Tetrads of lines spanning $\mathrm{PG}(7,2)$

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Abstract

Our starting point is a very simple one, namely that of a set $\mathcal{L}_4$ of four mutually skew lines in $\mathrm{PG}(7,2)$. Under the natural action of the stabilizer group $\mathcal{G}(\mathcal{L}_4) < \mathrm{GL}(8,2)$ the 255 points of $\mathrm{PG}(7,2)$ fall into four orbits $\omega_1, \omega_2, \omega_3, \omega_4$, of respective lengths $12, 54, 108, 81$. We show that the 135 points $\in \omega_2 \cup \omega_4$ are the internal points of a hyperbolic quadric $\mathcal{H}_7$ determined by $\mathcal{L}_4$, and that the 81-set ω_4 (which is shown to have a sextic equation) is an orbit of a normal subgroup $\mathcal{G}_{81} \cong (Z_3)^4$ of $\mathcal{G}(\mathcal{L}_4)$. There are 40 subgroups $\cong (Z_3)^3$ of $\mathcal{G}_{81}$, and each such subgroup $H < \mathcal{G}_{81}$ gives rise to a decomposition of ω_4 into a triplet $\{\mathcal{R}_H, \mathcal{R}'_H, \mathcal{R}''_H\}$ of 27-sets. We show in particular that the constituents of precisely 8 of these 40 triplets are Segre varieties $\mathcal{S}_3(2)$ in $\mathrm{PG}(7,2)$. This ties in with the recent finding that each $\mathcal{S} = \mathcal{S}_3(2)$ in $\mathrm{PG}(7,2)$ determines a distinguished Z_3 subgroup of $\mathrm{GL}(8,2)$ which generates two sibling copies $\mathcal{S}', \mathcal{S}''$ of $\mathcal{S}$.

1 Introduction

We work for most of the time over $\mathbb{F}_2 = \mathrm{GF}(2)$, and so we can then identify a projective point $\langle x \rangle \in \mathrm{PG}(n-1,2)$ with the nonzero vector $x \in V(n,2)$. In fact we will be dealing with vector space dimension $n = 8$, and we will start out from a(ny) direct sum decomposition

$$V_8 = V_a \oplus V_b \oplus V_c \oplus V_d \tag{1}$$

of $V_8 := V(8,2)$ into 2-dimensional spaces V_a, V_b, V_c, V_d. For $h \in \{a,b,c,d\}$ we will write

$$V_h = \{u_h(\emptyset), u_h(0), u_h(1), u_h(2)\}, \quad \text{with } u_h(\emptyset) = 0.$$

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(The reason for this labelling of the four elements of V_h is that in a later section we wish to use $0, 1, 2$ as the elements of the Galois field $\mathbb{F}_3$.) So $\mathbb{P}V_8 = \mathrm{PG}(7,2)$ is the span of the four projective lines

$$L_h := \mathbb{P}V_h = \{u_h(0), u_h(1), u_h(2)\},\ h \in \{a, b, c, d\}. \tag{2}$$

It is surprising, but gratifying, that from such a simple starting point so many interesting and intricate geometrical aspects quickly emerge, as we now describe.

1.1 $\mathcal{H}_7$-tetrads of lines in $\mathrm{PG}(7,2)$

For $v_h \in V_h$ let $(v_a, v_b, v_c, v_d) := v_a \oplus v_b \oplus v_c \oplus v_d$ denote a general element of V_8. Setting $U_{ijkl} := (u_a(i), u_b(j), u_c(k), u_d(l))$ then the 255 points of $\mathrm{PG}(7,2)$ are

$$\{U_{ijkl} |\ i, j, k, l \in \{\varnothing, 0, 1, 2\},\ ijkl \neq \varnothing\varnothing\varnothing\varnothing\}. \tag{3}$$

First observe that the subgroup $\mathcal{G}(\mathcal{L}_4)$ of $\mathrm{GL}(8,2)$ which preserves the direct sum decomposition (1), and hence the foregoing tetrad

$$\mathcal{L}_4 := \{L_a, L_b, L_c, L_d\} \tag{4}$$

of lines, has the semi-direct product structure

$$\mathcal{G}(\mathcal{L}_4) = \mathcal{N} \rtimes \mathrm{Sym}(4), \quad \text{where } \mathcal{N} := \mathrm{GL}(V_a) \times \mathrm{GL}(V_b) \times \mathrm{GL}(V_c) \times \mathrm{GL}(V_d),$$

and where $\mathrm{Sym}(4) = \mathrm{Sym}(\{a, b, c, d\})$. Hence $|\mathcal{G}(\mathcal{L}_4)| = 6^4 \times 24 = 31,104$.

The $\mathcal{G}(\mathcal{L}_4)$-orbits of points are easily determined. In addition to the weight $\mathrm{wt}(p) = \mathrm{wt}_{\mathcal{B}}(p)$ of a point $p \in \mathrm{PG}(7,2)$ with respect to a basis $\mathcal{B}$ for V_8, let us also define its *line-weight* $\mathrm{lw}(p)$ as follows:

$$\mathrm{lw}(U_{ijkl}) = r \quad \text{whenever precisely } r \text{ of } i, j, k, l \text{ are in } \{0, 1, 2\}.$$

Then the 255 points of $\mathrm{PG}(7,2)$ clearly fall into just four $\mathcal{G}(\mathcal{L}_4)$-orbits ω_1, ω_2, ω_3, ω_4, where

$$\omega_r = \{p \in \mathrm{PG}(7,2) |\ \mathrm{lw}(p) = r\}. \tag{5}$$

The lengths of these orbits are accordingly

$$|\omega_1| = 12, \quad |\omega_2| = \tbinom{4}{2} \times 3^2 = 54, \quad |\omega_3| = \tbinom{4}{3} \times 3^3 = 108, \quad |\omega_4| = 3^4 = 81.$$

Next take note that *there is a unique* $\mathrm{Sp}(8,2)$*-geometry on* $V_8 := V(8,2)$, given by a non-degenerate alternating bilinear form B, such that the subspaces V_a, V_b, V_c, V_d are hyperbolic 2-dimensional spaces which are pairwise orthogonal. If $\mathcal{B} = \{e_i\}_{i \in \{1,2,3,4,5,6,7,8\}}$ is any basis such that

$$V_8 = V_a \perp V_b \perp V_c \perp V_d = \prec e_1, e_8 \succ \perp \prec e_2, e_7 \succ \perp \prec e_3, e_6 \succ \perp \prec e_4, e_5 \succ \tag{6}$$

then the symplectic product $x \cdot y := B(x, y)$ is determined by its values on basis vectors:

$$\begin{aligned} &e_1 \cdot e_8 = e_2 \cdot e_7 = e_3 \cdot e_6 = e_4 \cdot e_5 = 1, \\ &e_i \cdot e_j = 0 \quad \text{for other values of } i, j, \end{aligned} \tag{7}$$

and so has the coordinate expression

$$x \cdot y = (x_1y_8 + x_8y_1) + (x_2y_7 + x_7y_2) + (x_3y_6 + x_6y_3) + (x_4y_5 + x_5y_4).$$

Perhaps less obvious is the fact that *the tetrad (4) also determines a particular non-degenerate quadric $\mathcal{Q}$ in* PG(7, 2). For, as we now show, such a quadric $\mathcal{Q}$ is uniquely determined by the two conditions

(i) it has equation $Q(x) = 0$ such that the quadratic form Q polarizes to give the foregoing symplectic form $B : Q(x+y) + Q(x) + Q(y) = x \cdot y$;

(ii) the 12-set of points

$$\mathcal{P}(\mathcal{L}_4) := \omega_1 = L_a \cup L_b \cup L_c \cup L_d \subset \mathrm{PG}(7,2)$$

supporting the tetrad $\mathcal{L}_4$ is external to $\mathcal{Q}$.

For it follows from (i) that the terms of degree 2 in Q must be $P_2(x) = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$, and then the eight conditions $Q(e_i) = 1$ entail that the linear terms in Q must be $P_1(x) = \sum_{i=1}^{8} x_i$, so that

$$Q(x) = P_2(x) + P_1(x) = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5 + u \cdot x, \tag{8}$$

where $u := \sum_{i=1}^{8} e_i$. Further Q in (8) is seen to satisfy also the four conditions $Q(e_i + e_j) = 1$, $ij \in \{18, 27, 36, 45\}$, so indeed $Q(p) = 1$ for all $p \in \omega_1$.

