Extinction and Decay Estimates of Solutions for a p-Laplacian Parabolic Equation with Nonlinear Source*

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Abstract

The extinction phenomenon of solutions for the the initial-boundary value problem of the p-Laplacian parabolic equation

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{m-1}u - \beta u$$

is studied. Sufficient conditions about the extinction and decay estimates of solutions are obtained by using L^p-integral model estimate methods and two crucial lemmas on differential inequality. Non-extinction results are obtained by super and sub-solution method.

1   Introduction

This paper is devoted to the extinction and decay estimates for the p-Laplacian parabolic equation

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{m-1}u - \beta u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0, & x \in \Omega \end{cases} \quad (1.1)$$

where $\Omega \in R^N, N \geq 2$ is an open bounded domain with smooth boundary, $1 < p < 2, m > 0, \lambda > 0, \beta \geq 0$ and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a nonzero nonnegative function.

This type of equations arises in biological and astrophysical context. In combustion theory, for instance, the term $\text{div}(|\nabla u|^{p-2}\nabla u)$ represents the thermal diffusion, the function $u(x,t)$ represents the temperature. Equations of the above form are mathematical models also occurring in studies of generalized reaction-diffusion theory [1], non-Newtonian fluid theory [2,3], non-Newtonian filtration theory [4,5] and the turbulent flow of a gas in porous medium [6]. In the non-Newtonian fluid theory, p is a characteristic quantity of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lose or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be found in [7,8].

In the present paper, our interest is in investigating the extinction of the nonnegative solution u of (1.1), i.e. there exists a finite time $T > 0$ such that the solution is nontrivial for $0 < t < T$, but $u(x,t) \equiv 0$ for all $(x,t) \in \Omega \times [T, +\infty)$. In this case, T is called the extinction time.

The phenomenon of extinction is an important property of solutions for many evolutionary equations which have been studied extensively by many researchers. The first result on extinction was due to Kalashnikov in 1974 (see [9]). Especially, there are also some papers concerning the extinction for the porous medium equation. For instance, in [10-12], the authors studied the extinction and large-time behavior of solution and in [13], the authors obtained conditions for the extinction of solutions of without absorption by using super and sub-solution methods and an eigenfunction argument. In [22-24], the authors discussed the extinction behavior and decay estimates of the solutions for a class of reaction-diffusion equations. But as far as we know, few works are concerned with the decay estimates of solutions for the p-Laplacian parabolic equation.

In [14], Gu gave a simple statement of the necessary and sufficient conditions of extinction of the solution to the following problem:

$$\begin{cases} u_t = \text{div}(|\nabla u|^{p-2}\nabla u) - \lambda u^q, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \qquad (1.2)$$

with $\lambda > 0$. He proved that if $p \in (1,2)$ or $q \in (0,1)$ the solutions of the problem vanish in finite time, but if $p \geq 2$ and $q \geq 1$, there is non-extinction. In the absence of absorption (i.e. $\lambda = 0$), Dibenedetto [15] proved that the necessary and sufficient conditions for the extinction to occur is $p \in (1,2)$.

In [16], the authors investigated the following p-Laplacian parabolic equation with nonlinear source:

$$\begin{cases} u_t = \text{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \qquad (1.3)$$

where $\Omega \in R^N, N \geq 2$ is an open bounded domain with smooth boundary, $1 < p < 2$, and $\lambda, q > 0$ and u_0 is a nonzero nonnegative function. They showed that $q = p - 1$ is the critical exponent of extinction for the weak solution. Furthermore, for $1 < p < 2$ and $q = p - 1$ they proved the extinction and non-extinction conditions.

Roughly speaking, for the problems (1.2), there is a comparison between the diffusion term and the absorption term, and fast diffusion or strong absorption will lead any bounded nonnegative solution to zero in finite time. But in (1.3), the nonlinear source u^q is physically called the "hot source", while in (1.2), the source $-u^q$ is called the "cool source"; the different sources have completely different influences on the properties of solutions (we refer the readers to [13,15-18]). But as far as we know, few works are concerned with problem (1.1) which both have "hot source" and "cool source". The purpose of the present paper is to establish sufficient conditions about the extinction and decay estimations of solutions for problem (1.1). For the proof of our result, we employ L^p-integral model estimate methods and two crucial lemmas on differential inequality.

