On locally convex weakly Lindelöf Σ-spaces

J. C. Ferrando J. Kąkol*

Abstract

A family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets covering a set E is called a resolution for E if $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$. A locally convex space (lcs) E is said to belong to class $\mathfrak{G}$ if there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for $(E', \sigma(E', E))$ such that each sequence in any A_α is equicontinuous. The class $\mathfrak{G}$ contains 'almost all' useful locally convex spaces (including (LF)-spaces and (DF)-spaces). We show that (i) every semi-reflexive lcs E in class $\mathfrak{G}$ is a Lindelöf Σ-space in the weak topology (this extends a corresponding result of Preiss-Talagrand for WCG Banach spaces) and the weak* dual of E is both K-analytic and has countable tightness, (ii) a barrelled space E has a weakly compact resolution if and only if E is weakly K-analytic, and (iii) if E is barrelled or bornological then E' has a weak* compact resolution if and only if it is weak* K-analytic. As an additional consequence we provide another approach to show that the weak* dual of a quasi-barrelled space in class $\mathfrak{G}$ is K-analytic. These results supplement earlier work of Talagrand, Preiss, Cascales, Ferrando, Kąkol, López Pellicer and Saxon.

1 Introduction

As mentioned in the abstract, a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets covering a set E is called a *resolution* for E if $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. This paper deals with the two following general problems: (i) for any locally convex space E, characterize in terms of E the existence of a non-empty set Σ in $\mathbb{N}^{\mathbb{N}}$ and an upper

*Supported for the second author by National Center of Science, Poland, Grant No. N N201 605340

Received by the editors in May 2012 - In revised form in November 2012.

Communicated by F. Bastin.

2010 *Mathematics Subject Classification* : 54D20, 46A03, 46A08.

Key words and phrases : K-analytic space, Lindelöf Σ-space, class $\mathfrak{G}$, compact resolution, barrelled space, bornological space, semi-reflexive space, (LF)-space.

semi-continuous compact-valued map T from Σ into $(E, \sigma(E, E'))$ covering E, i.e. such that $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = E$, and (ii) provide sufficient conditions on E to ensure that the existence of a compact resolution for $(E, \sigma(E, E'))$ (for $(E', \sigma(E', E))$) guarantees that $(E, \sigma(E, E'))$ (resp. $(E', \sigma(E', E))$) is K-analytic. There are several results motivating these problems. For example, concerning to the first one, in [3, Corollary 1.6] Cascales proved the following

Theorem 1 (Cascales). *For a semi-reflexive lcs E the following conditions are equivalent.*

1. *E has a bounded resolution, i.e. a resolution consisting of bounded sets.*
2. *E endowed with the weak topology $\sigma(E, E')$ is a K-analytic space.*
3. *$(E, \sigma(E, E'))$ is a quasi-Suslin space.*

On the other hand, relative to the second problem, in [11, Theorem 1] it is shown that

Theorem 2 (Ferrando-Kąkol-López Pellicer-Saxon). *Let E be an lcs. If $(E', \sigma(E', E))$ is quasi-Suslin, the following are equivalent.*

1. *The weak space $(E, \sigma(E, E'))$ is countably tight.*
2. *The weak* dual $(E', \sigma(E', E))$ is realcompact.*
3. *The weak* dual $(E', \sigma(E', E))$ is K-analytic.*
4. *The weak dual $(E', \sigma(E', E))$ is Lindelöf.*
5. *The Mackey space $(E, \mu(E, E'))$ is barrelled.*

A simple example of an lcs with a bounded resolution is provided by any lcs E admitting a stronger metrizable locally convex topology τ. Indeed, if $\{U_n : n \in \mathbb{N}\}$ is a decreasing base of τ-neighborhoods of the origin, for any $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ define $A_\alpha := \bigcap_{k=1}^{\infty} n_k U_k$.

In this paper we provide partial solutions concerning the above problems, see Theorem 11 and the consequences mentioned in the abstract.

Let Σ be a subset of $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ is equipped with the discrete topology, and let $\mathcal{A} := \{A_\alpha : \alpha \in \Sigma\}$ be a family of subsets of a set X. For each $\alpha \in \Sigma$ and $n \in \mathbb{N}$ define

$$A(\alpha|n) := \bigcup \{A_\beta : \beta \in \Sigma, \beta(i) = \alpha(i), 1 \leq i \leq n\}.$$