Theorem 1. *The quadric $\mathcal{Q}$ is a hyperbolic quadric $\mathcal{H}_7$; moreover $\mathcal{H}_7 = \omega_2 \cup \omega_4$.*

Proof. There exist just two kinds, $\mathcal{E}_7$ and $\mathcal{H}_7$, of non-degenerate quadrics in PG(7, 2). An elliptic quadric $\mathcal{E}_7$ has 119 points and a hyperbolic quadric $\mathcal{H}_7$ has 135 points; see [7, Theorem 5.21], [9, Section 2.2]. Since $\mathcal{Q}$ is uniquely determined, its internal points must be a union of the $\mathcal{G}(\mathcal{L}_4)$-orbits $\omega_2, \omega_3, \omega_4$, of respective lengths 54, 108, 81. So the only possibility is that $\mathcal{Q}$ is a hyperbolic quadric $\mathcal{H}_7 = \omega_2 \cup \omega_4$, having $54 + 81 = 135$ points. (So we will term such a tetrad $\mathcal{L}_4$ of lines in PG(7, 2) a *$\mathcal{H}_7$-tetrad.)* ∎

Corollary 2. *$\mathcal{G}(\mathcal{L}_4)$ is a subgroup of the isometry group $\mathcal{G}(Q) \cong \mathrm{O}^+(8,2) < \mathrm{Sp}(8,2)$ of the hyperbolic quadric $\mathcal{H}_7$.*

Remark 3. In fact $\mathcal{G}(\mathcal{L}_4)$ is a maximal subgroup of $\mathrm{O}^+(8,2) = \mathrm{O}_8^+(2) \cdot 2$; see [3, p. 85], where it is recorded as $S_3 \,\mathrm{wr}\, S_4$.

1.2 $\mathcal{G}(\mathcal{L}_4)$-invariant polynomials

The tetrad $\mathcal{L}_4 = \{L_a, L_b, L_c, L_d\}$ determines the following $\mathcal{G}(\mathcal{L}_4)$-invariant sets of flats in PG(7, 2) :

(i) four 5-flats : $\langle L_a, L_b, L_c\rangle$, $\langle L_a, L_b, L_d\rangle$, $\langle L_a, L_c, L_d\rangle$, $\langle L_b, L_c, L_d\rangle$;
(ii) six 3-flats : $\langle L_a, L_b\rangle$, $\langle L_a, L_c\rangle$, $\langle L_a, L_d\rangle$, $\langle L_b, L_c\rangle$, $\langle L_b, L_d\rangle$, $\langle L_c, L_d\rangle$;
(iii) four 1-flats : L_a, L_b, L_c, L_d.

Let (i) $F_{hkl} = 0$ be the quadratic equation of the 5-flat $\langle L_h, L_k, L_l\rangle$, (ii) $F_{hk} = 0$ be the quartic equation of the 3-flat $\langle L_h, L_k\rangle$ and (iii) $F_h = 0$ be the sextic equation of the line L_h. (See [11, Lemma 2].) Consequently the tetrad $\mathcal{L}_4$ determines the $\mathcal{G}(\mathcal{L}_4)$-invariant polynomials Q_2, Q_4, Q_6, of respective degrees $2, 4, 6$, defined as follows:

$$\begin{aligned}
\text{(i)}\ \ & Q_2 = F_{abc} + F_{abd} + F_{acd} + F_{bcd},\\
\text{(ii)}\ \ & Q_4 = F_{ab} + F_{ac} + F_{ad} + F_{bc} + F_{bd} + F_{cd},\\
\text{(iii)}\ \ & Q_6 = F_a + F_b + F_c + F_d.
\end{aligned}$$

Theorem 4. *The* 81*-set* ω_4 *has the sextic equation* $Q_{\omega_4}(x) = 0$*, where* $Q_{\omega_4} := Q_6 + Q_4 + Q_2$.

Proof. Setting $\psi_Q := \{p \in \mathrm{PG}(7,2) |\ Q(p) = 0\}$, the last entry in the following table follows from the three preceding entries.

		$Q(p)$ if $p \in$					
Q	$\deg Q$	ω_1	ω_2	ω_3	ω_4	ψ_Q	$\lvert\psi_Q\rvert$
Q_2	2	1	0	1	0	$\omega_2 \cup \omega_4$	135
Q_4	4	1	1	0	0	$\omega_3 \cup \omega_4$	189
Q_6	6	1	0	0	0	$\omega_2 \cup \omega_3 \cup \omega_4$	243
$Q_2 + Q_4 + Q_6$	6	1	1	1	0	ω_4	81

■

Remark 5. Of course $Q_2 = 0$ is, see Eq. (8), the $\mathcal{H}_7$ quadric $\mathcal{Q}$ of Theorem 1. Also Q_4 was denoted Q_4' in [11, Theorem 17] and Q_6 was denoted Q_6' in [11, Example 20]. The sextic terms in Q_6 are readily found, since

$$Q_6 = \Pi_{i\neq 1,8}(1 + x_i) + \Pi_{i\neq 2,7}(1 + x_i) + \Pi_{i\neq 3,6}(1 + x_i) + \Pi_{i\neq 4,5}(1 + x_i).$$

Consequently, in terms of the sextic monomials $\widehat{x_j x_k} := \Pi_{i\notin\{j,k\}} x_i$, we see that

$$Q_6 = \widehat{x_1x_8} + \widehat{x_2x_7} + \widehat{x_3x_6} + \widehat{x_4x_5} + \ (\text{terms of degree } < 6). \tag{9}$$

Remark 6. A sextic polynomial Q determines, via complete polarization, an alternating multilinear form $\times^6 V_8 \to \mathbb{F}_2$, and hence an element $b \in \wedge^6 V_8^* \cong \wedge^2 V_8$. (See [10, Section 1.1].) Since Q_6 is $\mathcal{G}(\mathcal{L}_4)$-invariant, and since there is a unique nonzero $\mathcal{G}(\mathcal{L}_4)$-invariant element of $\wedge^2 V_8^* \cong \mathrm{Alt}(\times^2 V_8, \mathbb{F}_2)$, namely B in Eq. (7), it follows, in the case $Q = Q_6$, that b must be the $\wedge^2 V_8$ image of $B \in \wedge^2 V_8^*$, namely

$$b = e_1 \wedge e_8 + e_2 \wedge e_7 + e_3 \wedge e_6 + e_4 \wedge e_5. \tag{10}$$

It follows from (10) that the sextic terms in Q_6 must be the four monomials in (9). So the result (9) could in fact have been deduced in this alternative manner.

From the foregoing it is not too difficult to find by hand the explicit coordinate form of the sextic polynomial Q_{ω_4}. In fact we used Magma, see [2], to obtain the result below. At times writing $1' = 8$, $2' = 7$, $3' = 6$, $4' = 5$, let us define the following polynomials:

$$P_1 = \textstyle\sum_{1\le i\le 8} x_i, \quad P_2 = \sum_{1\le i<j\le 8} x_i x_j, \quad P_3 = \sum_{1\le i<j<k\le 8} x_i x_j x_k,$$
$$P_4 = \textstyle\sum_{1\le k<l\le 4} x_k x_{k'} x_l x_{l'},$$
$$P_4' = \sum_{1\le m\le 4} x_m x_{m'} P_{mm'}, \quad \text{where } P_{mm'} = \sum_{\substack{k<l,\ l\neq k'\\ k,l\notin\{m,m'\}}} x_k x_l,$$
$$P_5 = \sum_{\substack{1\le k<l\le 4\\ m\notin\{k,k',l,l'\}}} x_k x_{k'} x_l x_{l'} x_m,$$
$$P_6 = \textstyle\sum_{1\le k<l<m\le 4} x_k x_{k'} x_l x_{l'} x_m x_{m'} = \widehat{x_1x_8} + \widehat{x_2x_7} + \widehat{x_3x_6} + \widehat{x_4x_5}\,.$$

Then, assisted by Magma, we found that

$$Q_{\omega_4} = P_6 + P_5 + P_4 + P_4' + P_3 + P_2 + P_1.$$

2 The eight distinguished spreads $\{\mathcal{L}_{85}^{ijk}\}_{i,j,k\in\{1,2\}}$

Next we show that the partial spread $\mathcal{L}_4$ of four lines determines a privileged set of eight extensions to a complete spread $\mathcal{L}_{85}$ of 85 lines in $\mathrm{PG}(7,2)$. To this end, for each $h \in \{a,b,c,d\}$ let us choose that element $\zeta_h \in \mathrm{GL}(V_h)$ of order 3 which effects the cyclic permutation $(u_h(0)u_h(1)u_h(2))$ of the points of L_h. Consider the eight Z_3-subgroups $\{Z_{ijk}\}_{i,j,k\in\{1,2\}}$ of $\mathcal{G}(\mathcal{L}_4)$ defined by

$$Z_{ijk} = \langle A_{ijk}\rangle, \quad \text{where } A_{ijk} := (\zeta_a)^i \oplus (\zeta_b)^j \oplus (\zeta_c)^k \oplus \zeta_d\,. \tag{11}$$