The existence of solutions to (1.1) has been obtained by Zeng [18]. It is well known that problem (1.1) is degenerate if $p > 2$ or singular if $1 < p < 2$, since the modulus of ellipticity is degenerate ($p > 2$) or blows up ($1 < p < 2$) at points where $\nabla u = 0$, and therefore there is no classical solution in general. For this, a nonnegative weak solution for problem (1.1) is defined as follows. For convenience, define $\Omega_T = \Omega \times (0, T)$, $T > 0$.

Definition 1.1. A nonnegative function u is called a weak solution of problem (1.1), if and only if $u \in L^\infty(\Omega_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$, $u_t \in L^2(\Omega_T)$, and there holds

$$\iint_{\Omega_T} (-u\varphi_t + |\nabla u|^{p-2}\nabla u \nabla \varphi - \lambda|u|^{m-1}u\varphi + \beta u\varphi)dxdt = 0, \qquad (1.4)$$

and

$$\lim_{t \to 0^+} \int_\Omega |u(x,t) - u_0(x)|dx = 0,$$

where the text function $\varphi(x,t) \in C_0^\infty(\Omega_T)$.

Remark 1.2. To define weak solutions for the problem with arbitrary nonnegative function $\psi \in L^\infty(\Omega_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$ as its boundary value, it suffices to require $u - \psi$ in $L^\infty(\Omega_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$ instead of $u \in L^\infty(\Omega_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$. Furthermore, because of the denseness of $C_0^\infty(\Omega_T)$ in $L^p(0, T; W_0^{1,p}(\Omega))$, one can assert that the above equality holds for any $\varphi \in L^\infty(\Omega_T) \cap L^p(0, T; W_0^{1,p}(\Omega))$.

Similarly, to define a subsolution (resp., supersolution) $\underline{u}(x,t)$ (resp., $\overline{u}(x,t)$), we need only demand $\underline{u}(x,0) \leq u_0(x)$ (resp., $\overline{u}(x,0) \geq u_0(x)$) in Ω, $\underline{u}(x,t) \leq 0$ (resp., $\overline{u}(x,t) \geq 0$) on $\partial\Omega \times [0, T]$, and equality in (1.4) is replaced by $\leq$ (resp., $\geq$) for every $\varphi(x,t) > 0$.

This work is organized as follows. In Section 2, we describe some necessary mathematical preliminaries which are required for establishing our results. Section 3 is devoted to extinction and decay estimates of the solution. We will prove non-extinction results in Section 4.

2 Preliminary results

Before studying our problem, we will give some lemmas, which will be useful tools in our later proofs. Firstly, we introduce the following lemma on the properties of the first eigenvalue λ_1 and the corresponding eigenfunction $\phi(x)$ to the problem

$$-\text{div}(|\nabla\phi|^{p-2}\nabla\phi) = \lambda|\phi|^{p-2}\phi, \quad \text{in } \Omega; \quad \phi|_{\partial\Omega} = 0. \tag{2.1}$$

In this paper, we choose $\phi(x) > 0$ in Ω and $\max_{x\in\Omega}\phi(x) = 1$.

Lemma 2.1. There exists a positive constant $\lambda_1(\Omega)$ with the following properties:

(i) For any $\lambda < \lambda_1(\Omega)$, the eigenvalue problem (2.1) has only the trivial solution $\phi(x) \equiv 0$.

(ii) There exists a positive solution $\phi \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ of (2.1) if and only if $\lambda = \lambda_1(\Omega)$.

(iii) The collection consisting of all solutions of (2.1) with $\lambda = \lambda_1(\Omega)$ is a one-dimensional vector space.

(iv) If Ω_1 and Ω_2 are bounded domains with smooth boundary satisfying $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$.

(v) Let $\{\Omega_n\}$ be a sequence of bounded domains with smooth boundaries such that $\Omega_n \subset \Omega_{n+1}$ and $\cup_{n=1}^{\infty}\Omega_n = \Omega$, then $\lim_{n\to\infty}\lambda_1(\Omega_n) = \lambda_1(\Omega)$.

This lemma follows from Lemmas 2.1, 2.2 in [19] and Lemma 1.1 in [20].