Clearly $A_\alpha \subseteq A(\alpha|n)$ for each $n \in \mathbb{N}$ and $A(\alpha|n+1) \subseteq A(\alpha|n)$ for all $(\alpha, n) \in \Sigma \times \mathbb{N}$. Since $A(\alpha|n) = A(\beta|n)$ whenever $\alpha(i) = \beta(i)$ for $1 \leq i \leq n$, the family $\mathcal{E} := \{A(\alpha|n) : \alpha \in \Sigma, n \in \mathbb{N}\}$ (called the *envelope* of $\mathcal{A}$) is countable. It is easy to see that if $\Sigma = \mathbb{N}^{\mathbb{N}}$ and $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution, i.e. a resolution consisting of bounded sets in a locally convex space E, then for each $\alpha \in \mathbb{N}$ and each neighborhood of zero U in E there exists $n \in \mathbb{N}$ such that $A(\alpha|n) \subseteq nU$. Indeed, otherwise there exist a neighborhood of zero V in E, β in $\mathbb{N}^{\mathbb{N}}$, and a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in A(\beta|n) \setminus nV$ for all $n \in \mathbb{N}$. Choose a sequence $\{\beta_n\}_{n=1}^{\infty}$ in $\mathbb{N}^{\mathbb{N}}$ with $\beta_n(i) = \beta(i)$ for $1 \leq i \leq n$ such that $x_n \in A_{\beta_n}$ for each $n \in \mathbb{N}$.

Then there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\beta_n \leq \gamma$ for each $n \in \mathbb{N}$. Hence $x_n \in A_{\beta_n} \subseteq A_\gamma$ for all $n \in \mathbb{N}$. Since $A_\gamma \subseteq mV$ for some $m \in \mathbb{N}$, we reach a contradiction. This motivates the following useful concept. Following [9] we will say that the envelope $\mathcal{E}$ of a family $\mathcal{A} = \{A_\alpha : \alpha \in \Sigma\}$ of subsets of an lcs E covering E (a Σ-*covering* henceforth) is *limited* if for each $\alpha \in \Sigma$ and a neighborhood of zero U in E there exists $n \in \mathbb{N}$ such that $A(\alpha|n) \subseteq nU$.

Recall that a completely regular Hausdorff topological space X is called *Lindelöf* Σ (or K-countably determined) if there is an upper semi-continuous compact-valued map T from a non-empty subset Σ of the product space $\mathbb{N}^{\mathbb{N}}$ into X (actually into the set $\mathcal{P}(X)$ of all subsets of X) covering X, i.e. such that $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = X$, see [1]. If the same holds for $\Sigma = \mathbb{N}^{\mathbb{N}}$, then X is called K-*analytic*. On the other hand, X is called *quasi-Suslin* if there exists a set-valued map T (called a quasi-Suslin map) from $\mathbb{N}^{\mathbb{N}}$ into X covering X which is quasi-Suslin, i.e. such that if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ then $\{x_n\}_{n=1}^{\infty}$ has a cluster point in $T(\alpha)$, see [23]. Alternatively, a completely regular space X is Lindelöf Σ if and only if there is a compact-valued mapping T from a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ into X such that $\{T(\alpha) : \alpha \in \Sigma\}$ covers X and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ the sequence $\{x_n\}_{n=1}^{\infty}$ has a cluster point *contained in* $T(\alpha)$. Note that K-analytic $\Leftrightarrow$ (Lindelöf $\wedge$ quasi-Suslin), and K-analytic $\Rightarrow$ Lindelöf Σ.

In what follows all vector spaces are supposed to be real. For the benefit of the reader we explicitly quote a number of results that will be used in what follows.

Theorem 3. ([1, Theorem IV.9.4]) *If the realcompactification* vX *of a completely regular Hausdorff space* X *is a Lindelöf* Σ*-space, then there exists a Lindelöf* Σ*-space* Z *such that* $C_p(X) \subseteq Z \subseteq \mathbb{R}^X$.

Although the next theorem was formulated for the original topology of E, the same proof yields the following

Theorem 4. ([9, Lemma 2]) *If an lcs* E *admits a* Σ*-covering* $\{A_\alpha : \alpha \in \Sigma\}$, $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$, *with limited envelope in the weak topology of* E*, then there exists a Lindelöf* Σ*-space* Z *such that* $(E', \sigma(E', E)) \subseteq Z \subseteq \mathbb{R}^E$*, where* $\mathbb{R}^E$ *is endowed with the product topology.*

We shall also need the following facts about Lindelöf Σ-spaces.

