On ϕ-biflat and ϕ-biprojective Banach algebras

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Abstract

In this paper, we introduce the new notions of ϕ-biflatness, ϕ-biprojectivity, ϕ-Johnson amenability and ϕ-Johnson contractibility for Banach algebras, where ϕ is a non-zero homomorphism from a Banach algebra A into $\mathbb{C}$. We show that a Banach algebra A is ϕ-Johnson amenable if and only if it is ϕ-inner amenable and ϕ-biflat. Also we show that ϕ-Johnson amenability is equivalent with the existence of left and right ϕ-means for A. We give some examples to show differences between these new notions and the classical ones. Finally, we show that $L^1(G)$ is ϕ-biflat if and only if G is an amenable group and $A(G)$ is ϕ-biprojective if and only if G is a discrete group.

1 Introduction

For the background theory of amenability of Banach algebras, see B. E. Johnson [11]. A Banach algebra A is amenable (contractible) if every continuous derivation from A into a dual Banach A-module X^* (Banach A-module X) is inner, for every Banach A-module X. Also in [12], Johnson showed that a Banach algebra A is amenable if and only if A has a virtual diagonal, that is, there exists an $m \in (A \otimes_p A)^{**}$ such that $a \cdot m = m \cdot a$ and $\pi^{**}(m)a = a$ for every $a \in A$, where $\pi : A \otimes_p A \to A$ is the product morphism, specified by $\pi(a \otimes b) = ab$.

There are some important homological notions which have direct relation with amenability and contractibility, such as biflatness and biprojectivity. Indeed, A is called biflat (biprojective), if there exists a bounded A-module morphism

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$\rho : A \to (A \otimes_p A)^{**}$ ($\rho : A \to A \otimes_p A$) such that $\pi^{**} \circ \rho$ is the canonical embedding of A into A^{**} (ρ is a right inverse for π), see [17]. In fact, a Banach algebra A is amenable if and only if A is biflat and has a bounded approximate identity.

Recently E. Kaniuth *et al.* in [13] have introduced and studied the notion of ϕ-amenability for Banach algebras. For a multiplicative linear functional ϕ on A, A is called ϕ-amenable if every continuous derivation from A into the dual Banach A-module X^* is inner, for every Banach A-module X such that $a \cdot x = \phi(a)x$. They showed that ϕ-amenability of A is equivalent with the existence of a bounded net $(a_\alpha)_{\alpha \in I}$ in A such that $aa_\alpha - \phi(a)a_\alpha \to 0$ and $\phi(a_\alpha) \to 1$, for every $a \in A$. Later on, this notion even has been generalized in [9], [14] and [15]. Motivated by these considerations, A. Jabbari *et al.* in [10], have introduced the ϕ-version of inner amenability, which is equivalent with the existence of a bounded net $(a_\alpha)_{\alpha \in I}$ in A such that $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) = 1$, for every $a \in A$.

The content of this paper is as follows. After recalling some background notations and definitions, we will define new notions of ϕ-Johnson amenability, ϕ-biflatness and ϕ-biprojectivity for Banach algebras and with some characterizations and some examples, we will show the differences between these new notions and the classical ones. It will be shown that A is ϕ-Johnson amenable if and only if A is ϕ-biflat and ϕ-inner amenable. Also, it will be shown that $L^1(G)$ is ϕ-biflat if and only if G is an amenable group. Also we will show that $A(G)$ is ϕ-biprojective if and only if G is a discrete group. The paper concludes with some examples about semigroup algebras.

We recall that if X is a Banach A-module, then with the following actions X^* is also a Banach A-module:

$$< a \cdot f, x > = < f, x \cdot a >, \quad < f \cdot a, x > = < f, x \cdot a > \qquad (a \in A, x \in X, f \in A^*).$$

The projective tensor product of A by A is denoted by $A \otimes_p A$. The Banach algebra $A \otimes_p A$ is a Banach A-module with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Throughout this paper, $\Delta(A)$ denotes the character space of A, that is, all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**}.

Now we will give the definition of our new notions.

Definition 1.1. A Banach algebra A is called ϕ-Johnson amenable, if there exists an element $m \in (A \otimes_p A)^{**}$ such that $a \cdot m = m \cdot a$ and $\tilde{\phi} \circ \pi^{**}(m) = 1$, for every $a \in A$, where $\tilde{\phi}$ is defined as above. Also, A is called a ϕ-Johnson contractible Banach algebra, if there exists an element $m \in A \otimes_p A$ such that $a \cdot m = m \cdot a$ and $\phi \circ \pi(m) = 1$, for every $a \in A$.

Definition 1.2. Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called ϕ-biprojective, if there exists a bounded A-module morphism $\rho : A \to A \otimes_p A$ such that $\phi \circ \pi \circ \rho = \phi$. Also A is called ϕ-biflat if there exists a bounded A-module morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi^{**} \circ \rho = \phi$.

2 Elementary properties

In this section, we prove some elementary lemmas to characterize the ϕ-Johnson amenability, the ϕ-biflatness and the ϕ-biprojectivity of Banach algebras.

