A p-Laplace equation with nonlocal boundary condition in a perforated-like domain*

Hailong Ye Yuanyuan Ke†

Abstract

In this paper, we consider the radially symmetric solutions for p-Laplacian with nonlocal boundary condition in a perforated-like domain. We obtain the existence, the uniqueness and some other properties of the radially symmetric solution. The nonexistence of solution is also studied.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, $0 \in \Omega$. We consider the radially symmetric solutions of the following p-Laplace equation in a perforated-like domain

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = h(x)f\left(x, u, |\nabla u|\right), \quad x \in \Omega \backslash \{0\}, \tag{1.1}$$

subject to the nonlocal boundary condition

$$\int_{\partial\Omega} |x|^{n-1} \left|\frac{\partial u}{\partial \nu}\right|^{p-2} \frac{\partial u}{\partial \nu}\,\mathrm{d}s = \int_{\Omega} |x|^{n-1} |\nabla u|^{p-2} \nabla u \cdot \nabla g\,\mathrm{d}x, \tag{1.2}$$

and

$$\lim_{x \to 0} u(x) = \alpha, \tag{1.3}$$

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†Corresponding author.

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where $\Omega\backslash\{0\}$ can be considered as the limit of $\Omega\backslash B_\varepsilon$, B_ε is a ball centered at the origin with radius ε small enough, $p > 2$, $\alpha \geq 0$, f, g and h are given functions and ν denotes the unit outward normal to the boundary $\partial\Omega$. In order to discuss the radially symmetric solutions, we assume that Ω is the unit ball B, $h(x)$, $g(x)$ and $f(x, u, |\nabla u|)$ are radially symmetric, namely,

$$h(x) = h(|x|), \quad g(x) = g(|x|), \quad f(x, u, |\nabla u|) = f(|x|, u, |\nabla u|).$$

Let $r = |x|$, then by a direct calculation, we can rewrite the problem (1.1)–(1.3) as

$$\left(r^{n-1}\phi_p(u')\right)' + r^{n-1}h(r)f(r, u, |u'|) = 0, \quad r \in (0, 1), \tag{1.4}$$

$$\phi_p\big(u'(1)\big) = \int_0^1 s^{n-1}\phi_p\big(u'(s)\big)\,\mathrm{d}g(s), \tag{1.5}$$

$$\lim_{r\to 0^+} u(r) = \alpha, \tag{1.6}$$

where $\phi_p(s) = |s|^{p-2}s$. Let $q = p/(p-1)$, then we have $\phi_p^{-1}(s) = \phi_q(s)$ for any $s \in \mathbb{R}$.

Nonlocal boundary value problem often occurs in the study of the electrochemistry, the thermal conduction problem, the semiconductor problem, etc., see [1]–[3]. This class of problems were first considered by Bitsadze [4] in the early 1960s. From then on, more and more workers take their notice of these problems, such as Il'in and Moiseev [5], Karakostas and Tsamatos [6]–[8], etc. Until now, the problems with nonlocal boundary value condition, as well as with the multipoint boundary value condition, also attract many authors to pay attention to, see [9]–[15], and the references cited therein. To the best of our knowledge, most works we mentioned above are focus on the discussion of the existence of the solutions, however, the works of studying the uniqueness of positive solutions for p-Laplacian are rather few in the literature.

In this paper, we study the existence, the uniqueness and some other properties of the radially symmetric solutions of the nonlocal boundary value problem (1.1)–(1.3), in which we extend the function $f(u)$ in [13] to the more general case $f(x, u, |\nabla u|)$. Since f is dependent on the first-order partial derivatives of $u(x)$, we have to find the variational relationship between $u(r)$ and $u'(r)$ under different conditions. And a fixed point result, called the nonlinear alternative of Leray-Schauder, which can be found in [16], would be used to obtain the existence of solutions.

The paper is organized as follows. In Section 2, we introduce some necessary preliminaries and give the statement of our main results. The proofs of the main results will be given in Section 3.

2 Preliminary and Statement of the Main Result

We firstly present the assumptions.

(H1) $f(r, s, t)$ is a continuous and positive function defined on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, which is strictly decreasing with respect to s for each fixed $(r, t) \in [0, 1] \times [0, +\infty)$ and t for each fixed $(r, s) \in [0, 1] \times [0, +\infty)$, respectively;

(H2) $h(r)$ is a positive and continuous function on $[0,1]$;
(H3) $g(r)$ is a nondecreasing function on $[0,1]$ with $0 = g(0) < g(1) < 1$.

Now we introduce the definition of the solution.

Definition 2.1. *A function $u(r)$ is said to be a solution of the equation (1.4), if $u(r) \in C([0,1]) \cap C^1((0,1])$, $u(r) \geq 0$ on $[0,1]$, and the integral equality*

$$\int_0^1 \left(r^{n-1}\phi_p(u')\varphi'(r) - r^{n-1}h(r)f(r,u,|u'|)\varphi(r) \right) \mathrm{d}r = 0$$

holds for any $\varphi \in C_0^\infty((0,1))$.