Theorem 5. ([9, Proposition 10]) *Let* E *be a linear subspace of an lcs* F *If there exists a Lindelöf* Σ*-space* X *such that* $E \subseteq X \subseteq F$*, then* E *admits a* Σ*-covering with limited envelope.*

Theorem 6. ([9, Theorem 3]) vX *is a Lindelöf* Σ*-space if and only if* $C_p(X)$ *admits a* Σ*-covering with limited envelope.*

Theorem 7. ([15, Proposition 9.15]) *If* X *is quasi-Suslin, the space* vX *is* K*-analytic.*

Let us recall that a topological space X is called *web-compact* if there is a map T from a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ into X such that $\overline{\bigcup \{T(\alpha) : \alpha \in \Sigma\}} = X$ and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X (this definition is equivalent to that given in [18, Definition]). Every Lindelöf Σ-space is web-compact and Lindelöf, but $\mathbb{R}^{\mathbb{R}}$ is a simple example of a web-compact space which is not Lindelöf. On the other hand a topological space X is *angelic* if relatively countably compact sets in X are relatively compact and for every relatively compact subset A of X each point of $\overline{A}$ is the limit of a sequence of A, [12]. The following two additional results will be used later.

Theorem 8. ([18, Theorem 3]) *If X is a web-compact space, then $C_p(X)$ is angelic.*

Theorem 9. ([3, Corollary 1.1]) *For an angelic space X the following are equivalent*:

1. *X has a compact resolution.*

2. *X is quasi-Suslin.*

3. *X is K-analytic.*

Let us recall that a locally convex space E belongs to *class* $\mathfrak{G}$ if there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in the weak* dual $(E', \sigma(E', E))$ of E such that each sequence in any A_α is equicontinuous, see [6]. Therefore every set A_α is relatively $\sigma(E', E)$-countably compact. The class $\mathfrak{G}$ is indeed large and contains 'almost all' important locally convex spaces (including (LF)-spaces and (DF)-spaces). Furthermore $\mathfrak{G}$ is stable by taking subspaces, Hausdorff quotients and countable direct sums and products.

Theorem 10. ([6, Theorem 13]) *If E is an lcs of the class $\mathfrak{G}$ that is weakly countably determined* (*i.e. a weakly Lindelöf Σ-space*)*, then the density character of E is equal to the density character of $(E', \sigma(E', E))$.*

2 Results

Before we state our first result let us recall that an lcs E is *barrelled* (*quasi-barrelled*) if every weak* bounded (resp. strongly bounded) set in E' is equicontinuous, hence relatively weak* compact. Let us point out that every barrelled space is quasi-barrelled; metrizable and bornological spaces are also examples of quasi-barrelled spaces.

Theorem 11. *Let E be a locally convex space such that every weak* bounded set in E' is relatively weak* compact. The space $(E, \sigma(E, E'))$ has a Σ-covering with limited envelope if and only if $(E', \sigma(E', E))$ is a Lindelöf Σ-space.*

Proof. If $(E, \sigma(E, E'))$ has a Σ-covering with limited envelope, by Theorem 4 there exists a Lindelöf Σ-space Z such that $(E', \sigma(E', E)) \subseteq Z \subseteq \mathbb{R}^E$. Hence there is $\Delta \subseteq \mathbb{N}^{\mathbb{N}}$ and a compact-valued upper semi-continuous map $S : \Delta \to Z$ such that $\bigcup \{S(\alpha) : \alpha \in \Delta\} = Z$. Given $\alpha \in \Delta$, the compactness of $S(\alpha)$ ensures that $S(\alpha) \cap E'$ is a closed bounded set in $(E', \sigma(E', E))$, so according to the hypotheses $S(\alpha) \cap E'$ is weak* compact.

Set $\Sigma = \{\alpha \in \Delta : S(\alpha) \cap E' \neq \emptyset\}$ and define $T : \Sigma \to (E', \sigma(E', E))$ by

$$T(\alpha) = S(\alpha) \cap E'.$$

Clearly T is compact-valued and $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = E'$. Let us show that T is upper semi-continuous.

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in Σ such that $\alpha_n \to \alpha$ in Σ and let $u_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$. Since $u_n \in S(\alpha_n)$ for every $n \in \mathbb{N}$ and S is upper semi-continuous there is a cluster point u of $\{u_n\}_{n=1}^{\infty}$ in Z such that $u \in S(\alpha)$. We claim that $\{u_n : n \in \mathbb{N}\}$ is a bounded set in $(E', \sigma(E', E))$. Otherwise there is an absolutely convex neighborhood of the origin U in $(E', \sigma(E', E))$ and a strictly increasing sequence $\{n_k\}$ of positive integers such that $u_{n_k} \notin kU$ for all $k \in \mathbb{N}$. Let V be a neighborhood of the origin in $\mathbb{R}^E$ such that $V \cap E' = U$. Since $\alpha_{n_k} \to \alpha$, then $\{u_{n_k}\}$ has a cluster point $v \in S(\alpha)$. Let $m \in \mathbb{N}$ be such that $S(\alpha) \subseteq mV$. Since $mV \cap Z$ is a neighborhood of v in Z, for each $k \in \mathbb{N}$ there is $k' \in \mathbb{N}$ with $k' \geq k$ such that $u_{n_{k'}} \in mV$. Particularly $u_{n_{m'}} \in mV \cap E' = mU$. But since $m' \geq m$ then $u_{n_{m'}} \in mU \subseteq m'U$, a contradiction.