Lemma 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. The Banach algebra A is ϕ-Johnson amenable if and only if there exists a bounded net $(m_\alpha)_{\alpha\in I}$ in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\phi \circ \pi(m_\alpha) \to 1$, for every $a \in A$.*

Proof. Let A be ϕ-Johnson amenable. Then there exists an $m \in (A \otimes_p A)^{**}$ such that $a \cdot m = m \cdot a$ and $\tilde{\phi} \circ \pi^{**}(m) = 1$. So by Goldstine's theorem m is a w^*-accumulation point of a bounded net $(m_\alpha)_{\alpha\in I} \subseteq A \otimes_p A$. Since π^{**} is w^*-continuous, hence $\pi(m_\alpha) \xrightarrow{w^*} \pi^{**}(m)$, $\pi(m_\alpha)(\phi) \to \tilde{\phi} \circ \pi^{**}(m)$, therefore $\phi \circ \pi(m_\alpha) \to 1$. Since $m_\alpha \xrightarrow{w^*} m$, for every $\psi \in (A \otimes_p A)^*$, we have $m_\alpha(a \cdot \psi) \to m(a \cdot \psi)$ and $m_\alpha(\psi \cdot a) \to m(\psi \cdot a)$. Therefore $m_\alpha \cdot a(\psi) \to m \cdot a(\psi)$, that is, $m_\alpha \cdot a \xrightarrow{w^*} m \cdot a$. Similarly, one can show that $a \cdot m_\alpha \xrightarrow{w^*} a \cdot m$. It is easy to verify that $a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{w} 0$. Consequently, one can assume that by Mazur's theorem, this limit holds even in the norm topology.

Conversely, let $(m_\alpha)_{\alpha\in I} \subseteq A \otimes_p A$ be a bounded net such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\phi \circ \pi(m_\alpha) \to 1$, for every $a \in A$. After passing to a subnet if necessary, let $m \in (A \otimes_p A)^{**}$ be a w^*-cluster point of the net $(m_\alpha)_{\alpha\in I}$. Since $a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{w^*} 0$, one can easily show that $a \cdot m = m \cdot a$, for every $a \in A$. Also the w^*-continuity of π^{**}, reveals that $\tilde{\phi} \circ \pi^{**}(m) = 1$ and the proof is complete. ∎

Recall that A is a left (right) ϕ-amenable Banach algebra, if there exists a bounded net $(m_\alpha)_{\alpha\in I}$ in A, such that $||am_\alpha - \phi(a)m_\alpha|| \to 0$ $\;(||m_\alpha a - \phi(a)m_\alpha|| \to 0)$, respectively and $\phi(m_\alpha) = 1$. For further details see [13].

Proposition 2.2. *Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. A is left and right ϕ-amenable if and only if A is ϕ-Johnson amenable.*

Proof. Suppose that $(m_\alpha)_{\alpha\in I}$ and $(m_\beta)_{\beta\in J}$ are bounded nets in A such that $\phi(m_\alpha) = \phi(m_\beta) = 1$, which satisfy $||am_\alpha - \phi(a)m_\alpha|| \to 0$ and $||m_\beta a - \phi(a)m_\beta|| \to 0$, respectively, for every $a \in A$. Define $m^\alpha_\beta = m_\alpha \otimes m_\beta \subseteq A \otimes_p A$, therefore $\phi \circ \pi(m^\alpha_\beta) = \phi(m_\alpha m_\beta) = \phi(m_\alpha)\phi(m_\beta) = 1$. On the other hand, for every $a \in A$, we have

$$||a \cdot (m_\alpha \otimes m_\beta) - (m_\alpha \otimes m_\beta) \cdot a|| \to 0.$$

To see this, by using the boundedness of $(m_\alpha)_{\alpha\in I}$ and $(m_\beta)_{\beta\in J}$, we obtain

$$\begin{aligned} ||a \cdot m^\beta_\alpha - m^\beta_\alpha \cdot a|| &= ||a \cdot (m_\alpha \otimes m_\beta) - (m_\alpha \otimes m_\beta) \cdot a|| \\ &\leq ||am_\alpha \otimes m_\beta - \phi(a)m_\alpha \otimes m_\beta|| + ||m_\alpha \otimes m_\beta\phi(a) - (m_\alpha \otimes m_\beta)a|| \\ &\leq ||am_\alpha - \phi(a)m_\alpha||\,||m_\beta|| + ||m_\alpha||\,||m_\beta a - \phi(a)m_\beta|| \to 0. \end{aligned}$$

So by Lemma 2.1, A is ϕ-Johnson amenable.

For converse, suppose that $(m_\alpha)_{\alpha\in I}$ is a bounded net in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\phi \circ \pi(m_\alpha) \to 1$. One can easily show that, there exists a bounded linear map $T : A \otimes_p A \to A$ defined by $T(a \otimes b) = \phi(b)a$, for every a and b in A. It is easy to see that $T(a \cdot m) = a \cdot T(m)$ and $T(m \cdot a) = \phi(a)T(m)$, where $m \in A \otimes_p A$. Now, consider the following

$$||T(a \cdot m_\alpha - m_\alpha \cdot a)|| \leq ||T|| \, ||a \cdot m_\alpha - m_\alpha \cdot a||,$$

therefore one can easily see that

$$||aT(m_\alpha) - \phi(a)T(m_\alpha)|| = ||T(a \cdot m_\alpha - m_\alpha \cdot a)|| \to 0.$$

Replacing m_α with $\phi(T(m_\alpha))^{-1}m_\alpha$ and using the fact $\phi(T(m_\alpha)) = \phi \circ \pi(m_\alpha) = 1$, we obtain a bounded net $(T(m_\alpha))_\alpha$ in A, which satisfies the hypotheses of [13, Theorem 1-4], hence A is left ϕ-amenable. Similarly, one can show that A is right ϕ-amenable. ■

Recall that, A is a ϕ-inner amenable Banach algebra, if A has a bounded net $(a_\alpha)_{\alpha \in I}$ such that $\phi(a_\alpha) \to 1$ and $aa_\alpha - a_\alpha a \to 0$, see [10, Theorem 2-1].

