Periodic forcing for some difference equations in Hilbert spaces

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Abstract

Let H be a real Hilbert space and let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal monotone operator. We are concerned with the difference equation

$$\Delta u_n + c_n A u_{n+1} \ni f_n, \qquad n = 0, 1, ...,$$

where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are p-periodic sequences for a positive integer p. We investigate the existence of periodic solutions to this equation as well as the weak or strong convergence of solutions to p-periodic solutions. The first result of this paper (Theorem 1) is a discrete analogue of the 1977 result by Baillon and Haraux (on the periodic forcing problem for the continuous counterpart of the above equation) and was essentially stated by Djafari Rouhani and Khatibzadeh in a recent paper [5]. Here we provide a simpler proof of this result that is based on old existing results due to Browder and Petryshyn [4] and Opial (see, e.g., [6], p.5). A strong convergence result is also given and some examples are discussed to illustrate the theoretical results.

1 Introduction

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and the induced Hilbertian norm $\| \cdot \|$. Let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal

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monotone operator. Consider the difference equation

$$\Delta u_n + c_n A u_{n+1} \ni f_n, \qquad n = 0, 1, ..., \tag{E}$$

where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are p-periodic sequences for a positive integer p and Δ is the difference operator defined as usual, i.e., $\Delta u_n = u_{n+1} - u_n$.
We shall investigate some conditions that guarantee the existence of periodic solutions to equation (E) as well as the weak or strong convergence of any solution to a periodic one, as $n \to \infty$.
The problem we investigate in the present paper is the discrete analogue of the periodic forcing problem for the "continuous" equation

$$u'(t) + Au(t) \ni f(t), \qquad t > 0,$$

studied by J. B. Baillon and A. Haraux [2].
Recently Djafari Rouhani and Khatibzadeh [5] formulated essentially Theorem 1 below that shows that the weak convergence result stated by Baillon and Haraux [2] for the case of a subdifferential operator A has a discrete counterpart for a general maximal monotone operator A. Here we provide a simpler proof of Theorem 1 by using old existing results due to Browder and Petryshyn [4] and Opial (see Lemma 3 below). In addition, we formulate a strong convergence result (Theorem 2) and discuss some examples to illustrate the two theorems.

2 Preparatory Lemmas

In what follows we need the following lemmas.

Lemma 1 ([4])**.** *Let X be a uniformly convex Banach space and let Q be a nonexpansive mapping of X into X (i.e., Q is Lipschitzian with constant 1). Then Q has a fixed point if and only if for any specific $x_0 \in X$ the sequence $x_n = Q^n x_0$) is bounded in X.*

Lemma 2 ([4])**.** *Let H be a Hilbert space and let $Q : H \to H$ be a nonexpansive mapping such that the set F of its fixed points is nonempty and Q is asymptotically regular (i.e., $Q^{n+1}x - Q^n x \to 0$ strongly in H as $n \to \infty$ for each $x \in H$). Then, $\forall x_0 \in H$, every weak cluster point of the sequence $x_n = Q^n x_0$ belongs to F.*

Lemma 3 (Opial's Lemma, see, e.g., [6], p. 5)**.** *Let H be a real Hilbert space and let F be a nonvoid subset of H. Assume that (x_n) is a sequence in H satisfying:*

(i) the $\lim \|x_n - q\| = \rho(q)$ exists, $\forall q \in F$;

(ii) any weak cluster point of (x_n) belongs to F.

Then, there exists a $p \in F$ such that $x_n \to p$ weakly in H.

Lemma 4 (see, e.g., [6], p. 42)**.** *If $A : D(A) \subset \mathbb{R} \to \mathbb{R}$ is maximal monotone, then there exists a lower semicontinuous (LSC) convex function $\varphi : \mathbb{R} \to (-\infty, +\infty]$ such that A is the subdifferential of φ: $A = \partial\varphi$.*

3 Main Results

The following theorem is the discrete analogue of the 1977 Baillon and Haraux result [2] and was essentially stated by Djafari Rouhani and Khatibzadeh in a recent paper [5].