Remark 2.1. *Let $u(r)$ be a solution of the equation (1.4), then u satisfies the equation (1.4) in $(0,1)$.*

Proof. According to Definition 2.1, we have

$$\left(r^{n-1}\phi_p(u') \right)' = -r^{n-1}h(r)f(r,u,|u'|)$$

in $(0,1)$ in the sense of distribution. Furthermore, by virtue of the assumptions (H1) and (H2), we obtain

$$\left(r^{n-1}\phi_p(u') \right)' \in C((0,1)),$$

hence,

$$r^{n-1}\phi_p(u') \in C^1((0,1)),$$

which implies that u satisfies the equation (1.4) in $(0,1)$. The proof is complete. ■

Next, we can derive the following properties of the solution u for the nonlocal boundary value problem (1.4)–(1.6), by using the assumptions above.

Proposition 2.1. *Let $u(r)$ be a solution of the nonlocal boundary value problem (1.4)–(1.6). Then*

(i) $u'(r) > 0$, $r \in (0,1]$;

(ii) $u(r) \in C^2((0,1))$;

(iii) $u''(r) < 0$, $r \in (0,1]$;

(iv) $0 < \lim\limits_{r \to 0^+} r^{n-1}\phi_p(u'(r)) < +\infty$.

Proof. **(i)** From the equation (1.4), we can see that

$$\left(r^{n-1}\phi_p(u') \right)' = -r^{n-1}h(r)f(r,u,|u'|) < 0, \quad r \in (0,1),$$

which implies that

$$r^{n-1}\phi_p(u'(r)) > \phi_p(u'(1)), \quad r \in (0,1).$$

If $u'(1) = 0$, from the above inequality, we have

$$0 = \phi_p\big(u'(1)\big) = \int_0^1 s^{n-1}\phi_p\big(u'(s)\big)\,\mathrm{d}g(s) > 0,$$

which is a contradiction. If $u'(1) < 0$, however, we see that

$$\int_0^1 s^{n-1}\phi_p\big(u'(s)\big)\mathrm{d}g(s) \geq \phi_p\big(u'(1)\big)\int_0^1 \mathrm{d}g(s) = \phi_p\big(u'(1)\big)g(1) > \phi_p\big(u'(1)\big),$$

which contradicts the nonlocal boundary value condition (1.5). Thus $u'(1) > 0$ and $\phi_p\big(u'(r)\big) > 0$, $r \in (0,1)$, that is,

$$u'(r) > 0, \quad r \in (0,1].$$

(ii) Recalling Remark 2.1, we have

$$r^{n-1}\phi_p(u') \in C^1((0,1)).$$

Since $u'(r) > 0$, $r \in (0,1]$, it follows

$$r^{n-1}\phi_p(u') = r^{n-1}(u')^{p-1} \in C^1((0,1)).$$

Note that $n \geq 2$ and $p > 2$, we have

$$u(r) \in C^2((0,1)).$$

(iii) The equation (1.4) implies that for any $r \in (0,1]$,

$$u''(r) = -\left(\frac{(n-1)u'(r)}{(p-1)r} + \frac{h(r)f(r,u,u')}{(p-1)(u')^{p-2}}\right) < 0.$$

(iv) Integrating the equation (1.4) from r to 1, it yields

$$\begin{aligned} r^{n-1}\phi_p\big(u'(r)\big) &= \phi_p\big(u'(1)\big) + \int_r^1 s^{n-1}h(s)f(s,u,u')\,\mathrm{d}s \\ &\leq \phi_p\big(u'(1)\big) + \int_0^1 s^{n-1}h(s)f(s,0,0)\,\mathrm{d}s \\ &< +\infty, \end{aligned}$$

uniformly in $r \in (0,1)$. Since

$$r^{n-1}\phi_p\big(u'(r)\big) > \phi_p\big(u'(1)\big) > 0,$$

$\lim_{r\to 0^+} r^{n-1}\phi_p\big(u'(r)\big)$ is existent and

$$0 < \phi_p\big(u'(1)\big) \leq \lim_{r\to 0^+} r^{n-1}\phi_p\big(u'(r)\big) < +\infty.$$

Summing up, we complete the proof of Proposition 2.1. ∎

Define a normed linear space X, which is the set of all real-valued functions defined in $C([0,1]) \cap C^1((0,1])$ with the norm

$$||u|| \triangleq \max\left\{ \sup_{0\le r\le 1} |u(r)|,\ \sup_{0<r\le 1} |ru'(r)| \right\}.$$

We can prove that X is a Banach space. Obviously, Proposition 2.1 indicates that for any solution u of the nonlocal boundary value problem (1.4)–(1.6), $u \in X$.

The main results in this paper are the following two theorems.