Lemma 2.3. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that A is ϕ-Johnson amenable. Then A is ϕ-inner amenable.*

Proof. Let $(m_\alpha)_{\alpha \in I} \subseteq A \otimes_p A$ be a bounded net such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\phi \circ \pi(m_\alpha) \to 1$. Now if we consider the net $(\pi(m_\alpha))_\alpha$ and since π is A-module morphism, then clearly,

$$a\pi(m_\alpha) - \pi(m_\alpha)a = \pi(a \cdot m_\alpha - m_\alpha \cdot a) \to 0$$

and $\phi \circ \pi(m_\alpha) \to 1$. Hence, A is a ϕ-inner amenable Banach algebra. ■

Now, we want to give an example which is ϕ-inner amenable but is not ϕ-Johnson amenable. Moreover, we give another example which is ϕ-biprojective, hence is ϕ-biflat but is not ϕ-Johnson amenable. Let I be a closed ideal of the Banach algebra A which $\phi|_I \neq 0$. Then I is left and right ϕ-amenable whenever A is left and right ϕ-amenable, see [13].

Example 2.4. Let A be a Banach algebra with $\dim(A) > 1$ such that $ab = \phi(a)b$ for every $a, b \in A$, where $\phi \in \Delta(A)$. Then A is weakly amenable, but not amenable [2, Proposition 2.13]. Also A is not a ϕ-inner amenable Banach algebra [5, Example 2-3]. Note that $A^\sharp = A \oplus \mathbb{C}$, the unitization of A, is a ϕ_e-inner amenable Banach algebra, where $\phi_e(a + \lambda) = \phi(a) + \lambda$, for every $a \in A$ and $\lambda \in \mathbb{C}$.

We claim that, this algebra is not ϕ_e-Johnson amenable. We go toward a contradiction and suppose that $A^\sharp$ is ϕ_e-Johnson amenable, where $\dim A > 1$. Since A is a closed ideal of $A^\sharp$ and $\phi_e|_A \neq 0$, A is ϕ-Johnson amenable. Hence, A is ϕ-inner amenable. So by [5, Example 2-3], $\dim(A) = 1$ which is a contradiction.

Furthermore, we show that $A^\sharp$ is not even a pseudo-amenable Banach algebra. To see this we go toward a contradiction, suppose that $A^\sharp$ is pseudo-amenable. Let $a_0 \in A$ be such that $\phi(a_0) = 1$. By [7, Theorem 3-1], clearly A is approximately amenable. Therefore A has an approximate identity say $(e_\alpha)_{\alpha \in I}$. Consider

$$a_0 = \lim_\alpha a_0 e_\alpha = \lim_\alpha \phi(a_0)e_\alpha = \lim_\alpha e_\alpha,$$

in other words, a_0 is a unit element for A. Then by the above considerations, one can easily see that

$$a = \lim ae_\alpha = a \lim e_\alpha = \phi(a)a_0$$

so $\dim(A) = 1$, which is a contradiction.

Note that, since $aa_0 = \phi(a)a_0$ and $\phi(a_0) = 1$, A is a left ϕ-amenable Banach algebra, so by [13, Lemma 3-2] $A^\sharp$ is left ϕ_e-amenable. Therefore by this example we have a Banach algebra which is ϕ_e-amenable and ϕ_e-inner amenable but is not ϕ_e-Johnson amenable.

We want to give an example which reveals differences of ϕ-biflatness and ϕ-biprojectivity with ϕ-Johnson amenability. Let A be a Banach algebra with $\dim(A) > 1$ such that $ab = \phi(b)a$, where $\phi \in \Delta(A)$. By [5, Example 2-3] A is not ϕ-inner amenable, so by previous lemma A is not ϕ-Johnson amenable. But we show that, A is ϕ-biprojective. Indeed, let $x_0 \in A$ be such that $\phi(x_0) = 1$. Define $\rho : A \to A \otimes_p A$ by $\rho(a) = a \otimes x_0$. One can easily see that ρ is a bounded A-module morphism and $\phi \circ \pi \circ \rho = \phi$. Then we have an example which is ϕ-biprojective and hence ϕ-biflat but is not ϕ-Johnson amenable.

Example 2.5. Let $A = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} | a, b \in \mathbb{C} \right\}$ and $\phi(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}) = b$. It is easy to see that ϕ is a character on A. By [18, page 3241] A is a biprojective Banach algebra, hence is ϕ-biprojective, therefore is ϕ-biflat. On the other hand, by [5, Example 2-3], this algebra is not ϕ-inner amenable, then by previous Lemma A is not ϕ-Johnson amenable.

3 Characterization of ϕ-biflatness and ϕ-biprojectivity

Lemma 3.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is ϕ-Johnson amenable, then A is ϕ-biflat.*

Proof. Let $m \in (A \otimes_p A)^{**}$ be such that $a \cdot m = m \cdot a$ and $\tilde{\phi} \circ \pi^{**}(m) = 1$. Define a map $\rho : A \to (A \otimes_p A)^{**}$ by $\rho(a) = a \cdot m$. Then ρ is an A-module morphism, since

$$b \cdot \rho(a) = b \cdot (a \cdot m) = ba \cdot m = \rho(ba), \quad \rho(a) \cdot b = (a \cdot m) \cdot b = ab \cdot m = \rho(ab).$$

On the other hand

$$\tilde{\phi} \circ \pi^{**} \circ \rho(a) = \tilde{\phi} \circ \pi^{**}(a \cdot m) = \tilde{\phi}(a\pi^{**}(m)) = \phi(a)\tilde{\phi} \circ \pi^{**}(m) = \phi(a).$$

Therefore A is a ϕ-biflat Banach algebra. ∎

Lemma 3.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is ϕ-Johnson contractible, then A is ϕ-biprojective. The converse holds, whenever A is either unital or a commutative Banach algebra.*

Proof. Let $m \in A \otimes_p A$ be such that $a \cdot m = m \cdot a$ and $\phi(\pi(m)) = 1$. Define $\rho : A \to A \otimes_p A$ by $\rho(a) = a \cdot m$. Then clearly ρ is a bounded A-module morphism and we have

$$\phi \circ \pi \circ \rho(a) = \phi(a\pi(m)) = \phi(a)\phi(\pi(m)) = \phi(a).$$

So A is ϕ-biprojective.