Theorem 1. *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences, i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, ...)$, for a given positive integer p. Then equation* (E) *has a bounded solution if and only if it has at least one p-periodic solution. In this case all solutions of* (E) *are bounded and for every solution (u_n) there exists a p-periodic solution (ω_n) of* (E) *such that*

$$u_n - \omega_n \to 0, \text{ weakly in } H, \text{ as } n \to \infty.$$

Moreover, every two periodic solutions differ by an additive constant vector.

Proof. Consider the initial condition

$$u_0 = x, \tag{IC}$$

for a given $x \in H$. We can rewrite equation (E) in the form:

$$u_{n+1} - u_n + c_n A u_{n+1} \ni f_n.$$

The solution of the problem (E)-(IC) is calculated successively from

$$u_{n+1} = \big(I + c_n A\big)^{-1}\big(u_n + f_n\big), \qquad n = 0, 1, \ldots,$$

in a unique manner, which will give a unique solution $(u_n)_{n\geq 0}$.
If a solution (u_n) of (E) is bounded (in particular periodic), then any other solution $(\tilde{u}_n)$ of (E) is bounded, because

$$\|u_n - \tilde{u}_n\| \leq \|u_0 - \tilde{u}_0\| \qquad \forall n = 0, 1, \ldots \tag{1}$$

Set $Q : H \to H$,

$$Qx = u_{p;x},$$

where $(u_{n;x})$ is the solution of (E) starting from x: $u_0 = x$. From (1) it follows that Q is nonexpansive. We also have $Q^n x = u_{np;x}$, $n = 0, 1, ...$ Thus $(Q^n x)_{n\geq 0}$ is a bounded sequence for all $x \in H$. Obviously H is uniformly convex so, by Lemma 1, there is an $x^* \in H$ such that $Qx^* = x^*$, i.e., $u^*_{p;x^*} = u^*_0 = x^*$, where (u^*_n) is the solution of (E) starting from x^*. In fact, $x^* \in D(A)$. Since both (c_n) and (f_n) are p-periodic sequences, this implies

$$u^*_{n+p} = u^*_n \qquad \forall n = 0, 1, \ldots$$

So the first part of the theorem is proved. For the second part we shall use Lemmas 2 and 3. Let F be the set of all fixed points of Q. According to the first part of the theorem, F is nonempty if and only if all the solutions of (E) are bounded. Assume that F is nonempty. Let $u_0 \in F$, i.e., the corresponding solution $(u_n)_{n\geq 0}$

of equation (E) is p-periodic. Let (z_n) be an arbitrary solution of (E) (which is bounded). We have

$$\|z_{kp+m} - u_m\| \leq \|z_{kp} - u_0\| = \|z_{(k-1)p+p} - u_p\| \\ \leq \|z_{(k-1)p+m} - u_m\| \leq \|z_{(k-1)p} - u_0\|$$

for each $m \in \{0, 1, \ldots, p-1\}$ and $\forall k = 1, 2, \ldots$.
Therefore,

$$\lim_{k\to\infty} \|z_{kp+m} - u_m\| = C, \qquad \text{for all } m \in \{0, 1, \ldots, p-1\},$$

where C is a constant, independent of m.
In particular (for $m = 0$) the sequence $(z_{kp})_{k\geq 0}$ satisfies the first condition of Opial's Lemma (Lemma 3). For the other condition of Lemma 3, we can use Lemma 2. Obviously, $w_n := z_n - u_n$ satisfies