When working using the basis $\mathcal{B}$ we will make the following choices for the four ζ_h in (11):

$$\begin{aligned} &\zeta_a : e_1 \mapsto e_8 \mapsto e_1 + e_8, \quad \zeta_b : e_7 \mapsto e_2 \mapsto e_2 + e_7,\\ &\zeta_c : e_3 \mapsto e_6 \mapsto e_3 + e_6, \quad \zeta_d : e_5 \mapsto e_4 \mapsto e_4 + e_5. \end{aligned} \tag{12}$$

We will also choose the $u_h(0)$ so that U_{0000} is the unit point u of the basis $\mathcal{B}$. Since $(A_{ijk})^2 + A_{ijk} + I = 0$, each Z_{ijk} acts fixed-point-free on $\mathrm{PG}(7,2)$ and gives rise to a spread $\mathcal{L}_{85}^{ijk}$ of lines in $\mathrm{PG}(7,2)$, with a point $p \in \mathrm{PG}(7,2)$ lying on the line

$$L^{ijk}(p) := \{p, A_{ijk}p, (A_{ijk})^2 p\} \in \mathcal{L}_{85}^{ijk}. \tag{13}$$

Note that if in (11) one or more of the ζ_h is replaced by the identity element $I_h \in \mathrm{GL}(V_h)$ then, although a Z_3-subgroup of $\mathcal{G}(\mathcal{L}_4)$ which preserves the lines is generated, it is not fixed-point-free on $\mathrm{PG}(7,2)$. So there exist precisely eight extensions of $\mathcal{L}_4$ to a Desarguesian spread of 85 lines in $\mathrm{PG}(7,2)$. Observe that in the case where p is the unit vector $u := \sum_{i=1}^{8} e_i$ of the basis $\mathcal{B}$ then the eight lines (13) are distinct: for, using $ijkl$ as shorthand for $e_i + e_j + e_k + e_l$, they are explicitly

$$\begin{aligned} &L^{111}(u) = \{u, 1357, 2468\}, \ L^{122}(u) = \{u, 1256, 3478\},\\ &L^{212}(u) = \{u, 5678, 1234\}, \ L^{221}(u) = \{u, 2358, 1467\},\\ &L^{222}(u) = \{u, 2568, 1347\}, \ L^{211}(u) = \{u, 3578, 1246\},\\ &L^{121}(u) = \{u, 1235, 4678\}, \ L^{112}(u) = \{u, 1567, 2348\}. \end{aligned} \tag{14}$$

Lemma 7. *For $p \in \mathrm{PG}(7,2)$ the eight lines $L^{ijk}(p)$ are distinct if and only if $p \in \omega_4$.*

Proof. We have in (14) just seen that the eight lines are distinct for the point $u \in \omega_4$, and hence for all $p \in \omega_4$. Consider a point $p = (0, v_b, v_c, v_d)$ of line-weight 3. Since $A_{1jk}p = A_{2jk}p$, and so $L^{1jk}(p) = L^{2jk}(p)$, the lines $L^{ijk}(p)$ coincide in pairs. Similarly for other points $p \in \omega_3$. For a point $p \in \omega_2$ of line-weight 2 the analogous reasoning shows that only two of the lines $L^{ijk}(p)$ are distinct. And of course if $p \in \omega_1$, that is if $p \in L_h$ for some $h \in \{a, b, c, d\}$, then $L^{ijk}(p) = L_h$ for all eight values of ijk. ∎

Recall that on a $\mathcal{H}_7$ quadric there exist two systems of *generators,* see [8, Section 22.4], elements of either system being solids (3-flats). Consequently it follows from the next theorem that *the foregoing eight Z_3-subgroups of $\mathcal{G}(\mathcal{L}_4)$ divide naturally into two sets of size four, namely $\mathbf{Z}$ and $\mathbf{Z}^*$* where

$$\mathbf{Z} = \{Z_{111}, Z_{122}, Z_{212}, Z_{221}\}, \quad \mathbf{Z}^* = \{Z_{222}, Z_{211}, Z_{121}, Z_{112}\}. \tag{15}$$

Theorem 8. *For $p \in \omega_4$ let $\Pi(p)$ denote the flat spanned by the four lines $L^{ijk}(p)$, $ijk \in \{111, 122, 212, 221\}$, and let $\Pi^*(p)$ denote the flat spanned by the four lines $L^{ijk}(p)$, $ijk \in \{222, 211, 121, 112\}$. Then $\Pi(p)$ and $\Pi^*(p)$ are generators of $\mathcal{H}_7$ which moreover belong to different systems.*

Proof. The flat $\Pi(p) = \langle L^{111}(p), L^{122}(p), L^{212}(p), L^{221}(p)\rangle$ for the point $p = (v_a, v_b, v_c, v_d) \in \omega_4$ is seen, upon using $(\zeta_h)^2 + \zeta_h = I_h$, to consist of the nine points $L^{111}(p) \cup L^{122}(p) \cup L^{212}(p) \cup L^{221}(p)$ of line-weight 4 together with the following six points of line-weight 2:

$$\begin{aligned}&(v_a, v_b, 0, 0),\ (v_a, 0, v_c, 0),\ (v_a, 0, 0, v_d),\\ &(0, 0, v_c, v_d),\ (0, v_b, 0, v_d),\ (0, v_b, v_c, 0).\end{aligned} \tag{16}$$

By Theorem 1, for each $p \in \omega_4$, the flat $\Pi(p)$ is in fact a solid on the quadric $\mathcal{H}_7$. Similarly the same applies to the flat $\Pi^*(p)$, whose six points of line-weight 2 moreover coincide with those of $\Pi(p)$. So $\Pi(p) \cap \Pi^*(p)$ is the isotropic plane consisting of $p = (v_a, v_b, v_c, v_d)$ together with the six points (16). Consequently, see [8, Theorem 22.4.12, Corollary], for each $p \in \omega_4$ the generators $\Pi(p)$ and $\Pi^*(p)$ belong to different systems. ∎

3 The normal subgroup $\mathcal{G}_{81}$ of $\mathcal{G}(\mathcal{L}_4)$

Let Z_h denote that Z_3 subgroup of $\mathcal{G}(\mathcal{L}_4)$ which fixes pointwise each of the three lines $\mathcal{L}_4 \setminus L_h$, $h \in \{a, b, c, d\}$. Then clearly the elementary abelian group

$$\mathcal{G}_{81} := Z_a \times Z_b \times Z_c \times Z_d \cong Z_3 \times Z_3 \times Z_3 \times Z_3$$

is a normal subgroup of $\mathcal{N} = \mathrm{GL}(V_a) \times \mathrm{GL}(V_b) \times \mathrm{GL}(V_c) \times \mathrm{GL}(V_d)$ and also of $\mathcal{G}(\mathcal{L}_4) = \mathcal{N} \rtimes \mathrm{Sym}(4)$. Observe that ω_4 is a single $\mathcal{G}_{81}$-orbit. One easily sees that $\mathcal{G}_{81}$ is equally well the direct product $Z_{111} \times Z_{122} \times Z_{212} \times Z_{221}$ of the four members of $\mathbf{Z}$, and also the direct product $Z_{222} \times Z_{211} \times Z_{121} \times Z_{112}$ of the four members of $\mathbf{Z}^*$.

Consider now *any* $Z_3 \times Z_3 \times Z_3$ subgroup $H < \mathcal{G}_{81}$. If $\mathcal{G}_{81} = H \cup H' \cup H''$ denotes the decomposition of $\mathcal{G}_{81}$ into the cosets of H then we define subsets $\mathcal{R} := \mathcal{R}_H, \mathcal{R}' := \mathcal{R}'_H, \mathcal{R}'' := \mathcal{R}''_H$ of ω_4 by

$$\mathcal{R} = \{hu, h \in H\},\ \mathcal{R}' = \{h'u, h' \in H'\},\ \mathcal{R}'' = \{h''u, h'' \in H''\}. \tag{17}$$

In particular $\mathcal{R} = \mathcal{R}_H$ is the orbit of u under the action of the group H. Each such subgroup $H < \mathcal{G}_{81}$ gives rise to a decomposition $\omega_4 = \mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''$ of ω_4 into a triplet of 27-sets.