The properties of the first eigenvalue θ_1 and the corresponding eigenfunction $\psi(x)$ of the eigenvalue problem

$$\triangle\psi = \theta\psi, \quad \text{in} \quad \Omega; \quad \psi|_{\partial\Omega} = 0,$$

are well known (see [21]). Moreover, we can define θ_1 using the "Rayleigh quotient":

$$\theta_1 = \inf_{u\in H_0^1(\Omega), u\neq 0} \frac{\int_{\Omega}|\nabla u|^2 dx}{\int_{\Omega} u^2 dx}.$$

We give a similar quotient for the first eigenvalue λ_1 of (2.1) as follows.

Lemma 2.2.
$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \frac{\int_{\Omega}|\nabla u|^p dx}{\int_{\Omega} u^p dx}.$$

Lemma 2.3.(Comparison Lemma) Suppose that $\underline{u}(x,t)$, $\overline{u}(x,t)$ are a subsolution and a supersolution of (1.1) respectively, then $\underline{u}(x,t) \leq \overline{u}(x,t)$ a.e. in Ω_T.

The proof of this Lemma is similar with the proof of Lemma 2.2 in [15], so we omit it here. The following two lemmas are of crucial importance in the proofs of decay estimates.

Lemma 2.4.(see [22]) Let $y(t) \geq 0$ be a solution of the differential inequality

$$\frac{dy}{dt} + Cy^k + \beta y \leq 0 \ \ (t \geq 0), \quad y(t_0) \geq 0, \tag{2.2}$$

where $C > 0$ is a constant and $k \in (0,1)$. Then one has the decay estimate

$$y(t) \leq [(y(T_0)^{1-k} + \frac{C}{\beta})e^{(k-1)\beta(t-T_0)} - \frac{C}{\beta}]^{1/(1-k)}, \ \ t \in [T_0, T_*), \tag{2.3}$$

$$y(t) \equiv 0, \quad t \in [T_*, +\infty), \tag{2.4}$$

where $T_* = (1/(1-k)\beta) \ln(1 + (\beta/C)y(T_0)^{1-k})$.

Lemma 2.5.(see [23]) Let $0 < k < p$, and let $y(t) \geq 0$ be a solution of the differential inequality

$$\frac{dy}{dt} + Cy^k + \beta y \leq \gamma y^p \ \ (t \geq 0), \quad y(0) \geq 0, \tag{2.5}$$

where $C, \gamma > 0$ and $k \in (0,1)$. Then there exist $\alpha > \beta$, $B > 0$, such that

$$0 \leq y(t) \leq Be^{-\alpha t}, \ \ t \geq 0. \tag{2.6}$$

3   Extinction and decay estimates

In this section, we consider the extinction of the solution to problem (1.1). Let $|| \cdot ||_p$ and $|| \cdot ||_{1,p}$ denote $L^p(\Omega)$ and $W^{1,p}(\Omega)$ norms, respectively, $1 \leq p < \infty$.

Theorem 3.1. Let $\beta = 0$ and $m > 1$, u be a weak solution of (1.1), then for sufficiently small initial data, there exists a finite time T such that $u \equiv 0$ for all $(x,t) \in \overline{\Omega} \times (T, +\infty)$.

Proof. Multiplying the first equation of (1.1) by u^s, $s > 0$, and integrating over Ω, we obtain

$$\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx + \frac{sp^p}{(s+p-1)^p}\int_\Omega |\nabla u^{\frac{p+s-1}{p}}|^p dx = \lambda \int_\Omega |u|^{m-1}u^{s+1}dx. \tag{3.1}$$

By Lemma 2.3, if $u_0 \leq k\phi(x)$ in Ω, it can be easily verified that $k\phi(x)$ is a supersolution of (1.1). Because $\max_{x\in\Omega}\phi(x) = 1$, we have $u(x,t) < k$ for all $(x,t) \in \Omega \times (0,\infty)$. From this, (3.1) can be rewritten as

$$\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx + \frac{sp^p}{(s+p-1)^p}\int_\Omega |\nabla u^{\frac{p+s-1}{p}}|^p dx \leq k^{m-1}\int_\Omega u^{s+1}dx.$$

(1)For $\frac{2N}{N+2} < p < 2$, let $s = 1$ in (3.1). By the Hölder's inequality and Sobolev embedding inequality, we have

$$\int_\Omega u^2 dx \leq |\Omega|^{1-\frac{2(N-p)}{Np}}(\int_\Omega u^{\frac{Np}{N-p}}dx)^{\frac{2(N-p)}{Np}} \leq \rho|\Omega|^{1-\frac{2(N-p)}{Np}}(\int_\Omega |\nabla u|^p dx)^{\frac{2}{p}},$$

where ρ is the embedding constant, depending only on p and N. Then we obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \rho^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} \leq k^{m-1}\int_\Omega u^2 dx.$$

