On a Hurwitz-Lerch zeta type function and its applications

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Abstract

We first define a generalized form of a Hurwitz-Lerch zeta type function and then use it in constructing certain classes of analytic functions in the unit disk. In our investigation, we obtain various results for the classes introduced, thereby, exhibiting their useful properties and characteristics by adopting the techniques of differential subordination. Several consequences of the main results are considered and relevant connections with some of the known results are also pointed out.

1 Introduction and Preliminaries

Let $\mathcal{H}$ denote a class of functions which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote a subclass of $\mathcal{H}$ comprising of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let Φ denote the class of all functions $\phi \in \mathcal{H}$ which are also univalent in $\mathbb{U}$ and $\phi(0) = 1$. By $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{C}(\alpha)$, we denote, respectively, the classes of starlike functions, convex functions and close-to-convex functions, each of order α $(0 \leq \alpha < 1)$ (see, for more details [15]). Given two functions $f \in \mathcal{H}$ and $g \in \mathcal{H}$, we say that g is subordinate to f in $\mathbb{U}$, and write $g(z) \prec f(z)$, if there exists a

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function $w \in \mathcal{H}$ with $|w(z)| < |z|$, $z \in \mathbb{U}$, such that $g(z) = f(w(z))$ in $\mathbb{U}$. In particular, if f is univalent in $\mathbb{U}$, then it follows that

$$g(z) \prec f(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad [g(0) = f(0) \quad \text{and} \quad g(\mathbb{U}) \subset f(\mathbb{U})].$$

For a bounded sequence $\{c(n)\}_{n=0}^{\infty}$ of real (or complex) numbers, we define a function $\Delta(z,s,b)$ by

$$\Delta(z,s,b) = \sum_{n=0}^{\infty} c(n) \frac{z^n}{(b+n)^s} \qquad (z \in \mathbb{U}; b \in \mathbb{R}_+; s \in \mathbb{C}/\mathbb{Z}_0^-). \tag{1.2}$$

It may be noted that for $c(n) = 1; n \in \mathbb{N}_0$, the function $\Delta(z,s,b)$ is the well known Hurwitz-Lerch zeta function [28]. Corresponding to the function $\Delta(z,s,b)$, we define a linear operator $\mathcal{T}_b^s f(z) : \mathcal{A} \to \mathcal{A}$, by

$$\mathcal{T}_b^s f(z) = \overline{\Delta}_{c(n)}(z,s,b) * f(z), \tag{1.3}$$

where $*$ denotes the Hadamard product (or convolution) and the function $\overline{\Delta}_{c(n)}(z,s,b)$ is given by

$$\overline{\Delta}(z,s,b) = \frac{(1+b)^s}{c(1)} \left(\Delta(z,s,b) - \frac{c(0)}{b^s} \right), \quad c(1) \neq 0. \tag{1.4}$$

If f is of the form (1.1), then it is easily seen from (1.2)-(1.4), that

$$\mathcal{T}_b^s f(z) = z + \sum_{n=2}^{\infty} \frac{c(n)}{c(1)} \left(\frac{b+1}{b+n} \right)^s a_n z^n, \quad z \in \mathbb{U}. \tag{1.5}$$

It is worth mentioning here that the operator $\mathcal{T}_b^s$ is a generalization of the various linear operators considered recently in many papers. In particular, we observe the following:

1. For $c(n) = 1$, the operator $\mathcal{T}_b^s$ is the well known Srivastava–Attiya operator [26] (see also [21]).

2. For $c(n) = 1$, the operator $\mathcal{T}_b^{-s}$ $(s > 0)$ is the Cho–Srivastava operator [3].

3. For $c(n) = n$ $(n \in \mathbb{N})$ the operator $\mathcal{T}_b^s$ $(s > 0, b \geq 0)$ is the Cho–Kim operator [4].

4. For

$$c(n) = \frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1}\, n!} \quad (\lambda, \mu \in \mathbb{C},\ \nu \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}), \tag{1.6}$$

where the symbol $(\lambda+1)_{n-1} = (\lambda+1)(\lambda+2)\cdots(\lambda+n-2)$ is the Pochhammer symbol, then the operator $\mathcal{T}_b^s$ converts into a recently defined operator $\mathcal{J}_{\lambda,\mu;\nu}^{s,b}$ studied by Prajapat and Bulboaca [23], which generalizes several previously investigated operators due to Cho and Srivastava [3], Choi-Saigo-Srivastava operator [5], Jung *et al.* [11], see also [24], Kwon and Cho [12], and Noor and Bukhari [17, p. 2, Eq.(1.3)], see also [20].

5. For
$$c(n) = \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}} \frac{1}{(n-1)!} \tag{1.7}$$
$$(\alpha_j > 0\,(j = 1,\ldots,q),\ \beta_j > 0\,(j = 1,\ldots,s),\ q \leq s+1;\ q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{1.8}$$
the operator $\mathcal{T}_b^0$ reduces to the well known Dziok-Srivastava operator [9].

6. For
$$c(n) = \frac{(-\mu+\eta+2)_{n-1}(2)_{n-1}}{(\lambda+\eta+2)_{n-1}(2-\mu)_{n-1}} \qquad (\lambda > 0; \mu,\eta \in \mathbb{R}), \tag{1.9}$$
the operator $\mathcal{T}_b^0$ reduces to the generalized fractional differintegral operator studied in [22], see also [6, 7, 8, 19, 25].