As we will now demonstrate, *the study of such triplets is greatly simplified by viewing $\mathcal{G}_{81}$ in a* GF(3) *light.*

3.1 A GF(3) view of $\mathcal{G}_{81}$

For $i, j, k, l \in \mathbb{F}_3 = \mathrm{GF}(3) = \{0, 1, 2\}$ define

$$A_{ijkl} := (\zeta_a)^i \oplus (\zeta_b)^j \oplus (\zeta_c)^k \oplus (\zeta_d)^l\ .$$

Note that if $i, j, k \in \{1, 2\}$ then $A_{ijk1} = A_{ijk}$, as previously defined in (11). In the following we will view $ijkl$ as shorthand for the element $(i, j, k, l) \in (\mathbb{F}_3)^4$. Since

$$A_\sigma A_\tau = A_{\sigma+\tau},\ \sigma, \tau \in (\mathbb{F}_3)^4, \tag{18}$$

observe that $A : \sigma \mapsto A_\sigma$, $\sigma \in (\mathbb{F}_3)^4$, is an isomorphism mapping the additive group $(\mathbb{F}_3)^4$ onto the multiplicative group $\mathcal{G}_{81}$. Now the orbit of any point $p \in \omega_4$ under the action of the group $\mathcal{G}_{81}$ is the whole of ω_4. In particular this is so for the unit point $u := \sum_{i=1}^{8} e_i$ of the basis $\mathcal{B}$. *Consequently the* 81*-set ω_4 is in bijective correspondence with $(\mathbb{F}_3)^4$ as given by the map $\theta_u : (\mathbb{F}_3)^4 \to \omega_4$* defined by

$$\theta_u(\sigma) = p_\sigma := A_\sigma u,\ \sigma \in (\mathbb{F}_3)^4. \tag{19}$$

Observe that the choices made in (2) and (12) imply that $\theta_u(ijkl) = U_{ijkl}$ for all $ijkl \in (\mathbb{F}_3)^4$.

In the GF(3) space $V(4,3) = (\mathbb{F}_3)^4$ we will chiefly employ the basis $\mathcal{B}_\varepsilon := \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, where

$$\varepsilon_1 = 1000,\ \varepsilon_2 = 0100,\ \varepsilon_3 = 0010,\ \varepsilon_4 = 0001, \tag{20}$$

and then write a general element $\xi = \sum_{r=1}^{4} \xi_r \varepsilon_r \in V(4,3)$ as $\xi = \xi_1\xi_2\xi_3\xi_4$. We denote the weight of $\xi \in (\mathbb{F}_3)^4$ in the basis $\mathcal{B}_\varepsilon$ by $\mathrm{wt}_\varepsilon(\xi)$.

Let us now study subgroups of $\mathcal{G}_{81}$ by viewing them in the light of their corresponding subspaces in the vector space $V(4,3) = (\mathbb{F}_3)^4$.

A Z_3 subgroup of $\mathcal{G}_{81}$ is of the form $\{I, A_\sigma, A_{2\sigma}\}$ for some non-zero $\sigma \in V(4,3)$. *So $\mathcal{G}_{81}$ contains* 40 *subgroups $\cong Z_3$ which are in bijective correspondence with the* 40 *points of the projective space* PG$(3,3) = \mathbb{P}V(4,3)$. If we denote by $\Xi \cup \Xi^*$ the following eight elements of $(\mathbb{F}_3)^4$:

$$\begin{aligned} \Xi :\quad & \alpha = 1111,\ \beta = 1221,\ \gamma = 2121,\ \delta = 2211, \\ \Xi^* :\quad & \alpha^* = 2221,\ \beta^* = 2111,\ \gamma^* = 1211,\ \delta^* = 1121, \end{aligned} \tag{21}$$

then observe that $A_\alpha, A_\beta, \dots, A_{\delta^*}$ are the respective generators of the eight Z_3-subgroups $Z_{111}, Z_{122}, \dots, Z_{112} \in \mathbf{Z} \cup \mathbf{Z}^*$ considered in (15). Now under the action by conjugacy of $\mathcal{G}(\mathcal{L}_4)$ on $\mathcal{G}_{81}$ the particular 4-set $\{Z_a, Z_b, Z_c, Z_d\} = \{\langle A_{\varepsilon_1}\rangle, \langle A_{\varepsilon_2}\rangle, \langle A_{\varepsilon_3}\rangle, \langle A_{\varepsilon_4}\rangle\}$ of Z_3 subgroups is fixed, whence

$$\mathcal{T}_\varepsilon := \{\langle\varepsilon_1\rangle, \langle\varepsilon_2\rangle, \langle\varepsilon_3\rangle, \langle\varepsilon_4\rangle\}$$

is a $\mathcal{G}(\mathcal{L}_4)$*-distinguished tetrahedron of reference in* $\mathrm{PG}(3,3)$. Consequently take note that *the eight Z_3 subgroups $\{\langle A_\rho\rangle\}_{\rho\in\Xi\cup\Xi^*}$ considered in (15) are picked out as the only Z_3 subgroups $\langle A_\rho\rangle$* of $\mathcal{G}_{81}$ *for which* $\mathrm{wt}_\varepsilon(\rho) = 4$.

Next let us consider subgroups $H \cong Z_3 \times Z_3 \times Z_3$ of $\mathcal{G}_{81}$.

Theorem 9. *The normal subgroup $\mathcal{G}_{81} < \mathcal{G}(\mathcal{L}_4)$ contains precisely* 40 *subgroups $H \cong Z_3 \times Z_3 \times Z_3$. These fall into four conjugacy classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of $\mathcal{G}(\mathcal{L}_4)$, of respective sizes* $8, 16, 12, 4$.

Proof. Each subgroup $\langle A_\rho, A_\sigma, A_\tau\rangle \cong Z_3 \times Z_3 \times Z_3$ arises as

$$\{A_\lambda \mid \lambda \in V_3 := \prec\rho, \sigma, \tau\succ\}$$

from a corresponding projective plane $\mathbb{P}V_3 = \langle\langle\rho\rangle, \langle\sigma\rangle, \langle\tau\rangle\rangle$ in $\mathrm{PG}(3,3)$. Now there exist precisely 40 planes in $\mathrm{PG}(3,3)$, and these fall into four kinds $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, where $\mathcal{P}_r$ denotes those planes in $\mathrm{PG}(3,3)$ which contain precisely r of the vertices $\langle\varepsilon_i\rangle$ of the tetrahedron of reference $\mathcal{T}_\varepsilon$. There are 8 planes of kind $\mathcal{P}_0$, namely those with one of the 8 equations

$$\xi_4 = c_1\xi_1 + c_2\xi_2 + c_3\xi_3, \quad c_1, c_2, c_3 \in \{1,2\}. \tag{22}$$

Similarly we see that there are, respectively, 16, 12, 4 planes of kinds $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$. The theorem now follows, since planes of the same kind are seen to correspond to conjugate $Z_3 \times Z_3 \times Z_3$ subgroups. ∎

Finally let us consider subgroups $H \cong Z_3 \times Z_3$ of $\mathcal{G}_{81}$. Such a subgroup $\langle A_\rho, A_\sigma\rangle$ arises from a corresponding line $\langle\langle\rho\rangle, \langle\sigma\rangle\rangle \subset \mathrm{PG}(3,3)$, and so we need to classify lines with respect to the $\mathcal{G}(\mathcal{L}_4)$-distinguished basis $\mathcal{B}_\varepsilon$. If n_w points of a line $L \subset \mathrm{PG}(3,3)$ have weight w, $w \in \{1,2,3,4\}$, with respect to the basis $\mathcal{B}_\varepsilon$ then we will say that L has *weight pattern* $\pi_\varepsilon(L) = (n_1, n_2, n_3, n_4)$.

Theorem 10. *The normal subgroup $\mathcal{G}_{81} < \mathcal{G}(\mathcal{L}_4)$ contains precisely* 130 *subgroups $\cong Z_3 \times Z_3$. These fall into seven conjugacy classes $\mathcal{K}_1, \dots, \mathcal{K}_7$ of $\mathcal{G}(\mathcal{L}_4)$, of respective sizes* $6, 24, 16, 12, 16, 48, 8$.

Proof. Each subgroup $\langle A_\rho, A_\sigma\rangle \cong Z_3 \times Z_3$ arises from a corresponding line $\langle\langle\rho\rangle, \langle\sigma\rangle\rangle \subset \mathrm{PG}(3,3)$, and there exist precisely 130 lines in $\mathrm{PG}(3,3)$. With respect to the $\mathcal{G}(\mathcal{L}_4)$-distinguished tetrahedron of reference $\mathcal{T}_\varepsilon$ the 130 lines L are of seven kinds $\Lambda_1, \dots, \Lambda_7$ as described in the following table.

	$\pi_e(L)$	$\lvert\Lambda_i\rvert$
$L \in \Lambda_1$	$(2,2,0,0)$	6
$L \in \Lambda_2$	$(1,1,2,0)$	24
$L \in \Lambda_3$	$(0,3,1,0)$	16
$L \in \Lambda_4$	$(0,2,0,2)$	12
$L \in \Lambda_5$	$(1,0,1,2)$	16
$L \in \Lambda_6$	$(0,1,2,1)$	48
$L \in \Lambda_7$	$(0,0,4,0)$	8

(23)

The theorem now follows since lines of the same kind correspond to conjugate $Z_3 \times Z_3$ subgroups. ■

3.2 The 27-set 'denizens' of ω_4

It follows from Theorem 9 that the 81-set ω_4 is populated by 40 triplets $\{\mathcal{R}, \mathcal{R}', \mathcal{R}''\}$ of 27-set 'denizens', as in (17), and that these triplets, and the 120 denizens of ω_4, can be classified into four kinds $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. One of our aims is to show that *precisely eight of these triplets are triplets of Segre varieties* $\mathcal{S}_3(2)$. So it helps to remind ourselves at this point about certain aspects of a Segre variety $\mathcal{S} = \mathcal{S}_3(2)$ in PG$(7,2)$, and to relate our present concerns to those in [4], [5] and [11].