Theorem 2.1. *Assume $p > n$ and* (H1)–(H3) *hold. For any $\alpha \ge 0$, the nonlocal boundary value problem (1.4)–(1.6) has a unique solution $u_\alpha(r) \in X$. Furthermore, if $0 \le \alpha_1 < \alpha_2$, then $y(r) = |u_{\alpha_1}(r) - u_{\alpha_2}(r)|$ has no local maximum value point in $(0,1)$.*

Theorem 2.2. *Assume $p \le n$ and* (H1)–(H3) *hold. For any $\alpha \ge 0$, the nonlocal boundary value problem (1.4)–(1.6) has no solution.*

According to Theorem 2.1, we have $y(r) < \max\{y(0),\, y(1)\}$. Therefore,

Remark 2.2. *If $p > n$ and $u_{\alpha_1}(r)$, $u_{\alpha_2}(r)$ are two solutions of the problem (1.4)–(1.6) with $u_{\alpha_1}(0) = \alpha_1$, $u_{\alpha_2}(0) = \alpha_2$, and $\alpha_1 \ne \alpha_2$. Then we have*

$$|u_{\alpha_1}(r) - u_{\alpha_2}(r)| < \max\left\{|\alpha_1 - \alpha_2|,\, |u_{\alpha_1}(1) - u_{\alpha_2}(1)|\right\},\ r \in (0,\,1).$$

3 Proofs of the Main Results

In this section, we give the proofs of the main results. In order to study the existence and uniqueness of solution of the nonlocal boundary value problem (1.4)–(1.6), we should first consider the following approximate problem, where we assume (H1)–(H3) hold true, and $\alpha \ge 0$, $\beta > 0$,

$$\begin{cases} \left(r^{n-1}\phi_p(u')\right)' + r^{n-1}h(r)f(r,u,|u'|) = 0, & r \in (0,1), \\ u(0) = \alpha, \\ u'(1) = \beta. \end{cases} \tag{3.1}$$

By applying some similar methods of the proof of Proposition 2.1, we can also obtain the following properties of the solutions for the boundary value problem (3.1).

Remark 3.1. *Assume $u(r)$ is a solution of the problem (3.1), then*

- **(i)** $u'(r) > 0$, $r \in (0,1]$;
- **(ii)** $u(r) \in C^2((0,1))$;
- **(iii)** $u''(r) < 0$, $r \in (0,1]$;
- **(iv)** $0 < \lim\limits_{r\to 0^+} r^{n-1}\phi_p\left(u'(r)\right) < \infty$.

In what follows, we shall show that the problem (3.1) admits one and only one solution in $C([0,1]) \cap C^1((0,1])$. We need the following lemmas.

Lemma 3.1. *Assume $u_1(r)$ and $u_2(r)$ are two solutions of the problem (3.1). If there exist a point $b \in (0,1]$, such that $u_1(b) = u_2(b)$, then for any $r \in [0,b]$, we have*

$$u_1(r) = u_2(r), \quad r \in [0,b].$$

Proof. We give the proof by contradiction. Without loss of generality, suppose that there exist a point $r_0 \in (0,b)$ such that $u_1(r_0) < u_2(r_0)$. Note that $u_1(0) = u_2(0)$ and $u_1(b) = u_2(b)$, we might as well take

$$b_1 = \inf\{r;\, 0 \le r < r_0,\, u_1(s) < u_2(s),\, s \in (r, r_0)\}$$

and

$$b_2 = \sup\{r;\, r_0 < r \le b,\, u_1(s) < u_2(s),\, s \in (r_0, r)\}.$$

Then for any $r \in (b_1, b_2)$, we have $u_1(r) < u_2(r)$ and $u_1(b_1) = u_2(b_1)$, $u_1(b_2) = u_2(b_2)$. Obviously, $u_1(r) - u_2(r)$ must have a minimal value point $c \in (b_1, b_2)$, such that $u_1'(c) = u_2'(c)$, $u_1''(c) > u_2''(c)$. A simple calculation for the first equation of the problem (3.1) shows that the solution u of the problem (3.1) satisfies

$$(n-1)r^{n-2}(u')^{p-1} + (p-1)r^{n-1}(u')^{p-2}u'' + r^{n-1}h(r)f(r,u,u') = 0.$$

Therefore, for the two solutions $u_1(r)$ and $u_2(r)$, we have

$$\begin{aligned}&(n-1)c^{n-2}(u_1')^{p-1}(c) + (p-1)c^{n-1}(u_1')^{p-2}(c)u_1''(c) + c^{n-1}h(c)f(c,u_1(c),u_1'(c))\\ &= (n-1)c^{n-2}(u_2')^{p-1}(c) + (p-1)c^{n-1}(u_2')^{p-2}(c)u_2''(c) + c^{n-1}h(c)f(c,u_2(c),u_2'(c)).\end{aligned}$$

By virtue of (H1) and (H2), we see that $f(c,u_1(c),u_1'(c)) > f(c,u_2(c),u_2'(c))$ and $h(c) > 0$, hence we conclude that $u_1''(c) < u_2''(c)$, which is a contradiction. The proof is complete. ∎

Lemma 3.2. *Assume $u_1(r)$ and $u_2(r)$ are two solutions of the problem (3.1) with boundary value conditions*

$$\begin{aligned}u_1(0) &= \alpha_1 \ge 0, \quad u_1'(1) = \beta_1 > 0;\\ u_2(0) &= \alpha_2 \ge 0, \quad u_2'(1) = \beta_2 > 0,\end{aligned}$$

respectively. If there exists a point $b \in (0,1)$ such that $u_1(b) < u_2(b)$, $u_1'(b) < u_2'(b)$, then for any $r \in (b,1]$, we have $u_1(r) < u_2(r)$ and $u_1'(r) < u_2'(r)$.