Choose k sufficiently small such that

$$k^{m+1-p} < \rho^{-\frac{p}{2}}|\Omega|^{-\frac{p}{N}},$$

then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \leq -\rho^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}}$$
$$+ k^{m-1}\left(\int_\Omega u^2 dx\right)^{1-\frac{p}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} \leq -C\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}},$$

in which

$$C = \rho^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - k^{m+1-p}|\Omega|^{1-\frac{p}{2}} > 0.$$

We thus have

$$\left(\int_\Omega u^2 dx\right)^{1-\frac{p}{2}} \leq \left(\int_\Omega u_0^2 dx\right)^{1-\frac{p}{2}} - (2-p)Ct,$$

as long as the right hand side is nonnegative. From this,

$$\int_\Omega u^2 dx \leq \int_\Omega u_0^2 dx\left[1 - \frac{(2-p)Ct}{\left(\int_\Omega u_0^2 dx\right)^{1-\frac{p}{2}}}\right]^{\frac{2-p}{2}}.$$

Then u vanishes in finite time for sufficiently small initial data.

(2)For $1 < p < \frac{2N}{N+2}$. Let $s = \frac{2N-(N+1)p}{p} > 1$ in (3.1). By the Sobolev embedding inequality and the choice of s, we have

$$\|u\|_{s+1}^{\frac{p+s-1}{p}} = \|u^{\frac{p+s-1}{p}}\|_{\frac{Np}{N-p}} \leq \rho\|\nabla u^{\frac{p+s-1}{p}}\|_p.$$

We conclude that

$$\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1} dx + \rho^{-p}\frac{sp^p}{(s+p-1)^p}\left(\int_\Omega u^{s+1} dx\right)^{\frac{p+s-1}{s+1}} \leq k^{m-1}\int_\Omega u^{s+1} dx.$$

By the similar argument as above, we can get u vanishes in finite time. The proof of Theorem 3.1 is complete. ∎

Theorem 3.2. Assume that $m \leq 1$, λ_1 is the first eigenvalue of (2.1) $\phi_1 > 0$ with $\|\phi_1\|_\infty = 1$ is the eigenfunction corresponding to the eigenvalue λ_1.

(1) If $m = p-1, \lambda \leq \lambda_1$, then the weak solution of problem (1.1) vanishes in the sense of $\|\cdot\|_2$ as $t \to \infty$.

(2) Let $m < p-1$, if $\frac{2N}{N+2} < p < 2$ with $|\Omega|^{\frac{N-p}{N}+\frac{m-p-1}{2}} > \lambda\kappa^{\frac{p}{2}}$ or $1 < p \leq \frac{2N}{N+2}$ with $\kappa^{-p}\frac{sp^p}{(s+p-1)^p} > \lambda|\Omega|^{1-\frac{m+s}{s+1}}$, then the weak solution of problem (1.1) vanishes in finite time, and

$$\|u\|_2 \leq \left[(\|u_0\|_2^{1-\frac{p}{2}} + \frac{C_1}{\beta_1})e^{(\frac{p}{2}-1)\beta_1 t} - \frac{C_1}{\beta_1}\right]^{2/(2-p)}, \quad \frac{2N}{N+2} < p < 2,$$

$$||u||_{s+1} \leq [(||u_0||_{s+1}^{\frac{2-p}{s+1}} + \frac{C_2}{\beta_2})e^{\frac{p-2}{s+1}\beta_2 t} - \frac{C_2}{\beta_2}]^{(s+1)/(2-p)}, \quad 1 < p \leq \frac{2N}{N+2},$$

for $t \in [0, T^*)$, where κ is the embedding constant defined below, $s = \frac{2N-(N+1)p}{p}$,

$$0 < T^* \leq \begin{cases} T_1, & \frac{2N}{N+2} < p < 2, \\ T_2, & 1 < p \leq \frac{2N}{N+2}, \end{cases}$$

and $C_1, C_2, T_1,$ and T_2 are given by (3.15), (3.16), (3.17), and (3.18), respectively.