First of all, $\mathcal{S}$ defines a $(27_3, 27_3)$ configuration, each of the 27 points of $\mathcal{S}$ lying on *precisely* 3 lines $\subset \mathcal{S}$, namely three of the 27 generators of $\mathcal{S}$. Moreover the stabilizer group $\mathcal{G}_{\mathcal{S}}$ of $\mathcal{S}$ contains as a normal subgroup a group $\langle A_1, A_2, A_3 \rangle \cong Z_3 \times Z_3 \times Z_3$ which acts transitively on the 27 points of $\mathcal{S}$, the three generators of $\mathcal{S}$ through a point $p \in \mathcal{S}$ being the lines

$$L^r(p) := \{p, A_r p, (A_r)^2 p\}, \; r = 1,2,3. \tag{24}$$

Here A_r satisfies $(A_r)^2 + A_r + I = 0$, each Z_3 group $\langle A_r \rangle$ acting fixed-point-free on PG$(7,2)$. Further, as noted in [11, Theorem 5], $\mathcal{S}$ determines a distinguished Z_3-subgroup $\langle W \rangle$ which also acts fixed-point-free on PG$(7,2)$, the distinguished tangent, see [11, Section 2.1], at $p \in \mathcal{S}$ being the line $\{p, Wp, W^2 p\}$. Moreover, see [11, Section 4.2], under the action of the distinguished Z_3-subgroup $\langle W \rangle$ the Segre variety $\mathcal{S}$ gave rise to a triplet $\{\mathcal{S},\ \mathcal{S}' = W(\mathcal{S}),\ \mathcal{S}'' = W^2(\mathcal{S})\}$ of Segre varieties. In [4, p. 82] (although without proof and using a different notation), [5, Proposition 5] and [11, Section 4.1] the five $\mathcal{G}_{\mathcal{S}}$-orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5$ of points were described, with $\mathcal{O}_5 = \mathcal{S}$ and $\mathcal{O}_4 = \mathcal{S}' \cup \mathcal{S}''$. These are related to the four $\mathcal{G}(\mathcal{L}_4)$-orbits (5) in the following simple manner:

$$\omega_1 = \mathcal{O}_1, \quad \omega_2 = \mathcal{O}_2, \quad \omega_3 = \mathcal{O}_3, \quad \omega_4 = \mathcal{O}_4 \cup \mathcal{O}_5 = \mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}''. \tag{25}$$

So ω_4 is a single orbit under the action of the group $\langle A_1, A_2, A_3, W \rangle \cong (Z_3)^4$, this last thus being the group $\mathcal{G}_{81}$ in our present context.

Lemma 11. *If $p \in \omega_4$ and $\lambda \in (\mathbb{F}_3)^4, \lambda \neq 0000$, then $L_p^\lambda := \{p, A_\lambda p, A_{2\lambda} p\}$ is a line in ω_4 if and only if $\pm\lambda \in \Xi \cup \Xi^*$.*

Proof. We already know, see (13), that L_p^λ is a line if $\lambda \in \Xi \cup \Xi^*$ or if $-\lambda \in \Xi \cup \Xi^*$. Also, as noted after equation (21), if $\pm\lambda \notin \Xi \cup \Xi^*$ then $\mathrm{wt}_\varepsilon(\lambda) < 4$, and so $\lambda = mnrs$ where at least one of m, n, r, s is 0. For example, suppose $\lambda = mnr0$, where $m, n, r \in \mathbb{F}_3$. Then $(I + A_\lambda + A_{2\lambda}) U_{ijkl} = U_{\emptyset\emptyset\emptyset l} \neq 0$. ■

Remark 12. A *partial affine space*, see [1, p. 35] or [6, p. 794], is an affine space from which some parallel classes have been removed. For example, the affine space on $(\mathbb{F}_3)^4$ turns into a partial affine space if we consider only affine lines with a direction vector $\pm\lambda \in \Xi \cup \Xi^*$ and restrict the parallelism of $(\mathbb{F}_3)^4$ to the set of those lines. Lemma 11 shows that ω_4 arises as the point set of an isomorphic partial affine space in the following way: The *lines* in ω_4 are of the form L_p^λ with $\pm\lambda \in \Xi \cup \Xi^*$. Two lines are *parallel* if they belong to the same distinguished spread $\mathcal{L}_{85}^{ijk}$.

Theorem 13. *A triplet of 27-sets $\{\mathcal{R}_H, \mathcal{R}'_H, \mathcal{R}''_H\}$ in (17) which arises from a $(Z_3)^3$ subgroup $H = \{A_\lambda |\ \lambda \in V_3\}$ will consist of Segre varieties $\mathcal{S}_3(2)$ if and only if the projective plane $P = \mathbb{P}V_3 \subset \mathrm{PG}(3,3)$ is of kind $\mathcal{P}_0$. So the 81-set ω_4 contains precisely 24 copies of a Segre variety $S_3(2)$.*

Proof. Since each point of a Segre $\mathcal{S} = \mathcal{S}_3(2)$ lies on *precisely* three generators of $\mathcal{S}$, see (24), it follows from the preceding lemma that in order for $\mathcal{R}_H$ to be a $\mathcal{S}_3(2)$ the subgroup H must be of the form $\langle A_\lambda, A_\mu, A_\nu\rangle$ for *precisely* three element $\lambda, \mu, \nu \in \Xi \cup \Xi^*$. But a straightforward check shows that planes of the kinds $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ contain, respectively, precisely $3, 2, 4, 0$ points $\langle\lambda\rangle$ with $\lambda \in \Xi \cup \Xi^*$. So only in the eight cases where the Z_3 subgroup $H < \mathcal{G}_{81}$ is of kind $\mathcal{C}_0$ can $\mathcal{R}_H$ be Segre variety $S_3(2)$. By (25) for one such subgroup a Segre variety arises and, by Theorem 9, the same holds for the remaining subgroups of kind $\mathcal{C}_0$. ∎

Remark 14. It is easy to check that if all three of λ, μ, ν are in Ξ, or if all three are in Ξ^*, then $\mathbb{P}(\prec\lambda, \mu, \nu\succ)$ is a plane in $\mathrm{PG}(3,3)$ of kind $\mathcal{P}_0$, thus accounting for all eight planes of kind $\mathcal{P}_0$. But if λ, μ, ν are split $2, 1$ or $1, 2$ between Ξ, Ξ^*, then we see that $H = \langle A_\lambda, A_\mu, A_\nu\rangle$ contains a fourth Z_3 subgroup $\langle A_\rho\rangle$ with $\rho \in \Xi \cup \Xi^*$: for example, observe results such as

$$\prec\alpha, \beta, \alpha^*\succ = \prec\alpha, \beta, \alpha^*, \beta^*\succ, \quad \text{and} \quad \prec\alpha, \beta, \gamma^*\succ = \prec\alpha, \beta, \gamma^*, \delta^*\succ. \tag{26}$$

So such a $(Z_3)^3$ subgroup H is of kind $\mathcal{C}_2$, not $\mathcal{C}_0$, and $\mathcal{R}_H$ gives rise to a $(27_4, 36_3)$ configuration in contrast to the $(27_3, 27_3)$ configuration arising from a $S_3(2)$.

It was proved in [11, Theorem 18] that a Segre variety $S_3(2)$ in $\mathrm{PG}(7,2)$ has a sextic equation. In fact, from our results in Section 1.2 we can deduce that if $\mathcal{S}$ is any of the 24 copies of a Segre variety $S_3(2)$ in ω_4 then $\mathcal{S}$ has a sextic equation of the form $Q_{\mathcal{S}}(x) = 0$ where, for some polynomial $F_{\mathcal{S}}$ of degree < 6,

$$Q_{\mathcal{S}} = \widehat{x_1x_8} + \widehat{x_2x_7} + \widehat{x_3x_6} + \widehat{x_4x_5} + F_{\mathcal{S}}. \tag{27}$$

For recall from (25) that $\omega_4 = \mathcal{O}_4 \cup \mathcal{O}_5$ where $\mathcal{O}_5 = \mathcal{S}$. Now, see [11, Theorem 17], the 201-set $(\mathcal{O}_4)^{\mathrm{c}} := \omega_1 \cup \omega_2 \cup \omega_3 \cup \mathcal{S}$ has a quartic equation, say $F'_{\mathcal{S}} = 0$, and, see Theorem 4, $\mathcal{O}_4 \cup \mathcal{S}$ has sextic equation $Q_6 + Q_4 + Q_2 = 0$. It follows that $\mathcal{S}$ has equation $Q_6 + Q_4 + Q_2 + F'_{\mathcal{S}} = 0$ which, see (9), is of the form (27).

