (8), in Theorem 4.1 holds, i.e., $S = f(\lambda^k) \leqslant f(t_1)$, we obtain from Lemma 3.2(1) again that $\lambda^k \leqslant t_1$, thus $P_{H_k}(\lambda^k) \leqslant 0$ by Lemma 3.1. In this case, $\frac{d^2\overline{w}}{ds^2} \geqslant 0$ and $\frac{d\overline{w}}{ds}$ is a monotone increasing function of s (while it is a monotone decreasing function in Case 1). The remain discussions are the same as that of Case 1. We complete the proof of Theorem 4.1. ■

4.2 The case of $H_k < 0$

Theorem 4.2. There is no complete spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ $(n \geqslant 4)$ with two distinct principal curvatures λ(multiplicity $n-1$), μ and constant k-th mean curvature $H_k < \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$ for $3 \leqslant k \leqslant n-1$.

Proof. If there exists such a hypersurface M with $H_k < \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$, then Lemma 3.1 tells us $P_{H_k}(t) = 0$ has no real root and $P_{H_k}(t) > 0$. Especially, $P_{H_k}(\lambda^k) > 0$ and then $\frac{d^2\overline{w}}{ds^2} < 0$ by virtue of (7). Henceforth, $\frac{d\overline{w}}{ds}$ is a strictly monotone decreasing function of s. Analogous to the proof of Lemma 3.3, a contradiction to the strictly monotonicity of $\frac{d\overline{w}(s)}{ds}$ occurs, which finishes the proof of Theorem 4.2. ■

Based upon the conclusion of Theorem 4.2, we only need to consider the case $H_k \geqslant \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$. According to Lemma 3.1, the function $P_{H_k}(t)$ has only one real positive zero point as $H_k = \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$, while it has only two distinct real positive zero points for $H_k > \frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}}$. We will state and prove our results with these two situations separately.

Theorem 4.3. Let M be an n-dimensional $(n \geqslant 4)$ complete spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with constant k-th $(3 \leqslant k \leqslant n-1)$ mean curvature H_k and two distinct principal curvatures one of which is simple. Assume that $\frac{2c}{n}(\frac{c(k-2)}{k-n})^{\frac{k-2}{2}} < H_k < 0$. If the squared norm S of the second fundamental form of M satisfies

$$S \geqslant \max_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\}$$

or

$$S \leqslant \min_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\},$$

then M is isometric to the Riemannian product $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $c_1 < 0$, $c_2 < 0$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and t_3, t_4 are two distinct positive real roots of the equation $P_{H_k}(t) = 0$.

Proof. Recall that $t_0' = \frac{n(1-k)H_k}{n-2k+k^2} > 0$ given in Lemma 3.2 and $S = f(\lambda^k)$. We first prove that λ and μ are constants on M.

Case 1 When $t_0' \leqslant t_3$, then $\lambda^k \geqslant t_0'$ according to Lemma 3.3(1) and $f(t_3) < f(t_4)$ by means of Lemma 3.2(2). Thus, if $S \geqslant \max_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\}$, then

$f(\lambda^k) = S \geqslant (n-1)t_4^{\frac{2}{k}} + c^2 t_4^{-\frac{2}{k}} = f(t_4)$. Consequently, $\lambda^k \geqslant t_4$ from the monotonicity of $f(t)$, and then $P_{H_k}(\lambda^k) \geqslant 0$ from Case 2(3) of Lemma 3.1. So we derive that $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} \leqslant 0$ from (7), this means that $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a monotone decreasing function of s. Similar to the proof of Theorem 4.1, we conclude that $\overline{w}(s)$ is a constant on M, so do λ and μ.

Case 2 When $t_0' \geqslant t_4$, we know $\lambda^k \leqslant t_0'$ from Lemma 3.3(2), and $f(t_4) < f(t_3)$ because of $f(t)$ be a decreasing function for $0 < t \leqslant t_0'$ by Lemma 3.2(2). The assumption $S \geqslant \max\limits_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\}$ is equivalent to $S = f(\lambda^k) \geqslant f(t_3)$. Therefore, $\lambda^k \leqslant t_3$ by means of Lemma 3.2(2) again, we get $P_{H_k}(\lambda^k) \geqslant 0$ from Case 2(3) of Lemma 3.1. So we have $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} \leqslant 0$ from (7). Taking similar arguments as Case 1, we show that λ and μ are constants on M.