Proof. (1) First of all, we show that

$$||u(x,t)||_\infty \leq ||u_0(x)||_\infty := M. \tag{3.2}$$

Multiplying (1.1) by $(u - M)_+$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u-M)_+^2 dx + \int_{A_M(t)} |\nabla u|^{p-2}\nabla u \nabla u dx$$
$$\leq \lambda \int_\Omega u^m(u-M)_+ dx - \beta \int_\Omega u(u-M)_+ dx \leq \lambda \int_{A_M(t)} u^p dx, \tag{3.3}$$

where $A_M(t) = \{x \in \Omega | u(x,t) > M\}$. Since λ_1 is the first eigenvalue, then we have

$$\int_\Omega |\nabla u|^p dx \geq \lambda_1 \int_\Omega u^p dx,$$

for any $u \in W_0^{1,p}(\Omega)$. We further have

$$\int_{A_M(t)} |\nabla u|^p dx \geq \lambda_1 \int_{A_M(t)} u^p dx. \tag{3.4}$$

Therefore, we have

$$\frac{d}{dt}\int_\Omega (u-M)_+^2 dx \leq 0. \tag{3.5}$$

Since $\int_\Omega (u-M)_+^2 dx = 0$, it follows that

$$\int_\Omega (u-M)_+^2 dx \equiv 0, \quad \forall \ t > 0, \tag{3.6}$$

which implies that $||u(x,t)||_\infty \leq ||u_0(x)||_\infty$.

Multiplying (1.1) by u and integrating over Ω, we conclude that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla u dx \leq \lambda \int_\Omega u^p dx - \beta \int_\Omega u^2 dx. \tag{3.7}$$

We further have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + (\lambda_1 - \lambda)\int_\Omega u^p dx + \beta \int_\Omega u^2 dx \leq 0. \tag{3.8}$$

Let $v = u/M$. Then, we have

$$\frac{d}{dt}\int_\Omega v^2 dx + 2M^{p-2}(\lambda_1 - \lambda)\int_\Omega v^p dx + 2\beta \int_\Omega v^2 dx \leq 0. \tag{3.9}$$

Since $1 < p < 2$, we have

$$\frac{d}{dt}\int_\Omega v^2 dx + 2M^{2(p-2)}(\lambda_1 - \lambda + \beta)\int_\Omega v^2 dx \le 0,$$

which implies that

$$\int_\Omega v^2 dx \le e^{-2M^{2(p-2)}(\lambda_1-\lambda+\beta)t}\int_\Omega v_0^2 dx, \tag{3.10}$$

that is,

$$\int_\Omega u^2 dx \le e^{-2||u_0||_\infty^{2(p-2)}(\lambda_1-\lambda+\beta)t}\int_\Omega u_0^2 dx. \tag{3.11}$$

Therefore, we conclude that $||u(x,t)||_2 \to 0$ as $t \to \infty$.

(2) Multiplying (1.1) by u^s and integrating over Ω, we have

$$\frac{1}{s+1}\frac{d}{dt}\int_\Omega u^{s+1}dx + \frac{sp^p}{(s+p-1)^p}\int_\Omega |\nabla u^{\frac{p+s-1}{p}}|^p dx =$$
$$\lambda\int_\Omega |u|^{m-1}u^{s+1}dx - \beta\int_\Omega u^{s+1}dx. \tag{3.12}$$

In the first case $\frac{2N}{N+2} < p < 2$, let $s = 1$ in (3.12). By the Hölder's inequality and Sobolev embedding inequality, we have

$$\int_\Omega u^2 dx \le |\Omega|^{1-\frac{2(N-p)}{Np}}\left(\int_\Omega u^{\frac{Np}{N-p}}dx\right)^{\frac{2(N-p)}{Np}} \le \kappa|\Omega|^{1-\frac{2(N-p)}{Np}}\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{2}{p}}. \tag{3.13}$$

where κ is the embedding constant, depending only on p and N. By (3.12) and (3.13), we obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \kappa^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} \le \lambda\int_\Omega |u|^{m+1}dx - \beta\int_\Omega u^2 dx. \tag{3.14}$$

Since $m \le 1$, $m + 1 < p$, according to Hölder's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \kappa^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} \le \lambda|\Omega|^{\frac{1-m}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{m+1}{2}} - \beta\int_\Omega u^2 dx$$
$$\le \lambda|\Omega|^{\frac{1-m}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} - \beta\int_\Omega u^2 dx.$$