It can easily be verified from (1.5) that
$$z\left(\mathcal{T}_b^{s+1} f(z)\right)' = (b+1)\mathcal{T}_b^s f(z) - b\mathcal{T}_b^{s+1} f(z), \tag{1.10}$$
and we also note that
$$\mathcal{T}_0^s f(z) = \lim_{b\to 0}\{\mathcal{T}_b^s f(z)\}.$$

For the purpose of this paper, we introduce the following subclasses of $\mathcal{A}$ by making use of the generalized linear operator $\mathcal{T}_b^s$.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}(s,b;\phi)$, $\phi \in \Phi$, if it satisfies the following subordination condition:*
$$\frac{\mathcal{T}_b^s f(z)}{\mathcal{T}_b^{s+1} f(z)} \prec \phi(z), \quad z \in \mathbb{U}. \tag{1.11}$$
We further, set
$$\mathcal{M}\left(s,b;1+\frac{A-B}{b+1}.\frac{z}{1+Bz}\right) = \widetilde{\mathcal{M}}\left(s,b;A,B\right),$$
and
$$\mathcal{M}\left(s,b;1+\frac{2(1-\alpha)}{b+1}.\frac{z}{1-z}\right) = \widetilde{\mathbb{M}}\left(s,b,\alpha\right).$$

Remark 1.1.

1. For $c(n) = 1,\ n \in \mathbb{N}$, the classes
$$\mathcal{M}(-1,0;\phi) = S^*(\phi) \quad \text{and} \quad \mathcal{M}(-2,0;\phi) = C(\phi),$$
with $\phi \in \Phi$ such that $\phi(\mathbb{U})$ is a convex region lying in the right half–plane which is a symmetric with respect to real axis have been studied by Ma and Minda [13].

2. For $c(n) = 1,\ n \in \mathbb{N}$, the class
$$\widetilde{\mathcal{M}}\left(-1,0;A,B\right) = S[A,B]$$
was studied by Janowski [10].

3. In particular, for $c(n) = 1,\ n \in \mathbb{N}$, the classes

$$\mathcal{M}\left(-1, 0; \frac{1+(1-2\alpha)z}{1-z}\right) = \widetilde{\mathcal{M}}\left(-1, 0; 1-2\alpha, -1\right) = \widetilde{\mathbb{M}}\left(-1, 0, \alpha\right) = \mathcal{S}^*(\alpha)$$

and

$$\mathcal{M}\left(-2, 0; \frac{1+(1-2\alpha)z}{1-z}\right) = \widetilde{\mathcal{M}}\left(-2, 0; 1-2\alpha, -1\right) = \widetilde{\mathbb{M}}\left(-2, 0, \alpha\right) = \mathcal{K}(\alpha)$$

are (as mentioned above), the well known classes of starlike functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$ and convex functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$, respectively. Further, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, are the classes of starlike functions in $\mathbb{U}$ and convex functions in $\mathbb{U}$, respectively.

In the present paper we derive various useful and interesting properties and characteristics of the above defined function classes by using some key results of the subordination theory. Several corollaries are deduced from the main results and relevant connections of some of the corollaries with known results are also pointed out.

2 Key Lemmas

In order to derive our main results, we recall here the following lemmas:

Lemma 2.1. *[15, p. 132]. Let the function q be analytic and univalent in $\mathbb{U}$, also let the functions θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z)\phi[q(z)], \quad h(z) = \theta[q(z)] + Q(z),$$

and suppose that

1. $\mathfrak{Re}\frac{zQ'(z)}{Q(z)} > 0$ *in* $\mathbb{U}$;
2. $\mathfrak{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \mathfrak{Re}\left(\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right) > 0, \ \ z \in \mathbb{U}.$

If p is analytic in $\mathbb{U}$, with $p(0) = q(0), p(\mathbb{U}) \subset \mathbb{D}$, and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant, in the sense that if there exists a function s such that $p(z) \prec s(z)$, then $q(z) \prec s(z)$.

Lemma 2.2. *[16]. If $-1 \leq B < A \leq 1$, $\beta > 0$ and the complex number γ satisfies $\mathfrak{Re}(\gamma) \geq -\beta(1-A)/(1-B)$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz}, \ \ z \in \mathbb{U},$$

has a univalent solution in $\mathbb{U}$ *given by*

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\mathrm{d}t} - \dfrac{\gamma}{\beta}, & B \neq 0, \\ \dfrac{z^{\beta+\gamma}\ \exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\ \exp(\beta At)\mathrm{d}t} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases} \tag{2.1}$$

If $\phi(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is analytic in* $\mathbb{U}$ *and satisfies*

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z)+\gamma} \prec \frac{1+Az}{1+Bz}, \ \ z \in \mathbb{U}, \tag{2.2}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz}, \ \ z \in \mathbb{U},$$

and q *is the best dominant.*

Lemma 2.3. *[30]. Let* ν *be a positive measure on* $[0,1]$. *Let* h *be a complex valued function defined on* $\mathbb{U} \times [0,1]$ *such that* $h(\cdot,t)$ *is analytic in* $\mathbb{U}$ *for each* $t \in [0,1]$, *and* $h(z,\cdot)$ *is* ν*-integrable on* $[0,1]$ *for all* $\mathbb{U}$. *In addition suppose that* $\Re\{h(z,t)\} > 0$, $h(-r,t)$ *is real and*

$$\Re\left(\frac{1}{h(z,t)}\right) \geq \frac{1}{h(-r,t)} \quad \textit{for} \quad |z| \leq r < 1 \quad \textit{and} \quad t \in [0,1].$$