The weak* boundedness of $\{u_n : n \in \mathbb{N}\}$ in E' implies that $K = \overline{\{u_n : n \in \mathbb{N}\}}^{\text{weak*}}$ is a weak* compact set in E', hence a compact set in Z. Therefore K contains all cluster points of $\{u_n\}$ in Z. This is tantamount to saying that $u \in K \subseteq E'$ and, consequently, that $u \in T(\alpha)$. So T is upper semi-continuous, which proves that $(E', \sigma(E', E))$ is a Lindelöf Σ-space, as stated.

For the converse set $X := (E', \sigma(E', E))$ and apply Theorem 6 to show that $C_p(X)$ has a Σ-covering $\{A_\alpha : \alpha \in \Sigma\}$ with limited envelope. Then $\{A_\alpha \cap E : \alpha \in \Sigma\}$ is a Σ-covering of $(E, \sigma(E, E'))$ with limited envelope. ■

Example 12. *Theorem 11 fails for quasibarrelled spaces E.*

Proof. Let $X := [0, \omega_1)$. Then X is sequentially compact non-compact and under (CH) it even has a compact resolution, see [21, Theorem 3.6]. Since vX is K-analytic, by [8, Theorem 3] the space $E := C_p(X)$ admits a Σ-covering with limited envelop. According to [8, Corollary 2] the weak* dual $L_p(X)$ of $C_p(X)$ is quasi-Suslin but not K-analytic. Hence $L_p(X)$ cannot be a Lindelöf Σ-space. On the other hand $C_p(X)$ is always quasibarrelled, see [14, Corollary 11.7.3]. ■

Corollary 13. *Let E be a barrelled space. The following conditions hold*

1. *If $(E, \sigma(E, E'))$ is a Lindelöf Σ-space, then $(E', \sigma(E', E))$ is a Lindelöf Σ-space.*
2. *$(E', \sigma(E', E))$ is quasi-Suslin if and only if $(E', \sigma(E', E))$ is K-analytic.*

Proof. (1) If $(E, \sigma(E, E'))$ is a Lindelöf Σ-space, by Theorem 5 it has a Σ-covering with limited envelope. Since very weak* bounded set in E' is relatively weak* compact, Theorem 11 ensures that $(E', \sigma(E', E))$ is a Lindelöf Σ-space. (2) It suffices to show that E has a Σ-covering with limited envelope, since in this case the statement is consequence of Theorem 11. In fact, if $Y = (E', \sigma(E', E))$ is quasi-Suslin then, according to Theorem 7, the space vY is K-analytic. Thus by Theorem 3 there exists a Lindelöf Σ-space Z such that $C_p(Y) \subseteq Z \subseteq \mathbb{R}^Y$. Since $(E, \sigma(E, E')) \subseteq C_p(Y)$, applying Theorem 5 with $X = Z$ and $F = \mathbb{R}^Y$, we get that E has a Σ-covering with limited envelope. ■

Clearly the converse in Corollary 13 (1) fails in general; any infinite dimensional Banach space which is not weakly Lindelöf provides such an example.

Condition (2) of Corollary 13 does not hold if E is only a quasi-barrelled space as the example 12 shows. The following result provides a variant of Theorem 1 for weakly Lindelöf Σ-spaces.

Proposition 14. *Let E be a semi-reflexive locally convex space. Then $(E, \sigma(E, E'))$ is a Lindelöf Σ-space if and only if $(E, \sigma(E, E'))$ admits a Σ-covering with limited envelope.*

Proof. If $(E, \sigma(E, E'))$ admits a Σ-covering with limited envelope, by Theorem 4 there exists a Lindelöf Σ-space Z such that $(E', \sigma(E', E)) \subseteq Z \subseteq \mathbb{R}^E$. So, according to Theorem 5, the space $(E', \sigma(E', E))$ has a Σ-covering with limited envelope. Since $E = (E', \sigma(E', E))'$ is semi-reflexive, Theorem 11 ensures that $(E, \sigma(E, E'))$ is a Lindelöf Σ-space. For the converse apply Theorem 5 with $E = X = F = (E, \sigma(E, E'))$. ∎

According to Talagrand [20] every Weakly Compactly Generated (WCG) Banach space is weakly Lindelöf. This fails however for (WCG) lcs in general, see [2]. Our next result provide a large class of weakly Lindelöf locally convex spaces.