So we can get

$$\frac{d}{dt}\int_\Omega u^2 dx + C_1\left(\int_\Omega u^2 dx\right)^{\frac{p}{2}} + \beta_1\int_\Omega u^2 dx \le 0, \tag{3.15}$$

where $C_1 = 2(\kappa^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - \lambda|\Omega|^{\frac{1-m}{2}}) > 0, \beta_1 = 2\beta > 0$. Setting $y(t) = ||u(x,t)||_2, y(0) = ||u_0(x)||_2$, by Lemma 2.4, we obtain

$$||u||_2 \le [(||u_0||_2^{1-\frac{p}{2}} + \frac{C_1}{\beta_1})e^{(\frac{p}{2}-1)\beta_1 t} - \frac{C_1}{\beta_1}]^{2/(2-p)}, \quad t \in [0, T_1),$$

$$||u||_2 \equiv 0, \quad t \in [T_1, +\infty),$$

where

$$T_1 = \frac{2}{(2-p)\beta_1} \ln(1 + \frac{\beta_1}{C_1}||u_0||_2^{1-\frac{p}{2}}). \tag{3.16}$$

We now turn to the case $1 < p \leq \frac{2N}{N+2}$, let $s = \frac{2N-(N+1)p}{p} > 1$ in (3.12). By the Sobolev embedding inequality and the choice of s, we have

$$||u||_{s+1}^{\frac{p+s-1}{p}} = ||u^{\frac{p+s-1}{p}}||_{\frac{NP}{N-p}} \leq \kappa ||\nabla u^{\frac{p+s-1}{p}}||_p,$$

$$\int_\Omega |u|^{m+s} dx \leq |\Omega|^{1-\frac{m+s}{s+1}} (\int_\Omega u^{s+1} dx)^{\frac{m+s}{s+1}} \leq |\Omega|^{1-\frac{m+s}{s+1}} (\int_\Omega u^{s+1} dx)^{\frac{p+s-1}{s+1}}.$$

We conclude that

$$\frac{d}{dt} \int_\Omega u^{s+1} dx + C_2 (\int_\Omega u^{s+1} dx)^{\frac{p+s-1}{s+1}} + \beta_2 \int_\Omega u^{s+1} dx \leq 0, \tag{3.17}$$

where $C_2 = (s+1)(\kappa^{-p} \frac{sp^p}{(s+p-1)^p} - \lambda |\Omega|^{1-\frac{m+s}{s+1}}) > 0$, $\beta_2 = (s+1)\beta > 0$. Setting $y(t) = ||u(x,t)||_{s+1}$, $y(0) = ||u_0(x)||_{s+1}$, by Lemma 2.4, we obtain

$$||u||_{s+1} \leq [(||u_0||_{s+1}^{\frac{2-p}{s+1}} + \frac{C_2}{\beta_2})e^{\frac{p-2}{s+1}\beta_2 t} - \frac{C_2}{\beta_2}]^{(s+1)/(2-p)}, \quad t \in [0, T_2),$$

$$||u||_{s+1} \equiv 0, \quad t \in [T_2, +\infty),$$

where

$$T_2 = \frac{s+1}{(2-p)\beta_2} \ln(1 + \frac{\beta_2}{C_2}||u_0||_{s+1}^{\frac{2-p}{s+1}}). \tag{3.18}$$

The proof of Theorem 3.2 is complete. ∎

Theorem 3.3. Let $p < m + 1$, then the weak solution u of (1.1) vanish in finite time, and

$$0 \leq ||u||_2 \leq Be^{-\alpha t}, \quad t \in [0, T_3),$$

$$||u||_2 \leq [(||u_0||_2^{1-\frac{p}{2}} + \frac{C_3}{\beta_3})e^{(\frac{p}{2}-1)\beta_3(t-T_3)} - \frac{C_3}{\beta_3}]^{2/(2-p)}, \quad t \in [T_3, T_4),$$

$$||u||_2 \equiv 0, \quad t \in [T_4, +\infty),$$

where C_3, T_3, and T_4 are given by (3.20), (3.21) and (3.22), respectively.