If the function $\mathcal{F}(z)$ *is defined by*

$$\mathcal{F}(z) = \int_0^1 h(z,t)\ \mathrm{d}\nu(t),$$

then

$$\Re\left(\frac{1}{\mathcal{F}(z)}\right) \geq \frac{1}{h(-r)}.$$

For real (or complex numbers) α, β and $\gamma(\neq 0,-1,-2,\ldots)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(\alpha,\beta;\gamma;z) = 1 + \frac{\alpha\,\beta}{\gamma}\,\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\,\frac{z^2}{2!} + \ \ldots.$$

We note that the above series converges absolutely for $z \in \mathbb{U}$, and hence, represents an analytic function in the unit disk $\mathbb{U}$ (see, for details, [31, Chapter 14]). Each of the following identities are fairly well known (cf., e.g., [31, Chapter 14]).

Lemma 2.4. *For real (or complex numbers)* α, β *and* $\gamma(\neq 0,-1,-1,\ldots)$:

$$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\mathrm{d}t = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}\ {}_2F_1(\alpha,\beta;\gamma;z), \ \ \gamma > \beta > 0, \tag{2.3}$$

$${}_2F_1(\alpha,\beta;\gamma;z) = {}_2F_1(\beta,\alpha;\gamma;z), \tag{2.4}$$

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}\ {}_2F_1\left(\alpha, c-\beta;\gamma;\frac{z}{z-1}\right). \tag{2.5}$$

3 Main Results

Our first main result is given by Theorem 3.1 below.

Theorem 3.1. *Let $\psi \in \Phi$. Suppose also that $z\psi'/\psi$ is starlike and $\mathfrak{Re}\{\psi(z)\} > 0$ in $\mathbb{U}$. If the function τ is defined by*

$$\tau(z) = \psi(z) + \frac{z\psi'(z)}{(b+1)\psi(z)}, \quad z \in \mathbb{U}, \tag{3.1}$$

and $f \in \mathcal{M}(s,b;\tau)$, then $f \in \mathcal{M}(s+1,b;\psi)$, whenever $(\mathcal{T}_b^{s+1}f)/(\mathcal{T}_b^{s+2}f)$ is an analytic function in $\mathbb{U}$.

Proof. Let $f \in \mathcal{M}(s,b;\tau)$. Define a function $p(z)$ by

$$p(z) = \frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)}, \quad z \in \mathbb{U}. \tag{3.2}$$

Differentiating both sides of (3.2) with respect to z, and making use of identity (1.10), we find that

$$p(z) + \frac{zp'(z)}{(b+1)p(z)} = \frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+1}f(z)}.$$

Thus by hypothesis, we have

$$(b+1)p(z) + \frac{zp'(z)}{p(z)} \prec (b+1)\psi(z) + \frac{z\psi'(z)}{\psi(z)}.$$

Let $\theta(w) = (b+1)w$ and $\phi(w) = 1/w$, then $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}/\{0\}$. Set

$$Q(z) = z\psi'(z)\phi(\psi(z)) = \frac{z\psi'(z)}{\psi(z)} \tag{3.3}$$

and

$$h(z) = \theta(\psi(z)) + Q(z) = (b+1)\psi(z) + \frac{z\psi'(z)}{\psi(z)}. \tag{3.4}$$

By the hypothesis of Theorem 3.1, $Q(z)$ is starlike and

$$\mathfrak{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathfrak{Re}\left\{\frac{\theta'[\psi(z)]}{\phi[\psi(z)]} + \frac{zQ'(z)}{Q(z)}\right\} =$$
$$\mathfrak{Re}\left\{(b+1)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0,$$

which shows that the function $h(z)$ is close-to-convex and thus univalent for all $z \in \mathbb{U}$. Therefore by virtue of Lemma 2.1, we conclude that $p(z) \prec \psi(z)$, i.e.

$$\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)} \prec \psi(z),$$

which implies that $f(z) \in \mathcal{M}(s+1,b;\psi)$. This proves Theorem 3.1. ∎

Corollary 3.1. *Assume that*

$$b \in \mathbb{R}, \quad -1 \leq B < A \leq 1, \quad b \geq -\frac{1-A}{1-B}, \quad C = \frac{A+bB}{1+b}$$

and

$$\eta(z) = \frac{1+Cz}{1+Bz} + \frac{(C-B)z}{(1+Bz)(1+Cz)}, \quad z \in \mathbb{U}.$$

If $f \in \mathcal{M}(s,b;\eta)$, then $f \in \widetilde{\mathcal{M}}(s+1,b;A,B)$, whenever $(\mathcal{T}_b^{s+1}f)/(\mathcal{T}_b^{s+2}f)$ is an analytic function in $\mathbb{U}$.