$$w_n - w_{n+1} \in c_n(Az_{n+1} - Au_{n+1}),\ n \geq 0.$$

Since A is monotone, we have

$$0 \leq (w_n - w_{n+1}, w_{n+1}),\ n \geq 0.$$

In particular, $(\|w_n\|)$ is nonincreasing. From

$$\|w_{n+1} - w_n\|^2 \leq \|w_n\|^2 - \|w_{n+1}\|^2$$

we derive

$$\textstyle\sum_{n=0}^{\infty} \|w_{n+1} - w_n\|^2 \leq \|w_0\|^2.$$

Therefore, $w_{n+1} - w_n \to 0$ as $n \to \infty$, which implies

$$Q^{k+1}z_0 - Q^k z_0 = z_{(k+1)p} - z_{kp} = w_{(k+1)p} - w_{kp} =$$

$$\textstyle\sum_{j=1}^{p} (w_{kp+j} - w_{kp+j-1}) \to 0,$$

so Q is asymptotically regular, since z_0 is an arbitrary vector. It follows by Lemma 2 that any weak cluster point of $z_{kp} = Q^k z_0$ belongs to F. Thus Lemma 3 implies that z_{kp} converges weakly to some $\omega_0 \in F$, as $k \to \infty$. Let (ω_n) be the periodic solution corresponding to ω_0. By the reasoning above $(z_{n+1} - \omega_{n+1}) - (z_n - \omega_n)$ converges strongly to 0, as $n \to \infty$. Therefore, $z_{kp+m} - \omega_m$ converges weakly to 0 as $k \to \infty$, for all $m \in \{0, 1, \ldots, p-1\}$.

In fact, for all $n = 0, 1, \ldots$, we have $n = kp + m$, with $m \in \{0, 1, \ldots, p-1\}$ and $k \to \infty$ as $n \to \infty$. Thus, $z_n - \omega_n = z_{kp+m} - \omega_m$ converges weakly to 0 as $n \to \infty$.

Now, let (ω'_n) be another periodic solution of (E). By the above reasoning, $(z_{kp+m+1} - \omega'_{m+1}) - (z_{kp+m} - \omega'_m) \to 0$ as $k \to \infty$, strongly in H, for all $m \in \{0, 1, \ldots, p-1\}$. Therefore, $\omega_{m+1} - \omega'_{m+1} = \omega_m - \omega'_m$ and thus $\omega_m - \omega'_m = \omega_0 - \omega'_0 = Const.$, for all $m \in \{1, \ldots, p-1\}$, showing that any two periodic solutions differ by an additive constant. The proof is complete. ∎

Open Problem 1. In general the periodic solution is not unique, i.e., F is not a singleton (see Example 1 below). Can one characterize the periodic solution (ω_n) associated with (u_n) in Theorem 1?

Open Problem 2. If in Theorem 1 A is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$ and F is nonempty (i.e., φ has at least a minimum point), then it is easy to see that

$$\varphi(u_{kp+m}) \to \varphi(\omega_m), \tag{2}$$

as $k \to \infty$, for $m = 0, 1, ..., p-1$. Indeed, assuming for simplicity that A is single-valued, we have for all $m \in \{1, 2, ..., p\}$

$$\varphi(u_{kp+m}) - \varphi(\omega_m) \leq (Au_{kp+m} - A\omega_m + A\omega_m, u_{kp+m} - \omega_m) =$$

$$\frac{1}{c_{m-1}}((u_{kp+m-1} - \omega_{m-1}) - (u_{kp+m} - \omega_m), u_{kp+m} - \omega_m) + (A\omega_m, u_{kp+m} - \omega_m),$$

which implies

$$\limsup_{k\to\infty} \varphi(u_{kp+m}) \leq \varphi(\omega_m).$$

Therefore (2) holds since φ is lower semicontinuous.

Question: What can one say about the rate of convergence in (2)?

Remark 1. Strong convergence in Theorem 1 is not true in general. Indeed, if $f_n = 0$ and $c_n = c > 0$, $\forall n \geq 0$, then (E) has a bounded solution if and only if all its solutions are bounded. In this case (E) has a 1-periodic solution, i.e. a constant solution, $u_n = u_0$: $0 \in Au_0$. It is known that if A is the subdifferential of a proper, convex, lower semicontinuous function, $A^{-1}0 \neq \emptyset$, then every solution $(z_n)_{n\geq 0}$ of

$$z_{n+1} - z_n + cAz_{n+1} \ni 0, \ n = 0, 1, ...$$

converges weakly to a point of $A^{-1}0$, but not strongly in general (see Baillon's counterexample [1]). However, strong convergence is possible in some cases, for instance, if either $(I + A)^{-1}$ is a compact operator or if A is strongly monotone, i.e., there is a constant $a > 0$, such that

$$(x_1 - x_2, y_1 - y_2) \geq a\|x_1 - x_2\|^2, \ \forall \, x_i \in D(A), \ y_i \in Ax_i, \ i = 1, 2.$$