Proof. Suppose to the contrary, that is, there exists a point $r_0 \in (b,1]$ such that $u_1'(r_0) \ge u_2'(r_0)$, then we take

$$r^* = \inf\{r;\, b < r \le r_0,\, u_1'(s) \ge u_2'(s),\, s \in (b,r]\}.$$

Obviously, r^* is exist, hence we have

$$\begin{aligned}u_1(r) &< u_2(r), \quad r \in [b, r^*],\\ u_1'(r) &< u_2'(r), \quad r \in [b, r^*),\end{aligned}$$

and

$$u_1'(r^*) = u_2'(r^*).$$

Then on the point r^*, by utilizing a similar method of the proof of Lemma 3.1, we obtain $u_1''(r^*) < u_2''(r^*)$, which implies that r^* is not a stationary point of $u_1'(r) - u_2'(r)$. Clearly, it is a contradiction since $u_1'(r^*) = u_2'(r^*)$. Thus we have

$$u_1'(r) < u_2'(r), \quad r \in [b, 1].$$

Since $u_1(b) < u_2(b)$, we further have

$$u_1(r) < u_2(r), \quad r \in [b, 1].$$

The proof is complete. ∎

Lemma 3.3. *For any fixed $\alpha \geq 0$ and $\beta > 0$, the problem (3.1) admits at most one solution.*

Proof. We give the proof by contradiction. Without loss of generality, assume that $u_1(r)$ and $u_2(r)$ are two solutions of the problem (3.1), and there exists one point $b \in (0, 1]$, such that $u_1(b) < u_2(b)$. By a simple analysis, we see that there must exist another point $r_0 \in (0, b]$ such that $u_1(r_0) < u_2(r_0)$ and $u_1'(r_0) < u_2'(r_0)$. Recalling Lemma 3.2, we can conclude that

$$u_1(r) < u_2(r) \quad \text{and} \quad u_1'(r) < u_2'(r), \quad \text{for any } r \in [r_0, 1).$$

Then from (H1) we have

$$\begin{aligned} 0 &> u_1'(r_0) - u_2'(r_0) \\ &= \phi_q \left[\frac{1}{r_0^{n-1}} \left(\int_{r_0}^1 r^{n-1} h(r) f\big(r, u_1(r), u_1'(r)\big) \, \mathrm{d}r + \beta^{p-1} \right) \right] \\ &\quad - \phi_q \left[\frac{1}{r_0^{n-1}} \left(\int_{r_0}^1 r^{n-1} h(r) f\big(r, u_2(r), u_2'(r)\big) \, \mathrm{d}r + \beta^{p-1} \right) \right] \\ &> 0. \end{aligned}$$

The contradiction implies that the lemma is proved. ∎

We use the following fixed point theorem, which can be found in [16], to obtain the solution of the problem (3.1).

Lemma 3.4. *Assume U is a relatively open subset of a convex set K in a Banach space X. Let $G\colon \bar{U} \to K$ be a compact map, $p \in U$, and $N_\lambda(u) = N(u, \lambda)\colon \bar{U} \times [0, 1] \to K$ a family of compact maps (i.e., $N(\bar{U} \times [0, 1])$ is contained in a compact subset of K and $N\colon \bar{U} \times [0, 1] \to K$ is continuous) with $N_1 = G$ and $N_0 = p$, the constant map to p. Then either*

- **(i)** *G has a fixed point in $\bar{U}$; or*
- **(ii)** *There is a point $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = N_\lambda u$.*

Lemma 3.5. *If $p > n$, then for any fixed $\alpha \geq 0$ and $\beta > 0$, the problem (3.1) has a solution $u(r) \in C([0,1]) \cap C^1((0,1])$.*

Proof. We first consider the following problem for any fixed parameter $\lambda \in (0,1)$,

$$\begin{cases} \left(r^{n-1}\phi_p(u')\right)' + \lambda r^{n-1}h(r)f(r,u,|u'|) = 0, & r \in (0,1), \\ u(0) = \alpha, \\ u'(1) = \beta. \end{cases} \tag{3.2}$$

Obviously, since Remark 3.1 also holds for the problem (3.2), we can see that $u \in X$ for any solution u of the problem (3.2). Therefore, solving the problem (3.2) is equivalent to finding a nonnegative solution $u(r) \in X$ with $u(r)$ satisfying

$$u(r) = \alpha + \int_0^r \phi_q \left[\frac{1}{s^{n-1}} \left(\lambda \int_s^1 t^{n-1}h(t)f\big(t,u,|u'|\big)\,\mathrm{d}t + \beta^{p-1} \right) \right] \mathrm{d}s. \tag{3.3}$$