Proof. Choose

$$\psi(z) = 1 + \frac{A-B}{1+b}\frac{z}{1+Bz},$$

in Theorem 3.1. It is then sufficient to show that $Q(z)$ defined by (3.3) is starlike and $h(z)$ defined by (3.4) is close-to-convex in $\mathbb{U}$. We observe that the function

$$Q(z) = \frac{z\psi'(z)}{\psi(z)} = \frac{(C-B)z}{(1+Bz)(1+Cz)}$$

is starlike in $\mathbb{U}$, because

$$\begin{aligned}
\mathfrak{Re}\left(\frac{zQ'(z)}{Q(z)}\right) &= -1 + \mathfrak{Re}\left(\frac{1}{1+Bz}\right) + \mathfrak{Re}\left(\frac{1}{1+Cz}\right) \\
&> -1 + \frac{1}{1+|B|} + \frac{1}{1+|C|} \\
&= \frac{1-|BC|}{(1+|B|)(1+|C|)} \geq 0,
\end{aligned}$$

for $z \in \mathbb{U}$. Also,

$$\begin{aligned}
\mathfrak{Re}\left(\frac{zh'(z)}{Q(z)}\right) &= (1+b)\mathfrak{Re}\left(\frac{1+Cz}{1+Bz}\right) + \mathfrak{Re}\left(\frac{z\,Q'(z)}{Q(z)}\right) \\
&> (1+b)\left(\frac{1-C}{1-B}\right) \geq 0, \quad z \in \mathbb{U},
\end{aligned}$$

which shows that the function h is close-to-convex and thus univalent for all $z \in \mathbb{U}$. This proves Corollary 3.1. ■

Remark 3.1.

(i) By setting

$$A = 1-2\alpha \ (0 \leq \alpha < 1), \ B = -1 \ \text{and} \ c(n) = 1,$$

and using (3) of Remark 1.1 in Corollary 3.1, we get an improvement of the main result by Srivastava and Attiya [26]. The improvement is with regard to the condition for α. While as, we have the condition $b \geq -\alpha$, but in [26], the condition $b \geq 0$ is imposed.

(ii) Next, by setting

$$c(n) = 1 \ \ s = \delta(\delta > 0), \ b = 1, \ A = 1-2\alpha \ (0 \leq \alpha < 1) \ \text{and} \ B = -1,$$

and using again (3) of Remark 1.1, then Corollary 3.1 corresponds to a known result due to Attiya [1, Theorem 2].

Theorem 3.2. *Let $\psi \in \Phi$ and $\lambda > -1$. Suppose also that $\mathfrak{Re}\{\lambda - b + (1+b)\psi(z)\} > 0$ and the function*

$$\Psi(z) = \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)}, \quad z \in \mathbb{U},$$

is starlike in $\mathbb{U}$. For the functions $\mathcal{F}_\lambda$ and $\mathcal{X}$ defined, respectively, by

$$\mathcal{F}_\lambda(z) = \frac{\lambda+1}{z^\lambda}\int_0^z t^{\lambda-1} f(t)\, \mathrm{d}t \qquad (f \in \mathcal{A}, z \in \mathbb{U}) \tag{3.5}$$

and

$$\mathcal{X}(z) = \psi(z) + \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)}, \quad z \in \mathbb{U}, \tag{3.6}$$

if $f \in \mathcal{M}(s,b;\mathcal{X})$, then $\mathcal{F}_\lambda \in \mathcal{M}(s,b;\psi)$, whenever, $(\mathcal{T}_b^s\mathcal{F}_\lambda)/(\mathcal{T}_b^{s+1}\mathcal{F}_\lambda)$ is an analytic function in $\mathbb{U}$.

Proof. Let $f \in \mathcal{M}(s,b;\mathcal{X})$. From (1.5) and (3.5), we obtain that

$$\begin{aligned}(1+\lambda)\mathcal{T}_b^{s+1}f(z) &= \lambda\mathcal{T}_b^{s+1}\mathcal{F}_\lambda(z) + z(\mathcal{T}_b^{s+1}\mathcal{F}_\lambda(z))' \\ &= (\lambda-b)\mathcal{T}_b^{s+1}\mathcal{F}_\lambda(z) + (1+b)(\mathcal{T}_b^{s}\mathcal{F}_\lambda(z)).\end{aligned} \tag{3.7}$$

Define a function $q(z)$ by

$$q(z) = \frac{\mathcal{T}_b^s\mathcal{F}_\lambda(z)}{\mathcal{T}_b^{s+1}\mathcal{F}_\lambda(z)}, \quad z \in \mathbb{U}, \tag{3.8}$$

then clearly q is analytic in $\mathbb{U}$. Now simple calculations shows that

$$\frac{\mathcal{T}_b^s f(z)}{\mathcal{T}_b^{s+1} f(z)} = q(z) + \frac{zq'(z)}{\lambda - b + (1+b)q(z)}. \tag{3.9}$$

Since $f \in \mathcal{M}(s,b;\mathcal{X})$, then we have from (3.9) that

$$q(z) + \frac{zq'(z)}{\lambda - b + (1+b)q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\lambda - b + (1+b)\psi(z)}.$$

Adopting now same procedure (as in the previous theorem), and using Lemma 2.1, we infer that $q(z) \prec \psi(z)$, which proves Theorem 3.2 ∎

Theorem 3.3. *Assume that $-1 \le B < A$ and $B \le A$. If $f \in \widetilde{\mathcal{M}}(s,b;A,B)$, then*

$$\frac{\mathcal{T}_b^{s+1} f(z)}{\mathcal{T}_b^{s+2} f(z)} \prec \frac{1}{(b+1)\mathcal{Q}(z)} = Q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.10}$$

whenever, $(\mathcal{T}_b^{s+1}f)/(\mathcal{T}_b^{s+2}f)$ is an analytic function in $\mathbb{U}$ and

$$\mathcal{Q}(z) = \begin{cases} \int_0^1 t^b \left(\frac{1+Btz}{1+Bz}\right)^{(b+1)(A-B)/B} \mathrm{d}t, & B \neq 0, \\ \\ \int_0^1 t^b exp((b+1)(t-1)Az)\, \mathrm{d}t, & B = 0. \end{cases} \tag{3.11}$$