Proposition 15. *Let E be an lcs in class $\mathfrak{G}$. If E is semi-reflexive then the following conditions hold.*

1. *E is a Lindelöf Σ-space in the weak topology $\sigma(E, E')$ of E.*
2. *The weak* dual of E is a K-analytic space with countable tightness.*
3. $\operatorname{dens}(E, \sigma(E, E')) = \operatorname{dens}(E', \sigma(E', E))$, *where* $\operatorname{dens}(\cdot)$ *means the density.*

Proof. If E belongs to class $\mathfrak{G}$ its weak* dual $Y = (E', \sigma(E', E))$ is quasi-Suslin, see [11, Theorem 4]. Hence, according to Theorem 7, the space vY is K-analytic. By Theorem 3 there is a Lindelöf Σ-space Z such that $C_p(Y) \subseteq Z \subseteq \mathbb{R}^Y$. Since $E \subseteq C_p(Y)$, applying again Theorem 5 with $X = Z$ and $F = \mathbb{R}^Y$, we get that E has a Σ-covering with limited envelope.

(1) Since E is semi-reflexive and admits a Σ-covering with limited envelope, part (1) follows from Proposition 14.

(2) By the previous condition $X := (E, \sigma(E, E'))$ is a Lindelöf Σ-space, so $C_p(X)$ is angelic by virtue of Theorem 8. Since $(E', \sigma(E', E))$ is linearly embedded in $C_p(X)$, it follows that $(E', \sigma(E', E))$ is angelic too. On the other hand, due to the fact that E belongs to class $\mathfrak{G}$ we know that its weak* dual $(E', \sigma(E', E))$ is quasi-Suslin. Therefore $(E', \sigma(E', E))$ being quasi-Suslin and angelic, it is K-analytic by virtue of Theorem 9. Concerning the second statement, according to Condition 1 any finite product $(E, \sigma(E, E'))^n$ is a Lindelöf space. So, applying [1, Theorem II.1.1], which ensures that if X^n, for X completely regular, is a Lindelöf space for each n then $C_p(X)$ has countable tightness, we get that $C_p(E, \sigma(E, E'))$ has countable tightness. Since $(E', \sigma(E', E))$ is embedded into $C_p(E, \sigma(E, E'))$, the conclusion follows.

(3) According to Condition 1 the space E is weakly Lindelöf Σ, so we may apply Theorem 10. The proof is complete. ∎

The proof of Proposition 15 uses the fact that E in class $\mathfrak{G}$ admits a Σ-covering with limited envelope. The converse statement is not true.

Example 16. *A locally convex space admitting a Σ-covering with limited envelope which does not belong to the class $\mathfrak{G}$.* If $X = \mathbb{R}^{\mathbb{N}}$ then $C_p(X)$ has a Σ-covering with limited envelope (see [9, Example 17]) but $C_p(X)$ is not in class $\mathfrak{G}$ since X is uncountable [5].

Remark 17. *Proposition 14 easily implies Theorem 1.* Let us see the only nontrivial implication 1 $\Rightarrow$ 2. In fact, if E has a bounded resolution $\{A_\alpha : \alpha \in N^{\mathbb{N}}\}$ then clearly $(E, \sigma(E, E'))$ admits a Σ-covering with limited envelope. So, if E is in addition semi-reflexive, Proposition 14 guarantees that $(E, \sigma(E, E'))$ is a Lindelöf Σ-space. The semi-reflexivity of E also guarantees that $\{\overline{A_\alpha}^{\sigma(E,E')} : \alpha \in N^{\mathbb{N}}\}$ is a weakly compact resolution (i.e. consisting of weakly compact sets) for E, so that $(E, \sigma(E, E'))$ is quasi-Suslin [3, Proposition 1]. Since every quasi-Suslin Lindelöf space is K-analytic, we are done.

Theorem 11 easily applies to provide another proof of the following results from [4, Theorem 4.6].

Proposition 18. *Let E be either a quasi-barrelled (DF)-space or an (LF)-space. Then the space $(E', \sigma(E', E))$ is K-analytic. In general, this holds for every quasi-barrelled lcs in class $\mathfrak{G}$.*

Proof. Let E be a quasi-barrelled (DF)-space and let $\{B_n : n \in \mathbb{N}\}$ be a fundamental sequence of absolutely convex closed bounded sets in $E = \bigcup_{n=1}^{\infty} B_n$. Let F be the completion of E and denote by K_n the closure of B_n in F for all $n \in \mathbb{N}$. Since E is quasi-barrelled, then by [19, Proposition 8.2.27] we have $F = \bigcup_{n=1}^{\infty} K_n$ and $\{K_n : n \in \mathbb{N}\}$ is a fundamental sequence of bounded sets in the barrelled (DF)-space F (recall that the completion of a quasi-barrelled space is barrelled). By Theorem 11 the space $(E', \sigma(E', F))$ is a Lindelöf Σ-space, hence Lindelöf. On the other hand, $(E', \sigma(E', E))$ is quasi-Suslin by [11]; hence $(E', \sigma(E', F))$ is K-analytic. Since $\sigma(E', E) \leq \sigma(E', F)$, the space $(E', \sigma(E', E))$ is K-analytic.