Since $p > n$, it follows that

$$u'(r) = \phi_q \left[\frac{1}{r^{n-1}} \left(\lambda \int_r^1 s^{n-1}h(s)f\big(s,u,|u'|\big)\,\mathrm{d}s + \beta^{p-1} \right) \right]$$

is integrable on $(0,1)$, then the right side of (3.3) is reasonable in $[0,1]$. Define the operator $N_\lambda : K \to K$ by

$$(N_\lambda u)(r) = \alpha + \int_0^r \phi_q \left[\frac{1}{s^{n-1}} \left(\lambda \int_s^1 t^{n-1}h(t)f\big(t,u,|u'|\big)\,\mathrm{d}t + \beta^{p-1} \right) \right] \mathrm{d}s, \tag{3.4}$$

where $K = \{u \in X;\, u(0) = \alpha,\, u'(1) = \beta\}$.

In what follows, we will show that the operator N_1 has a fixed point in X. The proof will be given in several steps.

Step 1: We shall show that there is a constant M^*, independent of λ, such that $||u|| \leq M^*$ for any solution $u(r)$ of the problem (3.2) for each $\lambda \in (0,1)$.

Let $u(r)$ be a solution of the problem (3.2). According to the equation (3.3), we have

$$\begin{aligned}
|u(r)| &\leq \alpha + \int_0^r \phi_q \left[\frac{1}{s^{n-1}} \left(\int_s^1 t^{n-1}h(t)f(r,0,0)\,\mathrm{d}t + \beta^{p-1} \right) \right] \mathrm{d}s \\
&\leq \alpha + \int_0^1 \phi_q \left[\frac{1}{s^{n-1}} \left(\int_0^1 t^{n-1}h(t)f(r,0,0)\,\mathrm{d}t + \beta^{p-1} \right) \right] \mathrm{d}s \\
&\leq \alpha + \phi_q(M_1) \int_0^1 s^{-(n-1)/(p-1)}\,\mathrm{d}s, \\
|ru'(r)| &= r\phi_q \left[\frac{1}{r^{n-1}} \left(\lambda \int_r^1 t^{n-1}h(t)f\big(t,u,|u'|\big)\,\mathrm{d}t + \beta^{p-1} \right) \right] \\
&\leq r\phi_q \left[\frac{1}{r^{n-1}} \left(\int_0^1 t^{n-1}h(t)f(r,0,0)\,\mathrm{d}t + \beta^{p-1} \right) \right] \\
&\leq r\phi_q \left(\frac{M_1}{r^{n-1}} \right),
\end{aligned}$$

where

$$M_1 > \int_0^1 r^{n-1}h(r)f(r,0,0)\,\mathrm{d}r + \beta^{p-1}$$

is a constant. Thus there is a constant M^*, independent of λ, such that $||u|| \le M^*$ for any solution $u(r)$ of the problem (3.2) for each $\lambda \in (0,1)$.

Step 2: It is easy to see that N_λ is continuous for any fixed λ. We will show that N_λ is even completely continuous for fixed λ by Arzela-Ascoli theorem.

Let Ω be a bounded subset of K, i.e., $||u|| \le C$ for all $u \in \Omega$. Here $C > 0$ is a constant. Firstly, following the proof in Step 1, we can see that $N_\lambda\Omega$ is closed and bounded, and there exist the following two inequalities

$$|(N_\lambda u)(r)| \le \alpha + \int_0^1 \phi_q\left(\frac{1}{s^{n-1}}M_2\right)\mathrm{d}s \tag{3.5}$$

and

$$|r(N_\lambda u)'(r)| \le \phi_q(M_2), \tag{3.6}$$

where

$$M_2 = \sup\left\{|f(r,u,0)|;\, 0 \le r \le 1,\, -C \le u \le C\right\}\int_0^1 t^{n-1}h(t)\,\mathrm{d}t + \beta^{p-1}.$$

We next show the equicontinuity of $N_\lambda\Omega$ on $[0,1]$. For $u \in \Omega$ and r_1, $r_2 \in [0,1]$, we have

$$\begin{aligned} &|(N_\lambda u)(r_1) - (N_\lambda u)(r_2)| \\ \le &\left|\int_{r_1}^{r_2}\phi_q\left[\frac{1}{s^{n-1}}\left(\lambda\int_s^1 t^{n-1}h(t)f\left(t,u,|u'|\right)\mathrm{d}t + \beta^{p-1}\right)\right]\mathrm{d}s\right| \\ \le &\,\phi_q(M_2)\left|\int_{r_1}^{r_2} s^{-(n-1)/(p-1)}\,\mathrm{d}s\right|, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\lim_{r\to 0^+} r(N_\lambda u)'(r) \\ = &\lim_{r\to 0^+} r\phi_q\left[\frac{1}{r^{n-1}}\left(\lambda\int_r^1 t^{n-1}h(t)f\left(t,u,|u'|\right)\mathrm{d}t + \beta^{p-1}\right)\right] \\ = &\lim_{r\to 0^+} r^{(p-n)/(p-1)}\phi_q\left(\lambda\int_0^1 t^{n-1}h(t)f\left(t,u,|u'|\right)\mathrm{d}t + \beta^{p-1}\right) \\ = &\,0. \end{aligned} \tag{3.8}$$

The equicontinuity of $N_\lambda\Omega$ on $[0,1]$ now follows from the inequality (3.7) and the equation (3.8). Therefore, the Arzela-Ascoli theorem implies that N_λ is completely continuous.