The function Q is the best dominant of (3.10).
Furthermore, if

$$-1 \leq B < 0 \quad \textit{and} \quad B < A \leq -\frac{B}{b+1},$$

then

$$\mathfrak{Re}\left\{\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)}\right\} > \left\{{}_2F_1\left(1, \frac{(b+1)(B-A)}{B}; b+2; \frac{B}{B-1}\right)\right\}^{-1}. \tag{3.12}$$

The result (3.12) *is best possible.*

Proof. Let p be given by (3.2), then by Theorem 3.1, we have

$$\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)} = p(z) + \frac{zp'(z)}{(b+1)p(z)}, \ \ z \in \mathbb{U}.$$

Since $f \in \widetilde{\mathcal{M}}(s, b; A, B)$, therefore, by applying Lemma 2.2 (for $\beta = b+1$ and $\gamma = 0$,) we obtain (3.10). Next, we show that

$$\inf_{|z|\leqslant 1} \mathfrak{Re}\{Q(z)\} = Q(-1). \tag{3.13}$$

If we set

$$\alpha = \frac{(b+1)(B-A)}{B}, \quad \beta = b+1 \quad \text{and} \quad \gamma = b+2,$$

so that $\gamma > \beta > 0$, then by applying Lemma 2.4, we find from (3.11) that for $B \neq 0:$

$$\mathcal{Q}(z) = (1+Bz)^{\alpha} \int_0^1 t^{\beta-1}(1+Btz)^{-\alpha}\mathrm{d}t = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\, {}_2F_1\left(1, \alpha; \gamma; \frac{Bz}{Bz+1}\right).$$

To prove (3.13), we show that

$$\mathfrak{Re}\left\{\frac{1}{\mathcal{Q}(z)}\right\} \geq \frac{1}{\mathcal{Q}(-1)}, \ \ z \in \mathbb{U}.$$

Again for $-1 \leq B < 0$ and $B < A \leq -B/(b+1)$ (so that $\gamma > \beta > 0$) can be written as

$$\mathcal{Q}(z) = \int_0^1 g(t,z)\mathrm{d}\mu(t),$$

where

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz}, \ 0 \leq t \leq 1 \ \text{ and } \ \mathrm{d}\mu(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\, t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\mathrm{d}t$$

is a positive measure on the closed interval [0,1].

For $-1 \leq B < 1$, we note that $\mathfrak{Re}\{g(t,z)\} > 0$, $g(t,-r)$ is real for $0 \leq r < 1$ and $t \in [0,1]$ and

$$\mathfrak{Re}\left\{\frac{1}{g(t,z)}\right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(t,-r)}$$

for $|z| \leq r < 1$ and $t \in [0,1]$ Thus, by making use of Lemma 2.3, and letting $r \to 1^-$, we get $\mathfrak{Re}\{1/\mathcal{Q}(z)\} \geq 1/\mathcal{Q}(-1)$, $z \in \mathbb{U}$. In the case, when $A = -B/(b+1)$, we obtain the required result by letting $A \to (-B/(b+1))^+$. The result is sharp because of the best dominant property of $Q(z)$. This complete the proof of Theorem 3.3. ∎

If we put $A = 1-2\delta$ $(0 \leq \delta < 1)$ and $B = -1$ in Theorem 3.3, we get

Corollary 3.2. *If $f \in \widetilde{\mathbb{M}}(s,b,\delta)$, then for $b/2(b+1) \leq \delta < 1$, we have*

$$\mathfrak{Re}\left(\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)}\right) > \left\{{}_2F_1\left(1, 2(b+1)(1-\delta); b+1; \frac{B}{B-1}\right)\right\}^{-1}, \quad z \in \mathbb{U}.$$

The result is best possible.

In the next result, we consider an inverse problem of Corollary 3.2.

Theorem 3.4. *If $f \in \widetilde{\mathbb{M}}(s+1,b,\delta)$ for some $\delta (0 \leq \delta < 1)$, then $f \in \widetilde{\mathbb{M}}(s,b,\delta)$ in $|z| < R(b,\delta)$, i. e.*

$$\frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+1}f(z)} \in \varphi(\mathbb{U}), \quad |z| < R(b,\delta),$$

where

$$\varphi(z) = 1 + \frac{2(1-\delta)}{b+1}\frac{z}{1-z}$$

and

$$R(b,\delta) = \begin{cases} \dfrac{(2-\delta) - \sqrt{(2-\delta)^2 - (b+1)(1-b-2\delta)}}{1-b-2\delta}, & \delta \neq \frac{1-b}{2}, \\ (1+b)/(3+b), & \delta = \frac{1-b}{2}. \end{cases} \tag{3.14}$$

The result is best possible.