In the latter case, we can state the following result:

Theorem 2. *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator, that is also strongly monotone (with a constant $a > 0$). Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences for a given positive integer p. Then Equation* (E) *has a unique p-periodic solution (ω_n) and for every solution (u_n) of* (E) *we have*

$$u_n - \omega_n \to 0, \text{ strongly in } H, \text{ as } n \to \infty.$$

Proof. Note that $c_n \geq \min\{c_k : 0 \leq k \leq p-1\} =: c > 0$. Since A is strongly monotone (hence coercive), it follows by Theorem 2 in [3] that all solutions of equation (E) are bounded. Therefore, by the argument used for the first part of

Theorem 1, there exists a p-periodic solution (ω_n) of equation (E). If (u_n) is an arbitrary solution of (E), we have

$$u_n - \omega_n \in u_{n+1} - \omega_{n+1} + c_n(Au_{n+1} - A\omega_{n+1}),\ n = 0, 1, ...$$

Multiplying this equation by $u_{n+1} - \omega_{n+1}$ we easily get

$$(1 + ac)\|u_{n+1} - \omega_{n+1}\| \le \|u_n - \omega_n\|,\ n = 0, 1, ...,$$

which implies

$$\|u_n - \omega_n\| \le (1 + ac)^{-n}\|u_0 - \omega_0\|,\ n = 0, 1, ...$$

Therefore, (ω_n) is the unique p-periodic solution of (E) and $u_n - \omega_n \to 0$, strongly in H, as claimed. ∎

4 Examples

If A is maximal monotone and coercive (i.e., there exists a $v^* \in H$ such that $(w, v - v^*)/\|v\| \to \infty$, for $v \in D(A)$, $w \in Av$, $\|v\| \to \infty$), then equation (E) has a periodic solution (equivalently, all its solutions are bounded) for all p-periodic sequences $(c_n) \subset (0, +\infty)$ and $(f_n) \subset H$. Indeed, in this case (f_n) is bounded, and $c_n \ge \min\{c_k : 0 \le k \le p-1\} =: c > 0$, so the assertion follows from Theorem 2 in [3]. This is not the case in general, as the following simple example shows:

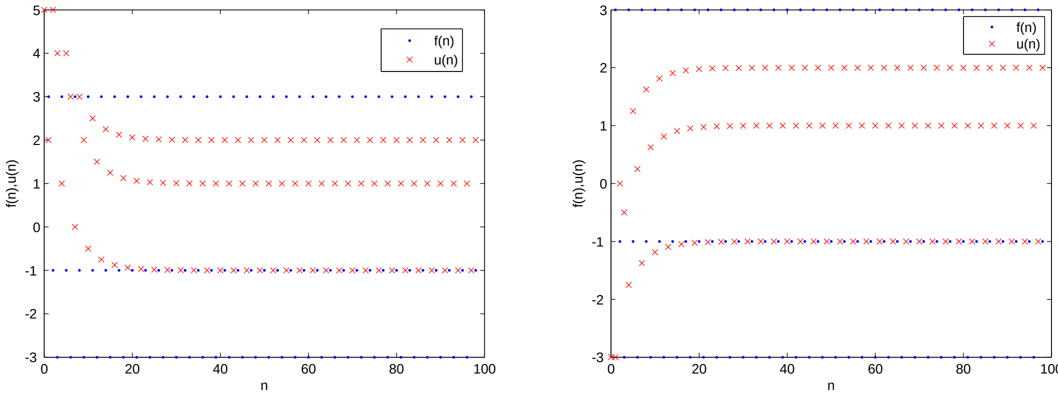
Example 1. *Let* $H = \mathbb{R}$, $A = \partial\varphi = \varphi'$, $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \le 0, \\ 0 & \text{if } x > 0, \end{cases}$$