Step 3: We shall show that $N(\bar{U} \times [0,1])$ is contained in a compact subset of K, where

$$U = \{u \in K;\, ||u|| \le M^* + 1\},$$
$$(N_0 u)(r) = \alpha + \beta\int_0^r \phi_q\left(\frac{1}{s^{n-1}}\right)\mathrm{d}s.$$

Let $N(u_n, \lambda_n)$ be any sequence in $N(\bar{U} \times [0,1])$. Then it is easy to see that $N(u_n, \lambda_n)$ is uniformly bounded and equicontinuous on $[0,1]$ since the inequality (3.5)–(3.7) and the equation (3.8) does not depend on the fixed λ, thus the Arzela-Ascoli theorem again yields the result. By Lemma 3.4, we deduce that N_1 has a fixed point, i.e., the problem (3.1) has a solution $u(r) \in C([0,1]) \cap C^1((0,1])$.

Summing up, we complete the proof of Lemma 3.5. ∎

Lemma 3.3 and Lemma 3.5 implies that the problem (3.1) has an unique solution. According to this, in what follows, by using the shooting method, precisely speaking, by selecting $\beta > 0$ suitably in the problem (3.1) such that the nonlocal boundary value condition (1.5) holds, we will show that the problem (1.4)–(1.6) admits a unique solution. Before going further, we need the following lemmas.

Lemma 3.6. *If $u_1(r)$, $u_2(r)$ are two solutions of the problem (3.1) with $u_1'(1) = \beta_1$, $u_2'(1) = \beta_2$, and $\beta_1 > \beta_2 > 0$, then*

$$u_1(r) \geq u_2(r), \quad r \in (0,1].$$

Proof. If the lemma were not true, there must exist one point $r_0 \in (0,1)$ such that $u_1(r_0) < u_2(r_0)$. Since $u_1(0) = u_2(0)$, it is easy to see that there exists another point $r_1 \in (0, r_0)$, such that $u_1(r_1) < u_2(r_1)$, $u_1'(r_1) < u_2'(r_1)$ by considering the continuity of the solution. According to Lemma 3.2, we conclude that $u_1'(1) < u_2'(1)$, namely, $\beta_1 < \beta_2$, which is a contradiction. The proof is complete. ∎

Lemma 3.7. *If $u_1(r)$, $u_2(r)$ are two solutions of the problem (3.1) with $u_1'(1) = \beta_1$, $u_2'(1) = \beta_2$, and $\beta_1 > \beta_2 > 0$, then*

$$u_1'(r) \geq u_2'(r), \quad r \in (0,1).$$

Proof. If the lemma were not true, there exists a point $r_0 \in (0,1)$ such that $u_1'(r_0) < u_2'(r_0)$. According to Lemma 3.6, we see that $u_1(r_0) \geq u_2(r_0)$. We will show that $u_1(r_0) > u_2(r_0)$. Otherwise, we have $u_1(r_0) = u_2(r_0)$. Since that $u_1'(r_0) < u_2'(r_0)$ and $u_1(0) = u_2(0)$, therefore, there must exist one point $b \in (0, r_0)$ such that $u_1(b) > u_2(b)$ and $u_1'(b) > u_2'(b)$ by considering the continuity of the solution. According to Lemma 3.2, we have $u_1(r_0) > u_2(r_0)$ which is a contradiction. Since $u_1(0) = u_2(0)$, $u_2(r) - u_1(r)$ must have a minimal value point $c \in (0, r_0)$, hence we have $u_1(c) > u_2(c)$, $u_1'(c) = u_2'(c)$. By utilizing a similar method in Lemma 3.1, we see that $u_1''(c) > u_2''(c)$, clearly, which contradicts the fact that c is a minimal value point of $u_2(r) - u_1(r)$. Therefore the proof is complete. ∎

Combining Lemma 3.6 and Lemma 3.7, we obtain the following conclusion.