Case 3 When $t_3 \leqslant t_0' \leqslant t_4$, there are two subcases: $\lambda^k \leqslant t_0'$ and $\lambda^k \geqslant t_0'$.

If $\lambda^k \leqslant t_0'$, using the assumption $S = f(\lambda^k) \geqslant (n-1)t_3^{\frac{2}{k}} + c^2 t_3^{-\frac{2}{k}} = f(t_3)$, combining with Lemma 3.2(2), we obtain $\lambda^k \leqslant t_3$. Therefore, $P_{H_k}(\lambda^k) \geqslant 0$ from Case 2(3) of Lemma 3.1. If $\lambda^k \geqslant t_0'$, making use of the assumption $S = f(\lambda^k) \geqslant (n-1)t_4^{\frac{2}{k}} + c^2 t_4^{-\frac{2}{k}} = f(t_4)$ and Lemma 3.2(2) once more, we have $\lambda^k \geqslant t_4$. Hence $P_{H_k}(\lambda^k) \geqslant 0$ from Case 2(3) of Lemma 3.1. The rest of proof is just the same as Case 1.

Summarizing, if the assumption $S \geqslant \max\limits_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\}$ holds, we prove that λ and μ are constants on M. Similarly, replacing the assumption $S \geqslant \max\limits_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\}$ with $S \leqslant \min\limits_{i=3,4} \{(n-1)t_i^{\frac{2}{k}} + c^2 t_i^{-\frac{2}{k}}\}$, we can also prove that λ and μ are constants on M. The same arguments as that of Theorem 4.1 complete the proof of Theorem 4.3. ∎

Theorem 4.4. Let M be an n-dimensional ($n \geqslant 4$) complete spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with constant k-th ($3 \leqslant k \leqslant n-1$) mean curvature $H_k = \frac{2c}{n}\left(\frac{c(k-2)}{k-n}\right)^{\frac{k-2}{2}}$ and two distinct principal curvatures, one of which is simple. If the squared norm S of the second fundamental form of M satisfies

$$S \geqslant (n-1)t_2^{\frac{2}{k}} + c^2 t_2^{-\frac{2}{k}}$$

or

$$S \leqslant (n-1)t_2^{\frac{2}{k}} + c^2 t_2^{-\frac{2}{k}},$$

then M is isometric to the Riemannian product $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$ for some constants $c_1 < 0$, $c_2 < 0$ with $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and t_2 is the positive real root of the equation $P_{H_k}(t) = 0$.

Proof. By using the same methods as the proof of Theorem 4.3, together with Lemma 3.4, Lemma 3.2(2) and Case 2(2) of Lemma 3.1, we complete the proof of Theorem 4.4. ∎

5 The case of $k = n$

The remain case to characterize hypersurfaces in $\mathbb{H}_1^{n+1}(c)$ with non-zero constant k-th mean curvature H_k is that of $k = n$, i.e. non-zero constant Gauss-Kronecker curvature H_n. Making a careful analysis of Theorem 4.1, we find that the assumption "$H_k > 0$", $k = 1, \dots, n-1$, ensures actually the existence of the zero point of the function $P_{H_k}(t)$. However, if the constant $H_n > 0$, then the equation $P_{H_n}(t) = 0$ really has no real positive root. In this section, we considered the case $k = n$ and proved that: if Gauss-Kronecker curvature H_n is a non-zero constant, then $H_n < 0$, which ensures again the function $P_{H_n}(t)$ has a unique zero point, and the main idea of the proof developed there can also follow. As a result, we finally characterize such hypersurfaces as hyperbolic cylinders, see Theorem 5.4 below.

When $k = n$, the formulas (4), (5) reduces to, respectively

$$\mu = \frac{H_n}{\lambda^{n-1}}, \quad \lambda - \mu = \frac{\lambda^n - H_n}{\lambda^{n-1}} \neq 0.$$

The second order differential equation (6) becomes

$$\frac{d^2\overline{w}}{ds^2} + \overline{w}\frac{c\lambda^{n-2} - H_n}{\lambda^{n-2}} = 0, \tag{10}$$

where $\overline{w}(s) = |\lambda^n - H_n|^{-\frac{1}{n}}$, $s \in (-\infty, +\infty)$. Let $P_{H_n}(t) = ct^{\frac{n-2}{n}} - H_n$, $t > 0$, then (10) can be rewritten as

$$\frac{d^2\overline{w}}{ds^2} + \overline{w}\frac{P_{H_n}(\lambda^n)}{\lambda^{n-2}} = 0. \tag{11}$$

Using the same analysis as Lemmas 3.1 and 3.2, we have the following lemmas for $k = n$.