Conversely, suppose that A is a ϕ-biprojective Banach algebra. Let $\rho : A \to A \otimes_p A$ be a bounded A-module morphism and e is an unit for A. Thus, $\rho(e) \in A \otimes_p A$ and $a \cdot \rho(e) = \rho(e) \cdot a$ and $\phi \circ \pi \circ \rho(e) = \phi(e) = 1$. Therefore A is ϕ-Johnson contractible. In the commutative case, let $x_0 \in A$ be such that $\phi(x_0) = 1$. For $\rho(x_0) \in A \otimes_p A$, we have $a \cdot \rho(x_0) = \rho(x_0) \cdot a$ and $\phi \circ \pi \circ \rho(x_0) = \phi(x_0) = 1$, for every $a \in A$. Then the proof is complete. ■

Proposition 3.3. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is ϕ-biflat and ϕ-inner amenable, then A is ϕ-Johnson amenable.*

Proof. Since A is a ϕ-biflat Banach algebra, there exists a bounded A-module morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi^{**} \circ \rho = \phi$. Suppose that $(a_\alpha)_{\alpha \in I}$ is a bounded net in A such that for each $a \in A$, $aa_\alpha - a_\alpha a \to 0$ and $\phi(a_\alpha) \to 1$. Thus, we have

$$||a \cdot \rho(a_\alpha) - \rho(a_\alpha) \cdot a|| \to 0$$

and

$$\tilde{\phi} \circ \pi^{**} \circ \rho(a_\alpha) \to 1.$$

We construct a bounded net $(b_\lambda) \subseteq A \otimes_p A$ such that $\phi \circ \pi(b_\lambda) \to 1$ and $||a \cdot b_\lambda - b_\lambda \cdot a|| \to 0$. Let $\epsilon > 0$, pick finite sets $F \subseteq A$ and $\Phi \subseteq (A \otimes_p A)^*$. Let

$$K = \{a \cdot \xi | a \in F, \xi \in \Phi\} \cup \{\xi \cdot a | a \in F, \xi \in \Phi\}.$$

Hence, there exists $v = v(\epsilon, F, \Phi)$ such that for every $a \in F$

$$||a \cdot \rho(a_v) - \rho(a_v) \cdot a|| < \frac{\epsilon}{3K_0}$$

and

$$|\tilde{\phi} \circ \pi^{**} \circ \rho(a_v) - 1| < \epsilon,$$

where $K_0 = \max\{||\xi|| : \xi \in \Phi\}$. By Goldstine's theorem, there exists a bounded net $(b_\lambda) \subseteq A \otimes_p A$ such that converges to $\rho(a_v)$ in the w^*-topology. Since π^{**} is w^*-continuous, $\pi(b_\lambda) \xrightarrow{w^*} \pi^{**}(\rho(a_v))$. Hence, there exists $\lambda_0 = \lambda_0(\epsilon, F, \Phi)$ such that

$$|\psi(b_{\lambda_0}) - \rho(a_v)(\psi)| < \frac{\epsilon}{3}$$

and

$$|\phi \circ \pi(b_{\lambda_0}) - \tilde{\phi} \circ \pi^{**} \circ \rho(a_v)| < \epsilon,$$

for all $\psi \in K$. Therefore for some $c \in \mathbb{R}$, we have

$$|\phi \circ \pi(b_{\lambda_0}) - 1| = |\phi \circ \pi(b_{\lambda_0}) - \tilde{\phi} \circ \pi^{**} \circ \rho(a_v) + \tilde{\phi} \circ \pi^{**} \circ \rho(a_v) - 1| < c\epsilon.$$

Since $|\psi(b_{\lambda_0}) - \rho(a_v)(\psi)| < \frac{\epsilon}{3}$,

$$|\xi(a \cdot b_{\lambda_0} - b_{\lambda_0} \cdot a)| \leq |\xi(a \cdot b_{\lambda_0}) - a \cdot \rho(a_v)(\xi)| + |a \cdot \rho(a_v)(\xi) - \rho(a_v) \cdot a(\xi)| + |\rho(a_v) \cdot a(\xi) - \xi(b_{\lambda_0} \cdot a)| < \epsilon.$$

Hence, we have $a \cdot b_\lambda - b_\lambda \cdot a \to 0$ in the w-topology. By Mazur's theorem, one can assume that $a \cdot b_\lambda - b_\lambda \cdot a \to 0$, with respect to the norm topology, as we desired. ■

Lemma 3.4. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Let I be a closed ideal of A such that $\phi|_I \neq 0$. If A is ϕ-biprojective, then I is $\phi|_I$-biprojective.*

Proof. Let $\rho : A \to A \otimes_p A$ be an A-module morphism such that $\phi \circ \pi \circ \rho = \phi$. Suppose that $i_0 \in I$ is such that $\phi(i_0) = 1$. Define $\eta : A \otimes_p A \to I \otimes_p I$ by $\eta(a \otimes b) = ai_0 \otimes i_0 b$ for every a and b in A. Since η is an A-module morphism, $\eta \circ \rho : A \to I \otimes_p I$ is an A-module morphism. Define $\hat{\rho} = \eta \circ \rho|_I$ which is an I-module morphism. It is easy to see that $\phi \circ \pi \circ \hat{\rho}(i) = \phi(i)$ for every $i \in I$. Then the proof is complete. ■

Similarly, one can see that the above lemma is also true for the ϕ-biflat case.