3.3 Non-Segre triplets in ω_4

When the authors first considered 27-sets such as $\mathcal{R}_{\alpha\beta\gamma^*} = \{A_\rho u\}_{\rho\in\prec\alpha,\beta,\gamma^*\succ}$ they were briefly misled into thinking that $\mathcal{R}_{\alpha\beta\gamma^*}$ was a Segre $\mathcal{S}_3(2)$. Upon discovering their error, see (26), they decided that all the 'deceitful' non-Segre 27-sets in ω_4 should be termed *'rogues'*. In this terminology we can summarize our foregoing results as follows.

> The 81-set ω_4 is populated by 120 denizen 27-sets which occur as 40 triplets $\{\mathcal{R}, \mathcal{R}', \mathcal{R}''\}$ such that $\mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}'' = \omega_4$. Of these triplets eight are of Segre varieties $\mathcal{S}_3(2)$, sixteen are of rogues of kind $\mathcal{C}_1$, twelve are of rogues of kind $\mathcal{C}_2$ and four are of rogues of kind $\mathcal{C}_3$.

Since a Segre variety $\mathcal{S}_3(2)$ spans PG$(7,2)$, the next two theorems confirm in a more vivid manner that rogues of kinds $\mathcal{C}_2$ and $\mathcal{C}_3$ are not Segre varieties.

Theorem 15. *Suppose that a 27-set $\mathcal{R} \subset \omega_4$ is a rogue of kind $\mathcal{C}_2$. Then $\langle \mathcal{R} \rangle$ is a 5-flat in* PG$(7,2)$.

Proof. Six of the twelve planes of kind $\mathcal{P}_2$ are those having equations of the kind $\xi_r = \xi_s$ and the other six are those having equations of the kind $\xi_r = 2\xi_s$. Consider a plane $P = \mathbb{P}V_3$ in the first six, say with equation $\xi_3 = \xi_4$. Then, see (3), the subset $\theta_u(V_3)$ consists of the 27 points

$$\mathcal{R} := \{U_{ijkk} | \ i, j, k \in \mathbb{F}_3 = \{0, 1, 2\}\}. \tag{28}$$

Now any two elements a, b of a line $L_h \in \mathcal{L}_4$ satisfy $a \cdot b = 1$ if $a \neq b$ and $a \cdot b = 0$ if $a = b$. So the three points of the line

$$L_{\mathcal{R}} := \{U_{\emptyset\emptyset 00}, U_{\emptyset\emptyset 11}, U_{\emptyset\emptyset 22}\} \subset \langle L_c, L_d \rangle$$

are perpendicular to every point of $\mathcal{R}$. However this is not the case for any other point of PG$(7,2)$, and so $\langle \mathcal{R} \rangle$ is the 5-flat $(L_{\mathcal{R}})^{\perp}$. A plane $P = \mathbb{P}V_3$ in the second six, say with equation $\xi_3 = 2\xi_4 = -\xi_4$, can be treated similarly, with the subset $\theta_u(V_3)$ consisting of the 27 points

$$\mathcal{R}_* := \{U_{ijk\bar{k}} | \ i, j, k \in \mathbb{F}_3 = \{0, 1, 2\}\}, \tag{29}$$

where, for $k \in \mathbb{F}_3$, $\bar{k}$ denotes $-k (= 2k)$. If $\mathcal{R}_*$ is as in (29) then we see that $\langle \mathcal{R}_* \rangle$ is the 5-flat $(L_{\mathcal{R}_*})^{\perp}$ where

$$L_{\mathcal{R}_*} := \{U_{\emptyset\emptyset 00}, U_{\emptyset\emptyset 12}, U_{\emptyset\emptyset 21}\} \subset \langle L_c, L_d \rangle. \qquad \blacksquare$$

Remark 16. Consider the triplet $\{\mathcal{R}, \mathcal{R}', \mathcal{R}''\}$ of kind $\mathcal{C}_2$ which contains $\mathcal{R}$ as in (28). Then $\mathcal{R}' = A_{\mu}(\mathcal{R})$ and $\mathcal{R}'' = A_{2\mu}(\mathcal{R})$ for suitable $\mu \in (\mathbb{F}_3)^4$, for example $\mu = 0001$. Consequently

$$L_{\mathcal{R}'} := \{U_{\emptyset\emptyset 01}, U_{\emptyset\emptyset 12}, U_{\emptyset\emptyset 20}\}, \quad L_{\mathcal{R}''} := \{U_{\emptyset\emptyset 02}, U_{\emptyset\emptyset 10}, U_{\emptyset\emptyset 21}\}.$$

Similarly, in the case of the triplet $\{\mathcal{R}_*, \mathcal{R}'_*, \mathcal{R}''_*\}$ of kind $\mathcal{C}_2$ which contains $\mathcal{R}_*$ as in (29) we see that

$$L_{\mathcal{R}'_*} := \{U_{\emptyset\emptyset 01}, U_{\emptyset\emptyset 10}, U_{\emptyset\emptyset 22}\}, \quad L_{\mathcal{R}''_*} := \{U_{\emptyset\emptyset 02}, U_{\emptyset\emptyset 11}, U_{\emptyset\emptyset 20}\}.$$

So the three lines $\{L_{\mathcal{R}}, L_{\mathcal{R}'}, L_{\mathcal{R}''}\}$ are a regulus in the 3-flat $\langle L_c, L_d \rangle$, and the three lines $\{L_{\mathcal{R}_*}, L_{\mathcal{R}'_*}, L_{\mathcal{R}''_*}\}$ are the opposite regulus, the 9-set supporting the two reguli being that hyperbolic quadric $\mathcal{H}_3$ in the 3-flat $\langle L_c, L_d \rangle$ which has L_c and L_d as its two external lines. Of course similar considerations apply to all twelve of the planes in PG$(3,3)$ of kind $\mathcal{P}_2$. Thus each of the six pairs of lines in $\mathcal{L}_4$ gives rise to a pair of opposite reguli and hence to a set of $6 \times 2 \times 3 = 36$ lines $L \subset \omega_2$. Each such L gives rise to a rogue $\mathcal{R}$ of kind $\mathcal{C}_2$, namely to $\mathcal{R} = L^{\perp} \cap \omega_4$.

Theorem 17. *Suppose that a* 27*-set* $\mathcal{R} \subset \omega_4$ *is a rogue of kind* $\mathcal{C}_3$*. Then* $\langle \mathcal{R} \rangle$ *is a* 6*-flat in* $\mathrm{PG}(7,2)$.

Proof. By Theorem 9 there are four triplets in ω_4 of kind $\mathcal{C}_3$, which arise from the four planes $\xi_r = 0$, $r \in \{1,2,3,4\}$. Consider the plane $P = \mathbb{P}V_3$ with equation $\xi_4 = 0$. Then, see (3), it gives rise to the following triplet of 27-sets

$$\mathcal{R} := \{U_{ijk0}\},\ \mathcal{R}' := \{U_{ijk1}\},\ \mathcal{R}'' := \{U_{ijk2}\},\quad i,j,k \in \mathbb{F}_3 = \{0,1,2\}.$$

It quickly follows that $\langle \mathcal{R} \rangle$, $\langle \mathcal{R}' \rangle$ and $\langle \mathcal{R}'' \rangle$ are 6-flats, namely

$$\langle \mathcal{R} \rangle = \langle U_{\emptyset\emptyset\emptyset 0} \rangle^{\perp},\ \langle \mathcal{R}' \rangle = \langle U_{\emptyset\emptyset\emptyset 1} \rangle^{\perp},\ \langle \mathcal{R}'' \rangle = \langle U_{\emptyset\emptyset\emptyset 2} \rangle^{\perp},$$

where $\{U_{\emptyset\emptyset\emptyset 0}, U_{\emptyset\emptyset\emptyset 1}, U_{\emptyset\emptyset\emptyset 2}\} = L_d$. Of course the other three triplets in ω_4 of kind $\mathcal{C}_3$ are associated in a similar way with the other three lines $L_a, L_b, L_c \in \mathcal{L}_4$. ∎

4 Intersection properties

4.1 Introduction

If Δ_1 and Δ_2 are any two distinct triplets of 27-set denizens of ω_4 note that

$$\mathcal{N}(\Delta_1, \Delta_2) := \{R_1 \cap R_2 : R_1 \in \Delta_1, R_2 \in \Delta_2\} \tag{30}$$

is an ennead of 9*-sets which provides a partition of* ω_4. For suppose that H_1 and H_2 are two $(Z_3)^3$ subgroups of $\mathcal{G}_{81}$ whose orbits in ω_4 yield the triplets Δ_1 and Δ_2. Then the $(Z_3)^2$ subgroup $H = H_1 \cap H_2$ yields one member $\{hu, h \in H\}$ of the ennead of 9-sets (30), the other members of the ennead being the other orbits of H in ω_4.