Proof. Let

$$\frac{z\left(\mathcal{T}_b^{s+2}f(z)\right)'}{\mathcal{T}_b^{s+2}f(z)} = \delta + (1-\delta)u(z), \quad z \in \mathbb{U}, \tag{3.15}$$

where $u(z) = 1 + u_1 z + u_2 z^2 + \ldots$ is analytic and has a positive real part in $\mathbb{U}$. Using (1.10) in (3.15), and differentiating logarithmically, we deduce that

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z\left(\mathcal{T}_b^{s+1}f(z)\right)'}{\mathcal{T}_b^{s+1}f(z)} - \delta\right\} &= (1-\delta)\mathfrak{Re}\left\{u(z) + \frac{zu'(z)}{\delta + b + (1-\delta)u(z)}\right\} \\ &\geq (1-\delta)\mathfrak{Re}\left\{u(z) - \frac{|zu'(z)|}{|\delta + b + (1-\delta)u(z)|}\right\}. \end{aligned} \tag{3.16}$$

Now using the well known estimates [14]:

$$|zu'(z)| \leq \frac{2r}{1-r^2}\mathfrak{Re}\{u(z)\} \quad \text{and} \quad \mathfrak{Re}\{u(z)\} \geq \frac{1-r}{1+r}, \quad |z| = r < 1,$$

in (3.16), and performing elementary calculations, we get

$$\Re\left\{\frac{z\left(\mathcal{T}_b^{s+1}f(z)\right)'}{\mathcal{T}_b^{s+1}f(z)}-\delta\right\}\geq$$
$$(1-\delta)\Re\{u(z)\}\left\{1-\frac{2r}{(\delta+b)(1-r^2)+(1-\delta)(1-r)^2}\right\},$$

which is positive if $r<R(b,\delta)$, where $R(b,\delta)$ is given by (3.14).

To show that the bound $R(b,\delta)$ is best possible, we consider the function $f\in\mathcal{A}$, defined by

$$\frac{z\left(\mathcal{T}_b^{s+2}f(z)\right)'}{\mathcal{T}_b^{s+2}f(z)}=\delta+(1-\delta)\,\frac{1+z}{1-z}\qquad(0\leq\delta<1, z\in\mathbb{U}).$$

Noting that

$$\begin{aligned}\frac{z\left(\mathcal{T}_b^{s+1}f(z)\right)'}{\mathcal{T}_b^{s+1}f(z)}-\delta &= (1-\delta)\left(\frac{1+z}{1-z}+\frac{2z}{(\delta+b)(1-z^2)+(1-\delta)(1+z)^2}\right)\\ &= 0\end{aligned}$$

for $\delta=-R(b,\delta)$, we conclude that the bound is best possible. This proves Theorem 3.4. ∎

Theorem 3.5. *Let $f\in\mathcal{A}$ and $\alpha,\beta\in\mathbb{R}$. Suppose also that ψ ia convex univalent function in $\mathbb{U}$ satisfying the inequality that*

$$\Re\left\{\frac{(1+b)(\alpha+2\beta\,\psi(z))}{\beta}+\frac{z\psi''(z)}{\psi'(z)}+1\right\}\geq 0,\ \ z\in\mathbb{U}. \qquad (3.17)$$

If

$$\alpha\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)}+\beta\frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+2}f(z)}\prec\alpha\psi(z)+\beta(\psi(z))^2+\frac{\beta}{1+b}z\psi'(z), \qquad (3.18)$$

then $f\in\mathcal{M}(s+1,b;\psi)$.

Proof. Let the function p be defined by (3.3), then simple calculations show that

$$\alpha\,\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)}+\beta\,\frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+2}f(z)}=\alpha p(z)+\beta(p(z))^2+\frac{\beta}{1+b}zp'(z).$$

By setting $\theta(w)=\alpha w+\beta w^2$ and $\phi(w)=\frac{\beta}{1+b}$, we find that the desired assertion of Theorem 3.5 follows by the application of Lemma 2.1. ∎

Corollary 3.3. *Let $f\in\mathcal{A}$; $\alpha,\beta\in\mathbb{R}$ and $-1\leq B<A\leq 1$. Suppose also that*

$$\alpha\ \geq\ -\frac{\beta}{1+b}\left(2\,b+\frac{2(1-A)}{1-B}+\frac{1-|B|}{1+|B|}\right).$$

If

$$\alpha\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+2}f(z)}+\beta\frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+2}f(z)} \prec \alpha+\beta+\frac{(\alpha+2\beta)(A-B)z}{(1+b)(1+Bz)}+\frac{\beta(A-B)^2z+\beta(A-B)z}{(1+b)^2(1+Bz)^2},$$

then $f \in \widetilde{\mathcal{M}}(s+1,b;A,B)$.

Proof. Choose

$$\psi(z)=1+\frac{A-B}{1+b}\frac{z}{1+Bz}, \quad -1 \le B < A \le 1,$$

in Theorem 3.5. Then, it is sufficient to show that ψ is convex univalent function in $\mathbb{U}$. We observe that

$$\mathfrak{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)=\mathfrak{Re}\left(\frac{1-Bz}{1+Bz}\right)>\frac{1-|B|}{1+|B|}\ge 0,$$

which shows that $\psi(z)$ is convex and univalent in $\mathbb{U}$. Also,

$$\begin{aligned}
&\mathfrak{Re}\left\{\frac{(1+b)(2\beta\psi(z)+\alpha)}{\beta}+\frac{z\psi''(z)}{\psi'(z)}+1\right\}\\
&\quad = 2b+\frac{\alpha(1+b)}{\beta}+2\mathfrak{Re}\left(\frac{1+Az}{1+Bz}\right)+\mathfrak{Re}\left(\frac{1-Bz}{1+Bz}\right)\\
&\quad = 2b+\frac{\alpha(1+b)}{\beta}+2\frac{1-A}{1+B}+\frac{1-|B|}{1+|B|} \ \ge\ 0.
\end{aligned}$$