Lemma 3.5. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A^{**} is $\tilde{\phi}$-biprojective, then A is ϕ-biflat.*

Proof. Let $\rho : A^{**} \to A^{**} \otimes_p A^{**}$ be an A^{**}-module morphism such that $\tilde{\phi} \circ \pi_{A^{**}} \circ \rho = \phi$. Define $\rho_0 = \rho|_A : A \to A^{**} \otimes_p A^{**}$. There exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,

(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,

(iii) $\pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m)$,

see [6, Lemma 1-7]. Clearly one can see that $\psi \circ \rho_0$ is an A-module morphism and $\tilde{\phi} \circ \pi_A^{**} \circ \psi \circ \rho_0 = \tilde{\phi} \circ \pi_{A^{**}} \circ \rho_0 = \phi$, the proof is complete. ■

The analogous result of [16, Proposition 2-4] holds for ϕ-biprojectivity.

Proposition 3.6. *Let A and B be Banach algebras and $\phi \in \Delta(A), \psi \in \Delta(B)$. Suppose that A and B are ϕ-biprojective and ψ-biprojective, respectively. Then $A \otimes_p B$ is $\phi \otimes \psi$-biprojective.*

Proof. Let $\rho_0 : A \to A \otimes_p A$ and $\rho_1 : B \to B \otimes_p B$ be such that $\phi \circ \pi_A \circ \rho_0 = \phi$ and $\psi \circ \pi_B \circ \rho_1 = \psi$. Define $\theta : (A \otimes_p A) \otimes_p (B \otimes_p B) \to (A \otimes_p B) \otimes_p (A \otimes_p B)$ by

$$(a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes b_2),$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Set $\rho = \theta \circ (\rho_0 \otimes \rho_1)$, for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$\pi_{A \otimes_p B} \circ \theta(a_1 \otimes a_2 \otimes b_1 \otimes b_2) = \pi_{A \otimes_p B}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = \pi_A(a_1 \otimes a_2)\pi_B(b_1 \otimes b_2),$$

then clearly one can show that $\pi_{A \otimes_p B} \circ \theta = \pi_A \otimes \pi_B$. Hence, $\pi_{A \otimes_p B} \circ \theta(\rho_0(a) \otimes \rho_1(b)) = \pi_A \circ \rho_0(a) \otimes \pi_B \circ \rho_1(b)$ and it is easy to see that

$$\phi \otimes \psi \circ \pi_{A \otimes_p B} \circ \theta(\rho_0 \otimes \rho_1)(a \otimes b) = \phi \otimes \psi(a \otimes b),$$

the proof is complete. ■

We now prove a partial converse to Proposition 3.6.

Proposition 3.7. *Let A and B be Banach algebras, $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Suppose that A is unital with unit e_A and B containing a non-zero idempotent x_0 such that $\psi(x_0) = 1$. If $A \otimes_p B$ is $\phi \otimes \psi$-biprojective, then A is ϕ-biprojective.*

Proof. Let A and B be Banach algebras. Then $A \otimes_p B$ becomes a Banach A-module with the actions given by

$$a_1 \cdot (a_2 \otimes b) = a_1 a_2 \otimes b, \quad a_2 \otimes b \cdot a_1 = a_2 a_1 \otimes b, \qquad (a_1, a_2 \in A,\, b \in B).$$

Suppose that $A \otimes_p B$ is $\phi \otimes \psi$-biprojective. Then there exists a bounded $A \otimes_p B$-module morphism $\rho_1 : A \otimes_p B \to (A \otimes_p B) \otimes_p (A \otimes_p B)$ such that $(\phi \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho_1 = \phi \otimes \psi$. By the above considerations, we have

$$\begin{aligned} \rho_1(a_1 a_2 \otimes x_0) = \rho_1((a_1 \otimes x_0) \otimes (a_2 \otimes x_0)) &= a_1 \otimes x_0 \cdot \rho_1(a_2 \otimes x_0) \\ &= a_1 \cdot (e_A \otimes x_0)\rho_1(a_2 \otimes x_0) \\ &= a_1 \rho_1(a_2 \otimes b_0). \end{aligned}$$

Similarly one can show that $\rho_1(a_2 a_1 \otimes x_0) = \rho_1(a_2 \otimes x_0) \cdot a_1$.

Define $T : (A \otimes_p B) \otimes_p (A \otimes_p B) \to A \otimes_p A$ by $T((a \otimes b) \otimes (c \otimes d)) = \psi(bd) a \otimes c$, where $a, c \in A$ and $b, d \in B$. Clearly T is a bounded linear operator and $\pi_A \circ T = (id_A \otimes \psi) \circ \pi_{A \otimes_p B}$ and also $\phi \circ (id_A \otimes \psi) = \phi \otimes \psi$, where $id_A \otimes \psi(a \otimes b) = \psi(b) a$ for $a \in A$ and $b \in B$.

Obviously the map $\rho : A \to A \otimes_p A$ defined by $\rho(a) = T \circ \rho_1(a \otimes x_0)$ is a bounded A-module morphism. Since $\psi(x_0) = 1$, we have

$$\begin{aligned} \phi \circ \pi_A \circ T \circ \rho(a) = \phi \circ \pi_A \circ T \circ \rho_1(a \otimes x_0) &= \phi \circ (id_A \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho_1(a \otimes x_0) \\ &= (\phi \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho_1(a \otimes x_0) \\ &= \phi(a) \end{aligned}$$

for all $a \in A$ and this completes the proof. ∎

4 Application to group algebras and Fourier algebras

Let G be a locally compact group and let $\hat{G}$ be its dual group, which consists of all non-zero continuous homomorphism $\zeta : G \to \mathbb{T}$. It is well-known that $\Delta(L^1(G)) = \{\phi_\zeta : \zeta \in \hat{G}\}$, where $\phi_\zeta(f) = \int_G \overline{\zeta(x)} f(x) dx$ and dx is a left Haar measure on G, for more details, see [8, Theorem 23-7].