$c_n = 1$ *for all* $n = 0, 1, \ldots$. *If* (f_n) *is the* 3*-periodic sequence defined by* $f_{3k} = -3$, $f_{3k+1} = 3$, $f_{3k+2} = -1$ *for* $k = 0, 1, \ldots$, *then for* $u_0 = 1$ *equation* (E) *has a* 3*-periodic solution* (ω_n), $\omega_{3k} = 1$, $\omega_{3k+1} = -1$, $\omega_{3k+2} = 2$, $k = 0, 1, \ldots$. *It turns out that* (ω_n) *is the unique* 3*-periodic solution of equations* (E)*. This follows easily by using the fact that any* 3*-periodic solution has the form* $(\omega_n + c)$, $c \in \mathbb{R}$. *By Theorem 1, every solution* (u_n) *of equation* (E) *tends asymptotically to* (ω_n)*:* $u_{3k} \to 1$, $u_{3k+1} \to -1$, $u_{3k+2} \to 2$ *as* $k \to \infty$ *(see Figure 1).*

On the other hand, if (f_n) *is the* 3*-periodic sequence defined by* $f_{3k} = 2$, $f_{3k+1} = 3$, $f_{3k+2} = -2$ *for* $k = 0, 1, \ldots$, *then all solutions of* (E) *are unbounded. Indeed, it is easy to show that there exists an unbounded solution of equation* (E) *and thus, according to Theorem 1, all solutions of* (E) *are unbounded.*

Now, let us show that for some periodic sequences (f_n) *the set of periodic solutions of* (E) *is not a singleton. For example, if* (f_n) *is the* 3*-periodic sequence defined by* $f_{3k} = 0.5$, $f_{3k+1} = 1.5$, $f_{3k+2} = -2$ *for* $k = 0, 1, \ldots$, *then the sequence* (ω_n) *defined by* $\omega_{3k} = c$, $\omega_{3k+1} = 0.5 + c$, $\omega_{3k+2} = 2 + c$, $k = 0, 1, \ldots$, *is a* 3*-periodic solution of* (E) *for all* $c \ge 0$.



(a) $u_0 = 5$ (b) $u_0 = -3$

Figure 1

In what follows we investigate some applications of the results presented in the previous section.

Example 2. *Consider in $\mathbb{R}$ the following parabolic type difference equation:*

$$\Delta_m u_{m,n} + f_{m+1}(u_{m+1,n}) \ni L_n u_{m+1,n} + q_{m,n}, \qquad m = 0, 1, ..., \ n = 1, 2, ... \qquad (E_p)$$

with the condition

$$u_{m,0} = 0, \qquad m = 0, 1, ..., \qquad (D)$$

where L_n denotes the discrete Laplace operator,

$$L_n u_{m+1,n} = \Delta_n^2 u_{m+1,n-1} = u_{m+1,n+1} - 2u_{m+1,n} + u_{m+1,n-1}, \qquad (3)$$

$q_{m,n}$ is a double real sequence, which is p-periodic with respect to m, and $f_m : D(f_m) \subset \mathbb{R} \to \mathbb{R}$ $(m = 1, 2, ...)$ are (possibly multivalued) maximal monotone mappings.

Consider the real Hilbert space

$$H = \ell^2(\mathbb{R}) = \{u = (u_1, u_2, \ldots) : \sum_{n=1}^{\infty} |u_n|^2 < \infty\}$$

with the usual inner product

$$(u, v) := \sum_{n=1}^{\infty} u_n v_n \qquad \forall u, v \in H.$$

Define on H the operator $A_1 := -L$, i.e., $A_1\big((v_n)_{n\geq 1}\big) = (-v_{n+1} + 2v_n - v_{n-1})_{n\geq 1}$, where $v_0 = 0$. We also define $A_2 : D(A_2) = \Pi_{n=1}^{\infty} D(f_n) \subset H \to H$,

$$A_2 v := \big(f_1(v_1), f_2(v_2), \ldots\big), \ v = (v_1, v_2, \ldots) \in D(A_2).$$