Thus, the proof of Corollary 2.3 is complete. ∎

On putting $c(n)=1$, $b=0$ and $s=-2$ in Corollary 3.3, and using (3) of Remark 1.1, we get

Corollary 3.4. *Let* $f \in \mathcal{A}$; $\alpha,\beta \in \mathbb{R}$ *and* $-1 \le B < A \le 1$. *Suppose also that*

$$\alpha \ge -\beta\left(\frac{2(1-A)}{1-B}+\frac{1-|B|}{1+|B|}\right).$$

If

$$\beta\frac{z^2f''(z)}{f'(z)}+(\alpha+\beta)\frac{zf'(z)}{f(z)} \prec \alpha+\beta+(\alpha+2\beta)\frac{(A-B)z}{1+Bz}+\frac{\beta(A-B)^2z^2+\beta(A-B)z}{(1+Bz)^2},$$

then $f \in \mathcal{S}^*[A,B]$.

Remark 3.2.

(i) Putting $\alpha=\lambda-1$ and $\beta=1$ in Corollary 3.4, we get a known result due to Xu and Yang [29, p. 581, Theorem 1].

(ii) If we choose $\alpha=1-\beta$ $(0<\beta\le 1)$, $A=1$ and $B=-1$ in Corollary 3.4, we get a known result of Padmanabham [18].

Theorem 3.6. *Let $f \in \mathcal{A}$; $\alpha, \beta \in \mathbb{C}$ and $\gamma, \delta \in \mathbb{C}/\{0\}$. Suppose also that $\psi(z)$ be analytic and univalent in $\mathbb{U}$ such that $\psi(z) \neq 0$, $z\psi'/\psi$ is starlike univalent in $\mathbb{U}$ satisfying the inequality that*

$$\mathfrak{Re}\left\{1 + \frac{\beta}{\gamma}\psi(z) - \frac{z\psi'(z)}{\psi(z)} + \frac{z\psi''(z)}{\psi'(z)}\right\} \geq 0, \ z \in \mathbb{U}. \tag{3.19}$$

If

$$\alpha + \beta\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\delta} + \gamma\delta(b+1)\left(\frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+1}f(z)} - 1\right) \prec \alpha + \beta\psi(z) + \gamma\frac{z\psi'(z)}{\psi(z)}, \tag{3.20}$$

then

$$\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\delta} \prec \psi(z), \ \ z \in \mathbb{U}/\{0\}, \tag{3.21}$$

and ψ is the best dominant.

Proof. Let the function $H(z)$ be defined by

$$H(z) = \left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\delta} \qquad (z \in \mathbb{U}/\{0\};\ f \in \mathcal{A}). \tag{3.22}$$

It is clear that H is analytic in $\mathbb{U}$. Differentiating (3.22) with respect to z and using identity (1.10), we find that

$$\alpha + \beta\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\delta} + \gamma\delta(b+1)\left(\frac{\mathcal{T}_b^{s}f(z)}{\mathcal{T}_b^{s+1}f(z)} - 1\right) = \alpha + \beta H(z) + \gamma\frac{zH'(z)}{H(z)}.$$

Now, by setting $\theta(w) = \alpha + \beta w$ and $\psi(w) = \gamma/w$, assertion (3.21) of Theorem 3.6 follows by application of Lemma 2.1. ∎

Setting

$$\alpha = \beta = 0, \quad \gamma = \frac{1}{\delta(b+1)} \quad \text{and} \quad \psi(z) = (1+Bz)^{\frac{\delta(A-B)}{B}}$$

in Theorem 3.6, we get

Corollary 3.5. *Let the constraints $-1 \leq B < A \leq 1, B \neq 0, \delta \in \mathbb{C}/\{0\}$ satisfy either*

$$\left|\frac{\delta(A-B)}{B} - 1\right| \leq 1 \qquad \text{or} \qquad \left|\frac{\delta(A-B)}{B} + 1\right| \leq 1.$$

If $f \in \widetilde{\mathcal{M}}(s, b; A, B)$, then

$$\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\delta} \prec \ (1+Bz)^{\frac{\delta(A-B)}{B}} = L(z), \tag{3.23}$$

where L is the best dominant.

Further, by putting $A = 1 - 2\alpha$ and $B = -1$ in Corollary 3.5, we get following result:

Corollary 3.6. *Let the constraints* $0 \leq \alpha < 1$, *and* $\delta \in \mathbb{C}/\{0\}$ *satisfy either*

$$|2\delta(1-\alpha)+1| \leq 1 \qquad or \qquad |2\delta(1-\alpha)-1| \leq 1.$$

If $f(z) \in \widetilde{\mathbb{M}}\,(s,b,\alpha)$, *then*

$$\mathfrak{Re}\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\frac{1}{2\delta(1-\alpha)}} > 2^{-\frac{1}{\delta}}, \;\; z \in \mathbb{U}. \tag{3.24}$$

The result is best possible.