Lemma 5.1. For $t > 0$, $c < 0$, $n \geqslant 3$, let $P_{H_n}(t) = ct^{\frac{n-2}{n}} - H_n$, where $H_n = \text{const.}$. If $H_n \geqslant 0$, then $P_{H_n}(t) = 0$ has no real root and $P_{H_n}(t) < 0$; if $H_n < 0$, then $P_{H_n}(t) = 0$ has a unique positive real root $\bar{t}$. Furthermore, $P_{H_n}(t) \geqslant 0$ for $0 < t \leqslant \bar{t}$, and $P_{H_n}(t) \leqslant 0$ for $t \geqslant \bar{t}$.

Lemma 5.2. For $t > 0$, $H_n = \text{const.} < 0$, $n \geqslant 3$, let $f(t) = \frac{1}{t^{\frac{2n-2}{n}}}\left\{(n-1)t^2 + H_n^2\right\}$, then $f(\bar{t}) = (n-1)\bar{t}^{\frac{2}{n}} + c^2\bar{t}^{-\frac{2}{n}}$, where $\bar{t}$ is the unique positive real root of $P_{H_n}(t) = 0$. Furthermore, $f(t)$ is a monotone increasing function for $t \geqslant -H_n$, while it is a monotone decreasing function for $0 < t \leqslant -H_n$.

Lemma 5.3. With the unique positive real root $\bar{t}$ of $P_{H_n}(t) = 0$ given in Lemma 5.1, we have

(1) If $H_n \geqslant -(-c)^{\frac{n}{2}}$, then $-H_n \geqslant \bar{t}$ and $\lambda^n \leqslant -H_n$.
(2) If $H_n \leqslant -(-c)^{\frac{n}{2}}$, then $-H_n \leqslant \bar{t}$ and $\lambda^n \geqslant -H_n$.

Proof. Since $\frac{dP_{H_n}(t)}{dt} = \frac{c(n-2)}{n}t^{-\frac{2}{n}} < 0$, so $P_{H_n}(t)$ is a strictly monotone decreasing function of t. Notice that

$$P_{H_n}(-H_n) = c(-H_n)^{\frac{n-2}{n}} - H_n = (-H_n)^{\frac{n-2}{n}}(c + (-H_n)^{\frac{2}{n}}).$$

If $H_n \geqslant -(-c)^{\frac{n}{2}}$, then $P_{H_n}(-H_n) \leqslant 0$ and $-H_n \geqslant \bar{t}$. Assume on the contrary that $\lambda^n > -H_n$, then $\lambda^n > \bar{t}$, $P_{H_n}(\lambda^n) < P_{H_n}(\bar{t}) = 0$ from Lemma 5.1. Therefore, (11) implies $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} > 0$, i.e. $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is strictly monotone increasing. Analogous to the proof of Lemma 3.3, a contradiction occurs which forces that $\lambda^n \leqslant -H_n$. Another case can be proved similarly. ∎

Theorem 5.4. Let M be an n-dimensional ($n \geqslant 3$) complete spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with constant Gauss-Kronecker curvature $H_n \neq 0$ and two distinct principal curvatures λ (with multiplicity $n-1$) and μ. The following holds:

(1) There is no such hypersurfaces with $H_n > 0$.

(2) When $H_n < 0$, if the squared norm S of the second fundamental form of M satisfies

$$S \geqslant (n-1)\bar{t}^{\frac{2}{n}} + c^2\bar{t}^{-\frac{2}{n}},$$

or

$$S \leqslant (n-1)\bar{t}^{\frac{2}{n}} + c^2\bar{t}^{-\frac{2}{n}},$$

then M isometric to the Riemannian product $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $c_1 < 0$, $c_2 < 0$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and $\bar{t}$ is a unique positive real root of the equation $P_{H_n}(t) = 0$ for $t > 0$.

Proof. Taking similar processing to the proof of Theorem 4.2, we can achieve our nonexistence conclusion. In order to characterize spacelike hypersurfaces with $H_n < 0$, using Lemma 5.3 we need to analysis under the case either $-(-c)^{\frac{n}{2}} \leqslant H_n < 0$ or $H_n \leqslant -(-c)^{\frac{n}{2}}$, and taking the same arguments as in the proof of Theorem 4.1, together with Lemmas 5.1, 5.2, we finish the proof of Theorem 5.4. ∎

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