Lemma 4.1. *For a locally compact group G, $L^1(G)$ is ϕ_ζ-biflat if and only if G is amenable.*

Proof. Let $L^1(G)$ be ϕ_ζ-biflat. Since $L^1(G)$ has a bounded approximate identity, then by Proposition 3.3 $L^1(G)$ is ϕ_ζ-Johnson amenable, hence by Proposition 2.2 $L^1(G)$ is left ϕ_ζ-amenable. Therefore by [1, Corollary 3-4] G is amenable. ∎

Lemma 4.2. *Let G be an infinite abelian discrete group. Then $\ell^1(G)$ is ϕ_ζ-biflat, but it is not ϕ_ζ-biprojective.*

Proof. Let G be an infinite abelian discrete group and let $\ell^1(G)$ be ϕ_ζ-biprojective. Since $\ell^1(G)$ is unital, Lemma 3.2 implies that $\ell^1(G)$ is ϕ_ζ-Johnson contractible. Using the same argument as in the proof of Proposition 2.2, we can show that $\ell^1(G)$ is ϕ_ζ-contractible, now by applying [15, Theorem 6-1] we see that G is compact which is a contradiction, so $\ell^1(G)$ is not ϕ_ζ-biprojective. But since an abelian group G is amenable, its group algebra $\ell^1(G)$ is amenable and so is ϕ_ζ-Johnson amenable. Thus by Lemma 3.1 $\ell^1(G)$ is ϕ_ζ-biflat. ∎

Lemma 4.3. *Let G be a compact group and $\phi_\zeta \in \Delta(L^1(G))$. Then $L^1(G)^{**}$ is $\tilde{\phi}_\zeta$-biprojective. If converse holds, then G is amenable.*

Proof. Since G is a compact group, then $\hat{G} \subseteq L^1(G)$. Suppose that $\phi_\zeta \in \Delta(L^1(G))$ where $\zeta \in \hat{G}$. Then ϕ_ζ has an extension to $L^1(G)^{**}$, which denoted by $\tilde{\phi}_\zeta$. Let $m = \zeta \otimes \zeta$. It is clear that $m \in L^1(G)^{**} \otimes_p L^1(G)^{**}$. We claim that, m is a $\tilde{\phi}_\zeta$-Johnson contraction for $L^1(G)^{**}$. Let $h \in L^1(G)^{**}$. Then there exists a net $(h_\alpha)_{\alpha \in I} \subseteq L^1(G)$ such that $h_\alpha \xrightarrow{w^*} h$. It is easy to verify that

$$h_\alpha \cdot \zeta \otimes \zeta = \tilde{\phi}_\zeta(h_\alpha)\zeta \otimes \zeta = \zeta \otimes \zeta\tilde{\phi}_\zeta(h_\alpha) = \zeta \otimes \zeta \cdot h_\alpha.$$

Since $h_\alpha \xrightarrow{w^*} h$,

$$\tilde{\phi}_\zeta(h_\alpha)\zeta \otimes \zeta \to \tilde{\phi}_\zeta(h)\zeta \otimes \zeta$$

and

$$\zeta \otimes \zeta\tilde{\phi}_\zeta(h_\alpha) \to \zeta \otimes \zeta\tilde{\phi}_\zeta(h).$$

Hence, it is clear that $\zeta \otimes \zeta \cdot h = h \cdot \zeta \otimes \zeta$ for $h \in L^1(G)^{**}$. Plainly one can show that $\tilde{\phi}_\zeta(\pi(\zeta \otimes \zeta)) = 1$, then m is a $\tilde{\phi}_\zeta$-Johnson contraction for $L^1(G)^{**}$, then $L^1(G)^{**}$ is ϕ_ζ-Johnson contractible, so by Lemma 3.2, it is $\tilde{\phi}_\zeta$-biprojective.

For converse, let $L^1(G)^{**}$ be ϕ_ζ-biprojective. Then by Lemma 3.5, $L^1(G)$ is ϕ_ζ-biflat. Hence Lemma 4.1 implies the amenability of G. ∎

Let A be a Banach algebra with norm $||\cdot||_A$. We recall that a Banach algebra B with norm $||\cdot||_B$ is called an abstract Segal algebra with respect to A if

(i) B is a dense left ideal in A,

(ii) there exists $M > 0$ such that $||b||_A \leq M||b||_B$ for every $b \in B$,

(iii) there exists $C > 0$ such that $||ab||_B \leq C||a||_A||b||_B$ for every $a \in A$ and $b \in B$.

Let G be a locally compact group and let $A(G)$ be its Fourier algebra. Then $\Delta(A(G))$ consists of all point evaluations ϕ_x $(x \in G)$ defined by $\phi_x(f) = f(x)$ for all $f \in A(G)$.

Lemma 4.4. *Let $A(G)$ be the Fourier algebra on a locally compact group G and let $SA(G)$ be an abstract Segal algebra with respect to $A(G)$. Suppose that $\phi_x \in \Delta(A(G))$ for some $x \in G$. Then $SA(G)$ is ϕ_x-biprojective if and only if G is a discrete group*

Proof. Suppose that $SA(G)$ is ϕ_x-biprojective. Since $SA(G)$ is a commutative Banach algebra, Lemma 3.2 implies that $SA(G)$ is ϕ_x-Johnson contractible. Hence, by similar arguments as in the proof of Proposition 2.2, $SA(G)$ is ϕ_x-contractible, then G is discrete, see [1, Theorem 3-5].