Thus Equation (E_p) *with condition* (D) *can be written in the form*

$$\Delta u_m + Au_{m+1} = q_m, \qquad m = 0, 1, ..., \tag{E_p^*}$$

with

$$u_m := (u_{m,n})_{n\geq 1}, \qquad q_m := (q_{m,n})_{n\geq 1},$$

where $A = A_1 + A_2$.
Operator A_1 *is everywhere defined, linear and strictly monotone (positive):*

$$(A_1 v, v) = |v_1|^2 + |v_1 - v_2|^2 + |v_2 - v_3|^2 + \ldots > 0,$$

for all v *different from zero. Moreover,* A_1 *is symmetric:*

$$(A_1 u, v) = (u, A_1 v) = \sum_{n=0}^{\infty} (u_{n+1} - u_n)(v_{n+1} - v_n),$$

where $u, v \in H$, $u_0 = v_0 = 0$. *Then* A_1 *is the subdifferential of* φ_1, $\varphi_1(v) = (1/2)(A_1 v, v)$, $v \in H$.
Assume that $0 \in D(f_m)$ *for all* $m = 1, 2, ...$ *Then,* A_2 *is maximal monotone in* H. *Moreover,* A_2 *is cyclically monotone in* H*, since all* f_m *are so in* $\mathbb{R}$ *(cf. Lemma 4), i.e.,* A_2 *is a subdifferential. Since* $D(A_1) = H$*, it follows that* $A = A_1 + A_2$ *is a maximal (cyclically) monotone operator, and furthermore* A *is strictly monotone. Therefore, if* $(q_m)_{m\geq 0}$ *is a* p*-periodic sequence in* H *then, the conditions specified in Theorem 1 are satisfied for equation* (E_p^*). *If a* p*-periodic solution of equation* (E_p^*) *exists, it is unique. Denote it* $(\omega_m)_{m\geq 0}$. *For any other solution* (u_m) *we have* $u_m - \omega_m \to 0$*, weakly in* H*, as* $m \to \infty$. *In particular, if* $u_m = (u_{m,n})_{n\geq 1}$*, we have* $u_{m,n} - \omega_{m,n} \to 0$*, as* $m \to \infty$*, for each* $n = 1, 2, ...$

Remark 2. Note that A_1 is just strictly monotone, not strongly monotone. Indeed, strong monotonicity would imply that A_1 is surjective, which is not the case
(e.g., the sequence $(-1/n) \in H$ does not belong to the range of A_1).

Example 3. *Consider in* $\mathbb{R}$ *the following difference equation:*

$$\Delta_m u_{m,n} + f_{m+1}(u_{m+1,n}) \ni L_n u_{m+1,n} + q_{m,n}, \qquad m = 0, 1, ..., \ n = 1, 2, ..., N, \tag{4}$$

with the Dirichlet type conditions

$$u_{m,0} = 0 = u_{m,N+1}, \qquad m = 0, 1, ..., \tag{5}$$

where N *is a positive integer,* $q_{m+p,n} = q_{m,n}$ *for all* $m = 0, 1, ..., \ n = 1, ..., N$ *for a given positive integer* p*, and* f_m *satisfy the same assumptions as in Example 2. We can choose* H *to be the Euclidean space* $\mathbb{R}^N$. A_1 *and* A_2 *are defined similarly. In this case,* A_1 *is even strongly monotone, so (according to Theorem 2) the above equation has a unique* p*-periodic solution,* $\omega_{m+p,n} = \omega_{m,n}$, $m = 0, 1, ..., \ n = 1, 2, ..., N$ *and any other solution* $u_{m,n}$ *converges to it:* $u_{m,n} - \omega_{m,n} \to 0$*, as* $m \to \infty$*, for each* $n = 1, 2, \cdots N$.

Remark 3. Choosing convenient mappings f_m we can obtain solutions with specific desired properties. E.g., if for all m f_m is the subdifferential of the indicator function of $[0, \infty)$, then the corresponding solutions have nonnegative components: $u_{m,n} \geq 0$.

Remark 4. If fact, in the above examples, A_2 could be a general maximal monotone operator from the corresponding space H into itself. Even more, if $\mathbb{R}$ is replaced by a general Hilbert space, then all the above reasonings work with slight modifications.

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