Let E be an (LF)-space, i.e. the inductive limit of a sequence $(E_n, \xi_n)_n$ of metrizable and complete lcs such that $\xi_{n+1}|E_n \leq \xi_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $(U_k^n)_k$ be a countable basis of absolutely convex neighborhoods of zero in E_n such that $U_k^n \subset U_k^{n+1}$ for all $k, n \in \mathbb{N}$, see [7] or [24]. For each $\alpha = (j_k) \in \mathbb{N}^{\mathbb{N}}$ set $A_\alpha^n := \bigcap_{k=1}^{\infty} j_k U_k^n$. Then $\{A_\alpha^n : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution in each (E_n, ξ_n). For $\alpha = (p_1, j_1, j_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ set $B_\alpha := A_{(j_1, j_2, \dots)}^{p_1}$. It is easy to see that $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bounded resolution in E. Since F is barrelled and $(E', \sigma(E', F))$ is quasi-Suslin, see [11, Theorem 4], we apply the same argument as above to show that $(E', \sigma(E', F))$ is K-analytic, which yields the conclusion.

Finally, assume that E is a quasi-barrelled lcs in class $\mathfrak{G}$. Since the completion F of E also belongs to class $\mathfrak{G}$, the space F is a barrelled space in class $\mathfrak{G}$. Then $Y = (E', \sigma(E', F))$ is quasi-Suslin, see again [11, Theorem 4]. Hence, according to Corollary 13 the space Y is K-analytic, consequently $(E', \sigma(E', E))$ is K-analytic. ∎

Let us recall that an lcs E is called *ℓ^∞-barrelled* if every weak* bounded sequence in E' is equicontinuous. Each barrelled space is ℓ^∞-barrelled and each metrizable ℓ^∞-barrelled space, as well as each separable ℓ^∞-barrelled space, is barrelled. There are locally convex spaces E equipped with the Mackey topology $\mu(E, E')$ which are ℓ^∞-barrelled but not barrelled [22].

Proposition 19. *Let E be an ℓ^∞-barrelled space. If E has a weak Σ-covering with limited envelope, then E is weakly angelic.*

Proof. If $(E, \sigma(E, E'))$ has a Σ-covering with limited envelope, a similar argument to that of the proof of Theorem 11 provides a Lindelöf Σ-space Z with $(E', \sigma(E', E)) \subset Z \subset \mathbb{R}^E$ and a compact-valued upper semi-continuous map $S : \Delta \to Z$ with $\Delta \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\bigcup \{S(\alpha) : \alpha \in \Delta\} = Z$. Setting again $\Sigma = \{\alpha \in \Delta : S(\alpha) \cap E' \neq \emptyset\}$ and $T(\alpha) = S(\alpha) \cap E'$ then clearly $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = E'$. We claim that $(E', \sigma(E', E))$ is a web-compact space. Indeed, if $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in Σ such that $\alpha_n \to \alpha$ in Σ and $u_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, then $\{u_n : n \in \mathbb{N}\}$ is a bounded set in $(E', \sigma(E', E))$, so that $K = \overline{\{u_n : n \in \mathbb{N}\}}^{\text{weak*}}$ is a weak* compact set in E'. Consequently the sequence $\{u_n\}_{n=1}^{\infty}$ has a cluster point in $(E', \sigma(E', E))$. Thus $Y := (E', \sigma(E', E))$ is web-compact and hence $C_p(Y)$ is angelic by Theorem 8, which implies that $(E, \sigma(E, E'))$ is also angelic. ∎

It is known that every K-analytic space admits a compact resolution, see [3]. The converse implication fails in general, although some positive results hold, see [15] as a source of several information. The following result provides another fact of this type.