For the converse, use the same argument as in the proof of [1, Theorem 3-5]. ∎

Corollary 1. *$A(G)$ is ϕ_x-biprojective for some $x \in G$ if and only if G is a discrete group.*

Corollary 2. *Let G be any non-discrete locally compact group and $\phi_x \in \Delta(A(G))$ for every $x \in G$. Then $A(G)$ is ϕ_x-biflat, but is not ϕ_x-biprojective.*

Proof. Let G be a locally compact group. By [13, Example 2-6] $A(G)$ is left ϕ_x-amenable for every $x \in G$. Since $A(G)$ is commutative, then $A(G)$ is right ϕ_x-amenable. Hence by Proposition 2.2 $A(G)$ is ϕ_x-Johnson amenable. Then by Lemma 3.1 $A(G)$ is ϕ_x-biflat for every locally compact group G. But by the above corollary $A(G)$ is not ϕ_x-biprojective. ∎

5 Example

Remark 5.1. Our standard reference for the following examples is [3]. Consider the semigroup $\mathbb{N}_\wedge$, with the semigroup operation $m \wedge n = \min\{m, n\}$, where m and n are in $\mathbb{N}$. $\Delta(\ell^1(\mathbb{N}_\wedge))$ consists precisely of the all functions $\phi_n : \ell^1(\mathbb{N}_\wedge) \to \mathbb{C}$ defined by $\phi_n(\sum_{i=1}^{\infty} \alpha_i \delta_i) = \sum_{i=n}^{\infty} \alpha_i$ for every $n \in \mathbb{N}$. It has been shown that $\mathbb{N}_\wedge$ is not a uniformly locally finite semigroup (see [16]).

Example 5.2. Let $\mathbb{N}_\wedge$ be as in the Remark 5.1. Since $\mathbb{N}_\wedge$ is not uniformly locally finite, $\ell^1(\mathbb{N}_\wedge)$ is neither biprojective nor biflat [16, Theorem 3-7]. But if we take $\phi_1 \in \Delta(\ell^1(\mathbb{N}_\wedge))$ and $m = \delta_1 \otimes \delta_1$, then we have $\phi_1(\pi(m)) = \phi_1(\pi(\delta_1 \otimes \delta_1)) = \phi_1(\delta_1) = 1$ and $a \cdot m = m \cdot a$, for every $a \in \ell^1(\mathbb{N}_\wedge)$. Therefore $\ell^1(\mathbb{N}_\wedge)$ is a ϕ_1-Johnson contractible Banach algebra. By Lemma 3.2, $\ell^1(\mathbb{N}_\wedge)$ is ϕ_1-biprojective and hence ϕ_1-biflat.

Example 5.3. Again let $\mathbb{N}_\wedge$ be as in the Remark 5.1 and let $\phi \in \Delta(\ell^1(\mathbb{N}_\wedge)^{**})$. Since $(\delta_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\ell^1(\mathbb{N}_\wedge)$ see [3, Proposition 3-3-1], $\ell^1(\mathbb{N}_\wedge)^{**}$ has a right unit E, which is a w^*-limit point of $(\delta_n)_{n\in\mathbb{N}}$. Since $\phi(E) = 1$, $\phi(\delta_n) \neq 0$ for sufficiently large n, hence $\phi|_{\ell^1(\mathbb{N}_\wedge)} \neq \{0\}$. So $\phi|_{\ell^1(\mathbb{N}_\wedge)}$ is a character on $\ell^1(\mathbb{N}_\wedge)$, by Remark 5.1 it has a form ϕ_n for some $n \in \mathbb{N}$, but every character ϕ_n on $\ell^1(\mathbb{N}_\wedge)$ has an unique extension $\tilde{\phi_n}$ on $\ell^1(\mathbb{N}_\wedge)^{**}$, that is, for some $n \in \mathbb{N}$ we have $\phi = \tilde{\phi_n}$.

Now if $\ell^1(\mathbb{N}_\wedge)^{**}$ is amenable, then by [6, Theorem 1-8] $\ell^1(\mathbb{N}_\wedge)$ is amenable, so by [4, Theorem 2] $\mathbb{N}_\wedge$ has a finite number of idempotents, which is impossible. Thus $\ell^1(\mathbb{N}_\wedge)^{**}$ is not amenable but we claim that it is $\tilde{\phi_1}$-Johnson contractible. To see this, let $a \in \ell^1(\mathbb{N}_\wedge)^{**}$. Then there exists a net $(a_\alpha)_{\alpha\in I}$ in $\ell^1(\mathbb{N}_\wedge)$ such that $a_\alpha \xrightarrow{w^*} a$. Hence,

$$a \cdot \delta_1 \otimes \delta_1 = w^* - \lim a_\alpha \delta_1 \otimes \delta_1 = \lim \phi_1(a_\alpha)\delta_1 \otimes \delta_1 = \tilde{\phi_1}(a)\delta_1 \otimes \delta_1$$

and similarly $\delta_1 \otimes \delta_1 \cdot a = \tilde{\phi_1}(a)\delta_1 \otimes \delta_1$. Moreover $\tilde{\phi_1}(\pi^{**}(\delta_1 \otimes \delta_1)) = \phi_1(\delta_1) = 1$, so $m = \delta_1 \otimes \delta_1 \in \ell^1(\mathbb{N}_\wedge)^{**} \otimes_p \ell^1(\mathbb{N}_\wedge)^{**}$ is a $\tilde{\phi_1}$-Johnson contraction for $\ell^1(\mathbb{N}_\wedge)^{**}$, that is, $\ell^1(\mathbb{N}_\wedge)^{**}$ is $\tilde{\phi_1}$-Johnson contractible. So by Lemma 3.2 it is $\tilde{\phi_1}$-biprojective. In the general case, for every $n > 1$, take $m = (\delta_n - \delta_{n-1}) \otimes (\delta_n - \delta_{n-1})$, it is easy to see that m is a $\tilde{\phi_n}$-Johnson contraction for $\ell^1(\mathbb{N}_\wedge)^{**}$. Hence, by Lemma 3.2 for every $n \in \mathbb{N}$, $\ell^1(\mathbb{N}_\wedge)^{**}$ is $\tilde{\phi_n}$-biprojective.