Corollary 3.1. *Let $u_1(r)$, $u_2(r)$ be two solutions of the problem (3.1) with $u_1'(1) = \beta_1$, $u_2'(1) = \beta_2$, and $\beta_1 > \beta_2 > 0$. Then there exists a point $b \in [0,1)$ such that*

$$u_1(r) = u_2(r), \quad 0 \leq r \leq b, \tag{3.9}$$

$$u_1(r) > u_2(r), \quad b < r \leq 1, \tag{3.10}$$

$$u_1'(r) > u_2'(r), \quad b < r \leq 1. \tag{3.11}$$

Proof. Let
$$b = \sup\{r;\, 0 \le r < 1,\, u_1(r) = u_2(r)\}.$$
By virtue of $u_1(0) = u_2(0)$ and Lemma 3.1, we have $0 \le b < 1$. Furthermore, together with Lemma 3.6, we also have

$$\begin{aligned} u_1(r) &= u_2(r), \quad && 0 \le r \le b, \\ u_1(r) &> u_2(r), \quad && b < r \le 1. \end{aligned}$$

So, it suffices to consider the inequality (3.11). If it were not true, then according to Lemma 3.7, there exists a point $r_0 \in (b, 1)$ such that $u_1'(r_0) = u_2'(r_0)$. By utilizing a similar method to the proof of Lemma 3.1, we see that $u_1''(r_0) > u_2''(r_0)$, which implies that the point r_0 is a minimal value point of $u_2(r) - u_1(r)$. Hence, there must exist a point $r_1 \in (b, r_0)$ such that $u_1'(r_1) < u_2'(r_1)$, which contradicts Lemma 3.7. The proof is complete. ∎

Combining with the above lemmas, we are now in a position to get the existence and uniqueness of solution of the problem (1.4)–(1.6) by the shooting methods. Define

$$k(\beta) = \phi_p\big(u_\beta'(1)\big) - \int_0^1 s^{n-1}\phi_p\big(u_\beta'(s)\big)\,\mathrm{d}g(s), \quad \beta > 0,$$

where u_β is the solution of the problem (3.1) with boundary value condition $u_\beta'(1) = \beta$. Then the desired unique solution of the problem (1.4)–(1.6) will be obtained by selecting an unique constant $\beta^* > 0$, such that $k(\beta^*) = 0$. First of all, we need to establish the strict monotonicity and continuity of the function $k(\beta)$ with respect to β.

Lemma 3.8. *$k(\beta)$ is continuous and strictly monotonous with respect to $\beta > 0$.*

Proof. Let $\beta_1 > \beta_2 > 0$. According to Corollary 3.1 and the first equation of the problem (3.1), we derive

$$\begin{aligned} 0 &\le \phi_p(u_{\beta_1}') - \phi_p(u_{\beta_2}') \\ &= \frac{1}{r^{n-1}}\Bigg[\bigg(\int_r^1 s^{n-1}h(s)f\big(s, u_{\beta_1}, u_{\beta_1}'\big)\,\mathrm{d}s + \beta_1^{p-1}\bigg) \\ &\quad - \bigg(\int_r^1 s^{n-1}h(s)f\big(s, u_{\beta_2}, u_{\beta_2}'\big)\,\mathrm{d}s + \beta_2^{p-1}\bigg)\Bigg] \\ &= \frac{1}{r^{n-1}}\Bigg[\bigg(\int_r^1 s^{n-1}h(s)\Big(f\big(s, u_{\beta_1}, u_{\beta_1}'\big) - f\big(s, u_{\beta_1}, u_{\beta_1}'\big)\Big)\,\mathrm{d}s \\ &\quad + (\beta_1^{p-1} - \beta_2^{p-1})\bigg)\Bigg], \end{aligned}$$

where u_{β_1} and u_{β_2} are the solutions of the problem (3.1) with $u_{\beta_1}'(1) = \beta_1$, $u_{\beta_2}'(1) = \beta_2$. By the assumption (H1), it is easy to see that

$$f\big(s, u_{\beta_1}, u_{\beta_1}'\big) - f\big(s, u_{\beta_2}, u_{\beta_2}'\big) \le 0.$$

Thus

$$0 \le \phi_p(u'_{\beta_1}) - \phi_p(u'_{\beta_2}) \le \frac{1}{r^{n-1}}(\beta_1^{p-1} - \beta_2^{p-1}),$$

namely,

$$0 \le r^{n-1}(\phi_p(u'_{\beta_1}) - \phi_p(u'_{\beta_2})) \le \beta_1^{p-1} - \beta_2^{p-1}.$$

By the above inequality, we obtain

$$\begin{aligned} & k(\beta_1) - k(\beta_2) \\ &= (\beta_1^{p-1} - \beta_2^{p-1}) - \int_0^1 s^{n-1}\left(\phi_p(u'_{\beta_1}(s)) - \phi_p(u'_{\beta_2}(s))\right) \mathrm{d}g(s) \\ &\ge (\beta_1^{p-1} - \beta_2^{p-1}) - \int_0^1 (\beta_1^{p-1} - \beta_2^{p-1}) \,\mathrm{d}g(s) \\ &= (\beta_1^{p-1} - \beta_2^{p-1})(1 - g(1)) \\ &> 0. \end{aligned}$$