Corollary 20. *If E is an ℓ^∞-barrelled lcs, particularly a barrelled space, the following conditions are equivalent.*

1. *E has a weakly compact resolution.*
2. *E is weakly quasi-Suslin.*
3. *E is weakly K-analytic.*

Proof. (1) ⇒ (2) is well known. Assume that E is weakly quasi-Suslin. Hence it admits a bounded resolution, hence $(E, \sigma(E, E'))$ admits a Σ-covering with limited envelope. By Proposition 19 the space $(E, \sigma(E, E'))$ angelic. $(E, \sigma(E, E'))$ being both quasi-Suslin and angelic, it is K-analytic by virtue of Theorem 9. This shows (2) ⇒ (3). Finally, if E is weakly K-analytic, then E has a weakly compact resolution. ∎

Remark 21. *Last corollary applies to provide another proof of Khurana's theorem [16], stating that every (WCG) Fréchet space is weakly K-analytic.* It suffices to show that E admits a weakly compact resolution. If $\{C_n : n \in \mathbb{N}\}$ is a sequence of weakly compact sets with $\overline{\text{span}\left(\bigcup_{n=1}^{\infty} C_n\right)} = E$, define $K_n = \overline{\text{abx}\left(\bigcup_{i=1}^{n} C_i\right)}$ for $n \in N$ and note by Krein's theorem that each K_n is a weakly compact subset of E. If $\{U_n : n \in \mathbb{N}\}$ is a decreasing base of absolutely convex neighborhoods of the origin in E, setting $A_\alpha := \bigcap_{i=1}^{\infty} \left(\alpha(i) K_{\alpha(i)} + U_i^{00}\right)$ for $\alpha \in \mathbb{N}^{\mathbb{N}}$, where U^{00} stands for the bipolar on U in E'', then every A_α is bounded in E and weak* closed in E'', hence weak* compact in E''. Moreover $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and, due to the fact that E is $\beta(E'', E')$-closed in E'' and $\{U_i^{00} : i \in \mathbb{N}\}$ is a base of neighborhoods of the origin in $(E'', \beta(E'', E'))$, we can see that $A_\alpha \subseteq E$ for all

$\alpha \in N^{\mathbb{N}}$. On the other hand, since $\bigcup_{n=1}^{\infty} nK_n$ is a dense linear subspace of E, for $x \in E$ and $i \in N$ there is $y \in \bigcup_{n=1}^{\infty} nK_n$ with $x - y \in U_i$. So if $y \in n_i K_{n_i}$ then $x \in n_i K_{n_i} + U_i$. Thus $x \in \bigcap_{i=1}^{\infty} (n_i K_{n_i} + U_i) \subseteq A_\gamma$ with $\gamma = (n_1, n_2, \ldots)$, which shows that $\bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = E$.

A different approach to those used so far allows us to supplement Corollary 13 for bornological spaces. Recall that an lcs E is *bornological* if every bounded linear mapping from E into any lcs F is continuous. Particularly every metrizable lcs is bornological.

Proposition 22. *If E is a bornological lcs, then $(E', \sigma(E', E))$ is quasi-Suslin if and only if $(E', \sigma(E', E))$ is K-analytic.*

Proof. Assume that $Y := (E', \sigma(E', E))$ is a quasi-Suslin space. Then by [8, Corollary 4] each compact set in $C_p(Y)$ is Talagrand compact. Since $(E, \sigma(E, E'))$ is linearly embedded into $C_p(Y)$, it follows that each weakly compact set of E is Talagrand compact as well. If $\mathcal{K}(E)$ denotes the family of all weakly compact subsets of E then the space $C_p(K)$ is K-analytic for each $K \in \mathcal{K}(E)$ equipped with the relative weak topology of E. Hence each $C_p(K)$ is Lindelöf and consequently $\prod_{K \in \mathcal{K}(E)} C_p(K)$ is realcompact. Note that $(E', \sigma(E', E))$ is realcompact. Indeed, set $X := (E, \sigma(E, E'))$ and consider the map $f \mapsto \{f|_K : K \in \mathcal{K}(E)\}$ from $C_p(X)$ into $\prod_{K \in \mathcal{K}(E)} C_p(K)$. Observe that this is an isomorphism (into) which embeds Y into $\prod_{K \in \mathcal{K}(E)} C_p(K)$. Indeed, if $\{y_d : d \in D\}$ is a net in Y (viewed as a subspace of $\prod_{K \in \mathcal{K}(E)} C_p(K)$), that converges to some $\{h_K : K \in \mathcal{K}(E)\} \in \prod_{K \in \mathcal{K}(E)} C_p(K)$, define $u(x) = h_K(x)$ whenever $x \in K$. Given $P, Q \in \mathcal{K}(E)$ such that $P \cap Q \neq \emptyset$, it follows from the nature of the embedding of Y into $\prod_{K \in \mathcal{K}(E)} C_p(K)$ that $y_d|_{P \cap Q}(x) \to h_{P \cap Q}(x)$, $y_d|_P(x) \to h_P(x)$ and $y_d|_Q(x) \to h_Q(x)$ for each $x \in P \cap Q$. This ensures that $h_P(x) = h_Q(x)$ for each $x \in P \cap Q$, which means that u is well-defined. Furthermore, the fact that $y_d|_K(x) \to h_K(x) = u(x)$ if $x \in K$ implies that $y_d \to u$ pointwise on E. Clearly u is a linear functional on E. Since $u|_K = h_K \in C(K)$, we can see that u is a sequentially continuous linear functional on E. Given that E is bornological, we get that $u \in Y$. This shows that Y is (homeomorphic to) a closed subspace of $\prod_{K \in \mathcal{K}(E)} C_p(K)$. Hence Y is realcompact. Since Y is quasi-Suslin, it is K-analytic. ∎