Since the origin of the present research arose from our interest in Segre varieties $\mathcal{S}_3(2)$ in $\mathrm{PG}(7,2)$, let us at least look at the different kinds of intersection $\mathcal{S} \cap \mathcal{R}$ of a Segre variety $\mathcal{S} \subset \omega_4$ with another 27-set denizen $\mathcal{R}$ of ω_4. Such an intersection we will term a *section* of the Segre $\mathcal{S}$. Recall from Theorem 13 that a Segre variety $\mathcal{S} \subset \omega_4$ arises as $\mathcal{S}_H := \{hu, h \in H\}$ from a $(Z_3)^3$ subgroup $H < \mathcal{G}_{81}$ which is of class $\mathcal{C}_0$, being the image, under the isomorphism A in (18), of a 3-dimensional subspace V_3 such that the projective plane $P = \mathbb{P}V_3 \subset \mathrm{PG}(3,3)$ is of kind $\mathcal{P}_0$.

Lemma 18. *Suppose that* $P = \mathbb{P}V_3 \subset \mathbb{P}V(4,3)$ *is a projective plane of kind* $\mathcal{P}_0$*. Then the* 13 *lines* $L \subset P$ *fall into three* $\mathcal{G}(\mathcal{L}_4)$*-orbits:*

(i) 3 *lines of kind* Λ_4*;*

(ii) 6 *lines of kind* Λ_6*;*

(iii) 4 *lines of kind* Λ_3.

Proof. Without loss of generality we may, see Remark 14, consider the particular plane

$$P = \mathbb{P}V_{\beta\gamma\delta}, \quad \text{where } V_{\beta\gamma\delta} = \prec\beta,\gamma,\delta\succ \subset V(4,3) = (\mathbb{F}_3)^4.$$

Observe that the 13 lines in the plane P are as follows:
(i) the 3 lines $\mathbb{P}\prec\beta,\gamma\succ, \mathbb{P}\prec\beta,\delta\succ, \mathbb{P}\prec\gamma,\delta\succ$ of weight pattern $(0,2,0,2)$;
(ii) the 6 lines $\mathbb{P}\prec\beta,\gamma\pm\delta\succ$, $\mathbb{P}\prec\gamma,\beta\pm\delta\succ$, $\mathbb{P}\prec\delta,\beta\pm\gamma\succ$ of weight pattern $(0,1,2,1)$;
(iii) the 4 lines $\mathbb{P}\prec\beta\pm\gamma,\beta\pm\delta\succ$ of weight pattern $(0,3,1,0)$.
Hence, see (23), the stated result holds. ∎

Equivalently expressed, the thirteen $Z_3 \times Z_3$ subgroups of $\langle A_\beta, A_\gamma, A_\delta\rangle < \mathcal{G}_{81}$ comprise: (i) three of class $\mathcal{K}_4$ (ii) six of class $\mathcal{K}_6$ (iii) four of class $\mathcal{K}_3$.

4.2 Sections of a Segre variety $\mathcal{S} \subset \omega_4$

Without loss of generality we may consider the particular Segre variety $\mathcal{S}_{\beta\gamma\delta} := \mathcal{S}_H$ where $H = \langle A_\beta, A_\gamma, A_\delta\rangle$:

$$\mathcal{S}_{\beta\gamma\delta} = \theta_u(V_{\beta\gamma\delta}), \quad \text{where } V_{\beta\gamma\delta} = \prec\beta,\gamma,\delta\succ \subset (\mathbb{F}_3)^4.$$

In detail the 27 elements of $V_{\beta\gamma\delta}$ are:

$$\boxed{\begin{matrix} 0000 & 1221 & 2112 \\ 2121 & 0012 & 1200 \\ 1212 & 2100 & 0021 \end{matrix}} \overset{T_\delta}{\longrightarrow} \boxed{\begin{matrix} 2211 & 0102 & 1020 \\ 1002 & 2220 & 0111 \\ 0120 & 1011 & 2202 \end{matrix}} \overset{T_\delta}{\longrightarrow} \boxed{\begin{matrix} 1122 & 2010 & 0201 \\ 0210 & 1101 & 2022 \\ 2001 & 0222 & 1110 \end{matrix}}. \tag{31}$$

In the display (31) the rows of each 9-set are orbits of $\langle T_\beta\rangle$ and the columns of each 9-set are orbits of $\langle T_\gamma\rangle$, where T_λ denotes the translation which maps $\mu \in (\mathbb{F}_3)^4$ to $\lambda + \mu \in (\mathbb{F}_3)^4$; of course $(T_\lambda)^2 = T_{2\lambda}$. Incidentally observe from (31) that, in conformity with (22), $V_{\beta\gamma\delta}$ is that 3-dimensional subspace of $V(4,3)$ having the equation

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0.$$

Using the basis $\mathcal{B}$, with ζ_h as in (12), we see, in the shorthand notation of Eq. (14), that $\mathcal{S}_{\beta\gamma\delta}$ thus consists of the following 27 points:

$$\boxed{\begin{matrix} u & 1256 & 3478 \\ 5678 & 56u & 78u \\ 1234 & 12u & 34u \end{matrix}} \overset{A_\delta}{\longrightarrow} \boxed{\begin{matrix} 2358 & 25u & 38u \\ 58u & 137u & 246u \\ 23u & 468u & 157u \end{matrix}} \overset{A_\delta}{\longrightarrow} \boxed{\begin{matrix} 1467 & 16u & 47u \\ 67u & 248u & 135u \\ 14u & 357u & 268u \end{matrix}}. \tag{32}$$

Observe that each 9-set in (32) is a Segre variety $\mathcal{S}_2(2)$, whose generators are the rows and columns in the display, these being orbits of, respectively, $\langle A_\beta\rangle$ and $\langle A_\gamma\rangle$.

Acting upon $\mathcal{S}_{\beta\gamma\delta}$ with a Z_3 subgroup of $\mathcal{G}_{81}$ which is not in $\langle A_\beta, A_\gamma, A_\delta\rangle$, for example with $\langle A_\alpha\rangle$, will produce the siblings $\mathcal{S}'_{\beta\gamma\delta}$, $\mathcal{S}''_{\beta\gamma\delta}$ of $\mathcal{S}_{\beta\gamma\delta}$, these siblings being the images under θ_u of the two affine subspaces in $V(4,3) = (\mathbb{F}_3)^4$ which are translates of $V_{\beta\gamma\delta}$.

Recalling the proof of Lemma 18, let us make the following choices of representatives for the three kinds of 2-dimensional subspaces $V_2 \subset V_{\beta\gamma\delta}$:

$$\text{(i)} \prec\beta,\gamma\succ, \quad \text{(ii)} \prec\delta,\beta+\gamma\succ, \quad \text{(iii)} \prec\beta-\gamma,\beta-\delta\succ.$$

For the choice (i) the nine elements of $V_2 = \prec\beta,\gamma\succ$ are those in the first 9-set in (31). The corresponding section $\theta_u(V_2)$ of $\mathcal{S}_{\beta\gamma\delta}$ is the first of the three $\mathcal{S}_2(2)$ varieties in (32).

For the choice (ii) the nine elements of $V_2 = \prec\delta,\beta+\gamma\succ$ satisfy $\xi_1 = \xi_2$ and are those underlined in:

$$\boxed{\begin{array}{ccc} \underline{0000} & 1221 & 2112 \\ 2121 & \underline{0012} & 1200 \\ 1212 & 2100 & \underline{0021} \end{array}} \xrightarrow{T_\delta} \boxed{\begin{array}{ccc} \underline{2211} & 0102 & 1020 \\ 1002 & \underline{2220} & 0111 \\ 0120 & 1011 & \underline{2202} \end{array}} \xrightarrow{T_\delta} \boxed{\begin{array}{ccc} \underline{1122} & 2010 & 0201 \\ 0210 & \underline{1101} & 2022 \\ 2001 & 0222 & \underline{1110} \end{array}}.$$

We will term the resulting section $\theta_u(V_2)$ of the $\mathcal{S}_3(2)$ a 3-*generator set:* it consists of three parallel generators of $\mathcal{S}_3(2)$ which meet a 'perpendicular' $\mathcal{S}_2(2)$ in three points no two of which lie on the same generator of the $\mathcal{S}_2(2)$.