Proof. For $A = 1 - 2\alpha$ and $B = -1$, (3.23) can be written as

$$\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\delta} = (1-w(z))^{-2\delta(1-\alpha)},$$

where w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$ This implies that

$$\left(\frac{\mathcal{T}_b^{s+1}f(z)}{z}\right)^{\frac{\delta}{2(1-\alpha)}} = (1-w(z))^{-\delta}, \;\; z \in \mathbb{U}.$$

In this last above equality, if we take real parts on both sides and use the following elementary inequality:

$$\mathfrak{Re}(w^{\frac{1}{m}}) \geq [\mathfrak{Re}(w)]^{\frac{1}{m}} \;\text{ for }\; \mathfrak{Re}(w) > 0 \;\text{ and }\; m \geq 1,$$

we are lead to the result (3.24). This completes the proof of Corollary 3.6. ∎

Remark 3.1. By setting

$$c(n) = 1, \;\; \psi(z) = \frac{1}{(1-z)^{2b}} (b \in \mathbb{C}/\{0\}),$$

$$b = 0, \;\; s = -1, \;\; \alpha = \delta = 1, \;\; \beta = 0, \;\; \gamma = \frac{1}{\eta},$$

and in the process using (3) of Remark 1.1, then Theorem 3.6, reduces to a known result due to Srivastava and Lashin [27].

Theorem 3.7. *Let* $f \in \mathcal{A}$, *and suppose also that* $g \in \mathcal{A}$ *satisfies the following inequality:*

$$\mathfrak{Re}\left(\frac{\mathcal{T}_b^{s+1}g(z)}{z}\right) > 0, \;\; z \in \mathbb{U}. \tag{3.25}$$

If

$$\left|\frac{\mathcal{T}_b^{s+1}f(z)}{\mathcal{T}_b^{s+1}g(z)} - 1\right| < 1, \;\; z \in \mathbb{U}, \tag{3.26}$$

and

$$\varphi(z) = 1 + \frac{2z}{(1+b)(1-z)} = \frac{1}{1+b}\left(\frac{1+z}{1-z} + b\right),$$

then for $|z| < (\sqrt{17}-3)/4$

$$\frac{\mathcal{T}_b^s f(z)}{\mathcal{T}_b^{s+1} f(z)} \in \varphi(\mathbb{U}),$$

or

$$f \in \widetilde{\mathbb{M}}\,(s,b,0) \quad in \quad |z| < (\sqrt{17}-3)/4.$$

Proof. Let

$$r(z) = \frac{\mathcal{T}_b^{s+1} f(z)}{\mathcal{T}_b^{s+1} g(z)} - 1 = c_1 z + c_2 z^2 + \ldots, \tag{3.27}$$

and we note that r is analytic in $\mathbb{U}$, with $r(0) = 0$ and $|r(z)| < 1$ for $z \in \mathbb{U}$. Thus, by applying the familiar Schwarz Lemma, we get

$$r(z) = z\psi(z),$$

where $\psi(z)$ is analytic in $\mathbb{U}$ and $|\psi(z)| \leq 1$ for $\mathbb{U}$. Therefore, (3.27) leads to

$$\mathcal{T}_b^{s+1} f(z) = (1 + z\,\psi(z))\mathcal{T}_b^{s+1} g(z), \;\; z \in \mathbb{U},$$

which on logarithmic differentiation gives

$$\frac{z(\mathcal{T}_b^{s+1} f(z))'}{\mathcal{T}_b^{s+1} f(z)} = \frac{z(\mathcal{T}_b^{s+1} g(z))'}{\mathcal{T}_b^{s+1} g(z)} + \frac{z\{\psi(z) + z\psi'(z)\}}{1 + z\psi(z)}. \tag{3.28}$$

Setting

$$\phi(z) = \frac{\mathcal{T}_b^{s+1} g(z)}{z},$$

we see that $\phi(z) = 1 + d_1 z + d_2 z^2 + \ldots$, is analytic in $\mathbb{U}$ with $\mathfrak{Re}(\phi(z)) > 0$ for $z \in \mathbb{U}$, and

$$\frac{z(\mathcal{T}_b^{s+1} g(z))'}{\mathcal{T}_b^{s+1} g(z)} = \frac{z\phi'(z)}{\phi(z)} + 1, \tag{3.29}$$

so by using the following well known estimates [2]:

$$\mathfrak{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\} \geq -\frac{2r}{1-r^2} \quad \text{and} \quad \mathfrak{Re}\left\{\frac{z\{\psi(z) + z\psi'(z)\}}{1 + z\psi(z)}\right\} \geq -\frac{1}{1-r},$$

for $|z| = r < 1$ in (3.29), we deduce that

$$\begin{aligned} \mathfrak{Re}\left(\frac{z(\mathcal{T}_b^{s+1} f(z))'}{\mathcal{T}_b^{s+1} f(z)}\right) &\geq 1 - \frac{2r}{1-r^2} - \frac{1}{1-r} \\ &= \frac{1-3r-2r^2}{1-r^2}, \;\; |z| = r < 1, \end{aligned}$$

which is certainly positive provided that $|z| < (\sqrt{17}-3)/4$. Therefore, for $|z| < (\sqrt{17}-3)/4$:

$$\begin{aligned} \frac{\mathcal{T}_b^s f(z)}{\mathcal{T}_b^{s+1} f(z)} &= \frac{1}{1+b}\left(\frac{z(\mathcal{T}_b^{s+1} f(z))'}{\mathcal{T}_b^{s+1} f(z)} + b\right) \\ &\in \frac{1}{1+b}\left(\frac{1+z}{1-z} + b\right)(\mathbb{U}) \\ &= \varphi(\mathbb{U}) \end{aligned}$$

This completes the proof of Theorem 3.7. ∎

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