Since every (LM)-space is bornological and its weak* dual is quasi-Suslin, Proposition 22 nicely applies to complement Proposition 18.

Acknowledgement. We are grateful to the referee for helpful suggestions that have improved the readability of the paper.

References

[1] A. V. Arkhangel'skiĭ, *Topological function spaces*, Math. and its Applications **78**. Kluwer Academic Publishers. Dordrecht Boston London, 1992.

[2] M. A. Canela, *K-analytic locally convex spaces*, Portugalia Math. **41** (1982), 105-117.

[3] B. Cascales, *On K-analytic locally convex spaces*, Arch. Math. **49** (1987), 232-244.

[4] B. Cascales, J. Kąkol and S. A. Saxon, *Weight of precompact subsets and tightness*, J. Math. Anal. Appl. **269** (2002) 500-518.

[5] B. Cascales, J. Kąkol and S. A. Saxon, *Metrizability vs. Fréchet-Urysohn property*, Proc. Amer. Math. Soc. **131** (2003) 3623-3631.

[6] B. Cascales and J. Orihuela, *On Compactness in Locally Convex Spaces*, Math. Z. **195** (1987), 365-381.

[7] N. De Grande-De Kimpe, J. Kąkol and C. Pérez-García, *Metrizability of compacts sets in non-Archimedian Hausdorff* (LM)*-spaces*, Contemporary Math. **319** (2003), 99–107.

[8] J. C. Ferrando, *Two new properties of the space* $C_p(X)$, Topology Appl. **154** (2007), 1799-1803.

[9] J. C. Ferrando, *Some characterizations for* vX *to be Lindelöf* Σ *or* K-analytic *in terms of* $C_p(X)$, Topology Appl. **156** (2009), 823-830.

[10] J. C. Ferrando and J. Kąkol, *A note on spaces* $C_p(X)$ *K-analytic-framed in* $\mathbb{R}^X$, Bull. Austral. Math. Soc. **78** (2008), 141-146.

[11] J. C. Ferrando, J. Kąkol, M. López Pellicer and S. A. Saxon, *Quasi-Suslin weak duals*, J. Math. Anal. Appl. **339** (2008), 1253-1263.

[12] K. Floret, *Weakly compact sets*, Lecture Notes in Math. **801**, Springer, Berlin Heidelberg New York, 1980.

[13] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, Berlin, Heidelberg, 1976.

[14] H. Jarchow, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981.

[15] J. Kąkol, W. Kubiś and M. López Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Springer, Developments in Math. **24**, New York Dordrecht Heidelberg London, 2011.

[16] S. S. Khurana, *Weakly compactly generated Fréchet spaces*, Internat. J. Math. Sci. **2** (1979), 721–726.

[17] O. G. Okunev, *On Lindelöf* Σ*-spaces of continuous functions in the pointwise topology*, Topology Appl. **49** (1993), 149-166.

[18] J. Orihuela, *Pointwise compactness in spaces of continuous functions*, J. London Math. Soc. **36** (1987), 143-152.

[19] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Mathematics Studies **131**, North-Holland, Amsterdam, 1987.

[20] M. Talagrand, *Espaces de Banach faiblement K-analytiques*, Ann. of Math. **110** (1979), 407–438.

[21] V. V. Tkachuk, *A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic*, Acta Math. Hungarica **107 (4)** (2005), 253-265.

[22] M. Valdivia, *A note on locally convex spaces*, Math. Ann. **201** (1973), 143-148.

[23] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland, Amsterdam (1982).

[24] D. Vogt, *Regularity properties of (LF)-spaces*, In 'Progress in Functional Analysis', 57–84, North Holland Math. Studies **170**, Amsterdam, 1992.

Centro de Investigación Operativa,
Universidad M. Hernández
E-03202 Elche (Alicante), Spain
email:jc.ferrando@umh.es

Faculty of Mathematics and Informatics,
A. Mickiewicz University, 61-614 Poznań, Poland
email:kakol@amu.edu.pl