Proof. Multiplying (1.1) by u, and by the embedding theorem and the Hölder's inequality, we can easily obtain

$$\frac{d}{dt} \int_\Omega u^2 dx + 2\kappa^{-\frac{p}{2}} |\Omega|^{\frac{N-p}{N}-\frac{p}{2}} (\int_\Omega u^2 dx)^{\frac{p}{2}} + 2\beta \int_\Omega u^2 dx \leq$$

$$2\lambda |\Omega|^{\frac{1-m}{2}} (\int_\Omega u^2 dx)^{\frac{m+1}{2}}. \tag{3.19}$$

By Lemma 2.5, there exist $\alpha > \beta$, $B > 0$, such that

$$0 \leq ||u||_2 \leq Be^{-\alpha t}, \quad t \geq 0.$$

Furthermore, there exist T_3, such that

$$2[\kappa^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}} - \lambda|\Omega|^{\frac{1-m}{2}}(\int_\Omega u^2 dx)^{\frac{m+1-p}{2}}] \geq 2[\kappa^{-\frac{p}{2}}|\Omega|^{\frac{N-p}{N}-\frac{p}{2}}$$

$$-\lambda|\Omega|^{\frac{1-m}{2}}(Be^{-\alpha T_3})^{m+1-p}] := C_3 \geq 0. \tag{3.20}$$

holds for $t \in [T_3, +\infty)$. Therefore, (3.19) turns to

$$\frac{d}{dt}\int_\Omega u^2 dx + C_3(\int_\Omega u^2 dx)^{\frac{p}{2}} + \beta_3 \int_\Omega u^2 dx \leq 0. \tag{3.21}$$

By Lemma 2.4, we can obtain the desire decay estimate for

$$T_4 = \frac{2}{(2-p)\beta_3}\ln(1 + \frac{\beta_3}{C_3}||u_0||_2^{\frac{2-p}{2}}). \tag{3.22}$$

The proof of Theorem 3.3 is complete. ∎

4   Non-extinction of the solution

In this section, we investigate the conditions under which the solution $u(x,t)$ of (1.1) cannot become extinction.

Theorem 4.1. For $\beta < 0$, $m = p - 1$, if one of the following cases holds:
(1) $\lambda > \lambda_1$;
(2) $\lambda = \lambda_1$, and $u_0(x)$ is identically positive,
then the weak solution u of (1.1) cannot vanish after finite time.
Proof. For $m = p - 1$, $\lambda > \lambda_1$, let $v(x,t) = g(t)\phi(x)$, and $g(t) = [(\lambda - \lambda_1)$
$(2-p)t]^{\frac{1}{2-p}}$, and $\phi(x)$ be the first eigenfunction of the eigenvalue problem (2.1).
We will show that $v(x,t)$ is a subsolution of (1.1).
Obviously, $g(t)$ satisfies the ordinary differential equation

$$g'(t) = (\lambda - \lambda_1)g^{p-1}(t), \tag{4.1}$$

$$g(0) = 0, \quad g(t) > 0, \quad t > 0.$$

Next, by applying (4.1) and $\phi(x) > 0$ in Ω, $\max_{x \in \Omega} \phi(x) = 1$, we can easily get

$$\int\int_{\Omega_T} v_t(x,s)\varphi(x,s)dxds + \int\int_{\Omega_T}(|\nabla v|^{p-2}\nabla v\nabla\varphi - \lambda v^m\varphi(x,s)$$

$$+\beta g(s)\phi(x)\varphi(x,s))dxds \leq \int\int_{\Omega_T}(\lambda - \lambda_1)g^{p-1}(s)\phi(x)\varphi(x,s)dxds$$

$$+\int\int_{\Omega_T}[\lambda_1 g^{p-1}(s)\phi^{p-1}(x) - \lambda g^m(s)\phi^m(x)]\varphi(x,s)dxds \leq 0.$$

Moreover, $v(x,0) = \phi(x)g(0) \leq u_0(x)$ in Ω, and $v|_\Omega = 0$. Then, according to Lemma 2.3, we have $u(x,t) \geq v(x,t) > 0$ in $\Omega \times (0,1)$.
Next, for the case $m = p - 1$, $\lambda = \lambda_1$, it is easily proved that $a\phi(x), a > 0$, is a steady state solution of (1.1). Then for any positive initial data, we can choose

a sufficiently small such that $u_0(x) \geq a\phi(x)$ in Ω. According to Lemma 2.3, we have that $a\phi(x)$ is a subsolution of (1.1). The proof of Theorem 4.1 is complete. ■

Theorem 4.2. For $\beta < 0, m < p - 1$, the weak solution u of (1.1) cannot vanish in finite time for any nonnegative initial data u_0.