Remark 5.4. Consider the semigroup $\mathbb{N}_\vee$, with semigroup operation $m \vee n = \max\{m, n\}$, where m and n are in $\mathbb{N}$. The character space $\Delta(\ell^1(\mathbb{N}_\vee))$ precisely consists of the all functions $\phi_n : \ell^1(\mathbb{N}_\vee) \to \mathbb{C}$ defined by $\phi_n(\sum_{i=1}^\infty \alpha_i\delta_i) = \sum_{i=1}^n \alpha_i$ for every $n \in \mathbb{N} \cup \{\infty\}$.

Example 5.5. Let $\mathbb{N}_\vee$ be as in the Remark 5.4 and let $\phi_n \in \Delta(\ell^1(\mathbb{N}_\vee))$ where $n \in \mathbb{N} \cup \{\infty\}$. We claim that $\ell^1(\mathbb{N}_\vee)$ is ϕ_n-biflat, for every n in $\mathbb{N} \cup \{\infty\}$. To see this, for every $n \in \mathbb{N}$, set $m = (\delta_n - \delta_{n+1}) \otimes (\delta_n - \delta_{n+1})$, then it is easy to see that $a \cdot m = m \cdot a$ and $\tilde{\phi_n}(\pi(m)) = 1$, where $a \in \ell^1(\mathbb{N}_\vee)$. In the case $n = \infty$, set $m = w^* - \lim \delta_k \otimes \delta_k$, then by the w^*-continuity of π^{**}, we have

$$\begin{aligned}\tilde{\phi_\infty}(\pi^{**}(m)) &= \tilde{\phi_\infty}(\pi^{**}(w^* - \lim \delta_k \otimes \delta_k))\\ &= \tilde{\phi_\infty}(w^* - \lim \pi^{**}(\delta_k \otimes \delta_k))\\ &= \tilde{\phi_\infty}(w^* - \lim \delta_k) = \lim \phi_\infty(\delta_k) = 1.\end{aligned}$$

For $\epsilon > 0$ and each $a = \sum_{i=1}^\infty \alpha_i\delta_i$ in $\ell^1(\mathbb{N}_\vee)$, pick $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0}^\infty |\alpha_i| < \epsilon$. Then for $k \geq n_0$, we have

$$||(\sum_{i=k}^\infty \alpha_i\delta_i) \otimes \delta_k - \delta_k \otimes (\sum_{i=k}^\infty \alpha_i\delta_i)|| \leq 2\sum_{i=k}^\infty |\alpha_i| < 2\epsilon.$$

Then clearly

$$(\sum_{i=k}^\infty \alpha_i\delta_i) \otimes \delta_k - \delta_k \otimes (\sum_{i=k}^\infty \alpha_i\delta_i) \xrightarrow{w^*} 0. \tag{5.1}$$

Now consider

$$\begin{aligned}a \cdot m - m \cdot a &= w^* - \lim(a\delta_k \otimes \delta_k - \delta_k \otimes \delta_k a)\\ &= w^* - \lim((\sum_{i=1}^\infty \alpha_i\delta_i\delta_k) \otimes \delta_k - \delta_k \otimes (\delta_k \sum_{i=1}^\infty \alpha_i\delta_i))\\ &= w^* - \lim((\sum_{i=1}^k \alpha_i\delta_i\delta_k) \otimes \delta_k + (\sum_{i=k+1}^\infty \alpha_i\delta_i\delta_k) \otimes \delta_k\\ &\quad - \delta_k \otimes (\delta_k \sum_{i=1}^k \alpha_i\delta_i) - \delta_k \otimes (\delta_k \sum_{i=k+1}^\infty \alpha_i\delta_i))\\ &= w^* - \lim(\phi_k(a)\delta_k \otimes \delta_k + \sum_{i=k+1}^\infty \alpha_i\delta_i \otimes \delta_k\\ &\quad - \delta_k \otimes \delta_k\phi_k(a) - \delta_k \otimes \sum_{i=k+1}^\infty \alpha_i\delta_i).\\ &= w^* - \lim(\sum_{i=k+1}^\infty \alpha_i\delta_i) \otimes \delta_k - \delta_k \otimes (\sum_{i=k+1}^\infty \alpha_i\delta_i).\end{aligned} \tag{5.2}$$

Then by (5.1) and (5.2), we have $a \cdot m = m \cdot a$. Therefore $\ell^1(\mathbb{N}_\vee)$ is ϕ_n-Johnson amenable for every $n \in \mathbb{N} \cup \{\infty\}$. Hence by Lemma 3.1 $\ell^1(\mathbb{N}_\vee)$ is ϕ_n-biflat for every $n \in \mathbb{N} \cup \{\infty\}$.

Moreover, let $\ell^1(\mathbb{N}_\vee)$ be biflat. Then since $\ell^1(\mathbb{N}_\vee)$ is unital with unit δ_1, so by [17, Exercise 4-3-15] $\ell^1(\mathbb{N}_\vee)$ is amenable. Hence by [4, Theorem 2] $\mathbb{N}_\vee$ has a finite number of idempotents which is impossible. Hence $\ell^1(\mathbb{N}_\vee)$ is not a biflat Banach algebra.

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