Furthermore, we can see the continuity of $k(\beta)$ when $\beta > 0$ from the following inequality

$$\begin{aligned} & |k(\beta_1) - k(\beta_2)| \\ &= \left|(\beta_1^{p-1} - \beta_2^{p-1}) - \int_0^1 s^{n-1}\left(\phi_p(u'_{\beta_1}(s)) - \phi_p(u'_{\beta_2}(s))\right) \mathrm{d}g(s)\right| \\ &\le \left|\beta_1^{p-1} - \beta_2^{p-1}\right| + \int_0^1 \left|\beta_1^{p-1} - \beta_2^{p-1}\right| \mathrm{d}g(s) \\ &\le \left|\beta_1^{p-1} - \beta_2^{p-1}\right| (1 + g(1)). \end{aligned}$$

Hence, the proof of Lemma 3.8 is complete. ■

Now, by virtue of the above established lemmas about the approximate problem (3.1), we are going to prove the main results in this paper.

Proof of Theorem 2.1. A simple calculation for the first equation of the problem (3.1) implies that

$$\int_0^1 s^{n-1}\phi_p(u'_\beta)\,\mathrm{d}g(s) = \int_0^1 \left(\beta^{p-1} + \int_s^1 t^{n-1}h(t)f(t, u_\beta, u'_\beta)\,\mathrm{d}t\right) \mathrm{d}g(s).$$

Noticing the assumption (H1)-(H3) and Fubini's Theorem, we see that, when $\beta > 1$,

$$\begin{aligned} k(\beta) &= (1 - g(1))\beta^{p-1} - \int_0^1 \int_s^1 t^{n-1}h(t)f(t, u_\beta, u'_\beta)\,\mathrm{d}t\,\mathrm{d}g(s) \\ &\ge (1 - g(1))\beta^{p-1} - g(1)\int_0^1 r^{n-1}h(r)f(r, 0, 0)\,\mathrm{d}r. \end{aligned} \tag{3.12}$$

On the other hand, when $\beta \le 1$, let δ be a positive constant, which is small enough, and $g(\delta) > 0$. From the proof of Lemma 3.5, we know that if u is a solution of the problem (3.1), then there exists a constant $M^* > 0$ such that $||u|| \le M^*$, that is,

$$u(r) \le M^* \quad \text{and} \quad u'(r) \le M^*/\delta, \qquad r \in [\delta, 1].$$

Applying the strict monotonicity of f and the positivity of f and h, we have

$$\begin{aligned} k(\beta) &= (1-g(1))\beta^{p-1} - \int_0^1 \int_s^1 t^{n-1} h(t) f(t, u_\beta, u'_\beta)\,\mathrm{d}t\,\mathrm{d}g(s) \\ &\le (1-g(1))\beta^{p-1} - \int_\delta^1 \int_s^1 t^{n-1} h(t) f(t, u_\beta, u'_\beta)\,\mathrm{d}t\,\mathrm{d}g(s) \\ &\le (1-g(1))\beta^{p-1} - g(\delta) \int_\delta^1 r^{n-1} h(r) f(r, M^*, M^*/\delta)\,\mathrm{d}r. \end{aligned} \tag{3.13}$$

Combining the inequality (3.12) with the inequality (3.13), it follows

$$\lim_{\beta\to 0^+} k(\beta) < 0, \qquad \lim_{\beta\to+\infty} k(\beta) = +\infty.$$

Then recalling the strict monotonicity and continuity of $k(\beta)$, there is one and only one $\beta^* > 0$, such that $k(\beta^*) = 0$, and u_{β^*} is the unique positive solution of nonlocal boundary value problem (1.4)–(1.6).

Finally, we consider the property of the solution of the problem (1.4)–(1.6). Suppose that there exists a point $r_0 \in (0, 1)$ such that $y(r_0)$ is the local maximum value of $y(r)$. Without loss of generality, we assume that $u_{\alpha_1}(r_0) > u_{\alpha_2}(r_0)$, then we see that $u''_{\alpha_1}(r_0) < u''_{\alpha_2}(r_0)$. Following the proof of Lemma 3.1, however, we can obtain $u''_{\alpha_1}(r_0) > u''_{\alpha_2}(r_0)$, which is a contradiction.

Summing up, we complete the Proof of Theorem 2.1. ∎

Proof of Theorem 2.2. Suppose that the nonlocal boundary value problem (1.4)–(1.6) admits a solution $u \in C([0,1]) \cap C^1((0,1])$. Then by a simple calculation to the equation (1.4), we see that $u'(r)$ satisfies

$$u'(r) = r^{-(n-1)/(p-1)} \phi_q \left(\int_r^1 s^{n-1} h(s) f(s, u, u')\,\mathrm{d}s + \phi_p(u'(1)) \right), \; r \in (0,1].$$

Obviously, $u'(r)$ is not integrable since $p \le n$, which contradicts the definition of u. Hence, we complete the proof of Theorem 2.2. ∎

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School of Mathematical Sciences, South China Normal University,
Guangzhou 510631, P.R. China

School of Information, Renmin University of China,
Beijing 100872, P.R. China
Email: ke_yy@163.com.