Proof. For $m < p - 1$, define

$$h(t) = (\frac{\lambda}{\lambda_1})^{\frac{1}{p-m-1}}(2 - \exp(-Mt))^{\frac{1}{1-m}},$$

in which $M \in (0, (p - m - 1)(\frac{\lambda_1^{1-m}}{\lambda^{2-p}})^{\frac{1}{p-m-1}})$.

Let $v(x, t) = h(t)\phi(x)$, and $\phi(x)$ be the first eigenfunction of the eigenvalue problem (2.1). We will show that $v(x, t)$ is a subsolution of (1.1).

Then we can get

$$h'(t) = \frac{M}{1 - m}(\frac{\lambda}{\lambda_1})^{\frac{1}{p-m-1}}(2 - \exp(-Mt))^{\frac{m}{1-m}}\exp(-Mt), \qquad (4.2)$$

$$-\lambda_1 h^{p-1} + \lambda h^m =$$
$$(\frac{\lambda^{p-1}}{\lambda_1^m})^{\frac{1}{p-m-1}}(2 - \exp(-Mt))^{\frac{m}{1-m}}[2 - (2 - \exp(-Mt))^{\frac{p-m-1}{1-m}}]. \quad (4.3)$$

For $0 < \frac{p-m-1}{1-m} < 1$, we have

$$(2 - \exp(-Mt))^{\frac{p-m-1}{1-m}} \leq -\frac{p - m - 1}{1 - m}\exp(-Mt) + 2. \qquad (4.4)$$

By (4.2)-(4.4), the following equation is satisfied,

$$-\lambda_1 h^{p-1} + \lambda h^m \geq$$
$$\frac{p - m - 1}{1 - m}(\frac{\lambda^{p-1}}{\lambda_1^m})^{\frac{1}{p-m-1}}(2 - \exp(-Mt))^{\frac{m}{1-m}}\exp(-Mt)$$
$$> \frac{M}{1 - m}(\frac{\lambda}{\lambda_1})^{\frac{1}{p-m-1}}(2 - \exp(-Mt))^{\frac{m}{1-m}}\exp(-Mt) = h'(t).$$

Applying the same argument as in the proof of Theorem 4.1, we can easily prove that $v(x, t)$ is a subsolution of (1.1), then extinction in finite time cannot occur. ■

5 Conclusion

The boundary value problems of quasilinear differential equation (1.1) are mathematical models occurring in the studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, p is a characteristic quantity of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. In

1979, Gidas, Ni and Nirenberg [26] proved some very interesting facts, for example, they showed that all positive solution in $C^2(B_R)$ of the problem

$$\begin{cases} \triangle u + f(u) = 0 & \text{in } B_R \\ u(x) = 0 & \text{on } \partial B_R \end{cases} \tag{5.1}$$

are radially symmetric solutions for very general f. They also proved that no such results can automatically apply to the annulus (see also [27]). Unfortunately, this result does not apply to the case $p \neq 2$. Kichenassary and Smoller showed that there exist many positive nonradial solutions of the above problem for some f (see [25]). The major stumbling block in the case of $p \neq 2$ is that certain nice features inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be found in [7,8]. In the case of (5.1), the solutions are classical (that is, smooth), but in the case of the problem

$$\begin{cases} \text{div}(|\nabla u|^{p-2}\nabla u) + f(u) = 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial \Omega \end{cases} \tag{5.2}$$

belong to $C^{1+\alpha}(\overline{\Omega})$ for some $\alpha (0 < \alpha < 1)$ but not always to $C^2(\overline{\Omega})$. For example, when the domain Ω is a ball centered at the origin 0, the function $u(x)$ defined by

$$u(x) = a|x|^{p/(p-1)} + b,$$

with constants a and b $(a < 0, b > 0)$, is a solution to the problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial \Omega. \end{cases} \tag{5.3}$$

Moreover, the solutions of (5.2) on a symmetric domain are not necessarily radially symmetric. Tolksdorf [28] showed that there exist solutions of the problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } R^2$$

that are of the form

$$u(x) = r^\lambda \phi(\theta),$$

where $r = |x|$.

In this paper, based on the L^p-integral model estimate methods and two crucial lemmas on differential inequality, we study the extinction phenomenon of solutions to problem (1.1) which the right hand side functions are more general. Finally, non-extinction results are obtained by super and sub-solution method.

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