Finally the nine elements of $V_2 = \prec\beta-\gamma,\beta-\delta\succ$ satisfy $\xi_4 = 0$ and are those underlined in:

$$\boxed{\begin{array}{ccc} \underline{0000} & 1221 & 2112 \\ 2121 & 0012 & \underline{1200} \\ 1212 & \underline{2100} & 0021 \end{array}} \xrightarrow{T_\delta} \boxed{\begin{array}{ccc} 2211 & 0102 & \underline{1020} \\ 1002 & \underline{2220} & 0111 \\ \underline{0120} & 1011 & 2202 \end{array}} \xrightarrow{T_\delta} \boxed{\begin{array}{ccc} 1122 & \underline{2010} & 0201 \\ \underline{0210} & 1101 & 2022 \\ 2001 & 0222 & \underline{1110} \end{array}}.$$

We will term the resulting section $\theta_u(V_2)$ of the $\mathcal{S}_3(2)$ a *fan:*

Definition 19. A subset $\mathcal{F}$ of nine points of a $\mathcal{S}_3(2)$ is a *fan* (= *f*ar-*a*part *n*ine) if no two points of $\mathcal{F}$ lie on the same generator. (So if $\mathcal{F}$ is a fan for a $\mathcal{S}_3(2)$ then the 3 generators through each of the 9 points of $\mathcal{F}$ account for all $3 \times 9 = 27$ generators of $\mathcal{S}_3(2)$.)

We may summarize the foregoing as follows.

Theorem 20. *A section of a Segre variety $\mathcal{S}_3(2)$ in ω_4 is either (i) a $\mathcal{S}_2(2)$, or (ii) a 3-generator set, or (iii) a fan.*

4.3 Hamming distances and troikas

In the GF(3) space $V(4,3) = (\mathbb{F}_3)^4$ we have been employing the basis $\mathcal{B}_\varepsilon := \{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}$, see (20). Using this basis it helps at times to make use of the associated *Hamming distance* $\mathrm{hd}_\varepsilon(\rho,\sigma)$ between two elements $\sigma,\tau \in (\mathbb{F}_3)^4$, as defined by

$$\mathrm{hd}_\varepsilon(\rho,\sigma) = \mathrm{wt}_\varepsilon(\rho-\sigma).$$

Remark 21. If ρ,σ belong to the same row in (21) observe that $\mathrm{hd}_\varepsilon(\rho,\sigma) = 2$, while if ρ,σ belong to different rows then $\mathrm{hd}_\varepsilon(\rho,\sigma)$ is odd.

The next lemma demonstrates that some aspects of orthogonality in the GF(2) space PG(7,2) can be neatly dealt with in GF(3) terms.

Lemma 22. *Two points* $p_\rho, p_\sigma \in \omega_4$ *are orthogonal or non-orthogonal according as* $\mathrm{hd}_\varepsilon(\rho,\sigma)$ *is even or odd.*

Proof. For two points $u_h(i), u_h(j) \in L_h$ we have $u_h(i) \cdot u_h(j) = 1 + \delta_{ij}$. Hence if $\rho = ijkl$ and $\sigma = i'j'k'l'$ it follows that $p_\rho \cdot p_\sigma = \delta_{ii'} + \delta_{jj'} + \delta_{kk'} + \delta_{ll'}$, whence the stated result. ∎

A description of the different sections of a Segre variety $\mathcal{S} \subset \omega_4$ can sometimes be helped by the use of the alternative basis $\mathcal{B}_\Xi := \{\beta, \gamma, \delta, \alpha\}$ for $V(4,3) = (\mathbb{F}_3)^4$. Here, as in (21), $\beta = 1221,\ \gamma = 2121,\ \delta = 2211,\ \alpha = 1111$. The change of basis equations are therefore:

$$
\begin{aligned}
\beta &= \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4, & \gamma &= -\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4,\\
\delta &= -\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4, & \alpha &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4,\\
\varepsilon_1 &= \beta - \gamma - \delta + \alpha, & \varepsilon_2 &= -\beta + \gamma - \delta + \alpha,\\
\varepsilon_3 &= -\beta - \gamma + \delta + \alpha, & \varepsilon_4 &= \beta + \gamma + \delta + \alpha.
\end{aligned} \tag{33}
$$

Observe that the chosen ordering of the elements of the basis $\mathcal{B}_\Xi$ results in the change of basis matrix $\mathbf{M}$ having the simple properties $\mathbf{M}^{\mathrm{t}} = \mathbf{M} = \mathbf{M}^{-1}$.

So in $V(4,3)$ we now have available the Hamming distance in the basis $\mathcal{B}_\Xi$, namely

$$\mathrm{hd}_\Xi(\rho,\sigma) := \mathrm{wt}_\Xi(\rho - \sigma), \quad \rho, \sigma \in (\mathbb{F}_3)^4,$$

where $\mathrm{wt}_\Xi(\lambda)$ denotes the weight of an element $\lambda \in V(4,3)$ in the basis $\mathcal{B}_\Xi$.

Lemma 23.

$$
\begin{aligned}
&\mathrm{wt}_\Xi(\lambda) = 2 \iff \mathrm{wt}_\varepsilon(\lambda) = 2, \quad \mathrm{wt}_\Xi(\lambda) = 3 \iff \mathrm{wt}_\varepsilon(\lambda) = 3,\\
&\mathrm{wt}_\Xi(\lambda) = 1 \iff \mathrm{wt}_\varepsilon(\lambda) = 4, \quad \mathrm{wt}_\Xi(\lambda) = 4 \iff \mathrm{wt}_\varepsilon(\lambda) = 1.
\end{aligned} \tag{34}
$$

Proof. The results (34) follow immediately from (33). ∎

Let us also define the Hamming distance $\mathrm{hd}(p_\rho, p_\sigma)$ between two points $p_\rho, p_\sigma \in \omega_4$ to be

$$\mathrm{hd}(p_\rho, p_\sigma) := \mathrm{hd}_\Xi(\rho, \sigma).$$

Observe that this definition does not depend upon the choice of the point u used in the bijective correspondence θ_u in (19), due to the invariance of hd_Ξ under translations: $\mathrm{hd}_\Xi(\lambda + \rho, \lambda + \sigma) = \mathrm{hd}_\Xi(\rho, \sigma)$. Suppose that we confine our attention to points $p, p', \ldots$ on a particular Segre variety in ω_4, say $\mathcal{S} = \mathcal{S}_{\beta\gamma\delta}$ in (32). Then observe that $\mathrm{hd}(p, p') = d,\ d \in \{1,2,3\}$, provided that p' can be obtained from p only by the use of at least d of the generating subgroups $\langle A_\beta \rangle, \langle A_\gamma \rangle, \langle A_\delta \rangle$ of $\mathcal{S}$. In particular distinct points p and p' lie on the same generator of $\mathcal{S}$ if and only if $\mathrm{hd}(p, p') = 1$. This Hamming distance on $\mathcal{S}$ appears also in [5].

Definition 24. (i) A *troika* on a given $\mathcal{S}_3(2)$ variety $\mathcal{S}$ in ω_4 is a set of three points of $\mathcal{S}$ which are $\mathrm{hd}_\Xi = 3$ apart.

(ii) The *centre* of a troika $t = \{p_1, p_2, p_3\}$ is the point $c(t) = p_1 + p_2 + p_3$.

For example, the three points $u, 246u, 357u \in \mathcal{S}_{\beta\gamma\delta}$ in (32) are $\mathrm{hd}_\Xi = 3$ apart and so form a troika. Alternatively this follows from (34) since $0000, 0111, 0222$ in (31) are $\mathrm{hd}_\varepsilon = 3$ apart.

Theorem 25. *(i) A fan $\mathcal{F}$ for a Segre variety $\mathcal{S}$ in ω_4 can be uniquely expressed as the union $\mathcal{F} = t \cup t' \cup t''$ of a triplet of troikas.*

(ii) The three troikas t, t', t'' in $\mathcal{F}$ share the same centre, say $c_{\mathcal{F}}$.

(iii) Moreover a fan $\mathcal{F}$ for $\mathcal{S}$ determines uniquely a triplet $\mathcal{T} = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$ of fans such that $\mathcal{F} \cup \mathcal{F}' \cup \mathcal{F}'' = \mathcal{S}$. Further, $L(\mathcal{T}) := \{c_{\mathcal{F}}, c_{\mathcal{F}'}, c_{\mathcal{F}''}\}$ is one of the lines of the $\mathcal{H}_7$-tetrad $\mathcal{L}_4$.

Proof. Suppose that $\mathcal{S} = \theta_u(V_3)$ and that $\mathcal{F} \subset \mathcal{S}$ is a fan which contains u.

(i) So $\mathcal{F} = \theta_u(V_2)$, $V_2 \subset V_3$, where the projective line $L = \mathbb{P}V_2$ is of kind Λ_3, with line pattern $(0,3,1,0)$. Consequently L has a unique point $\langle\lambda\rangle$ with $\mathrm{wt}(\lambda) = 3$. Hence the element $0000 \in (\mathbb{F}_3)^4$ has the unique extension $V_1 := \{0000, \lambda, 2\lambda\}$ to a 3-set of elements of V_2 which are Hamming distance 3 apart. So the point u belongs to a unique troika $t \subset \mathcal{F}$, namely $t := \theta_u(V_1) = \{u, A_\lambda u, A_{2\lambda} u\}$. If $V_2 = \prec\lambda, \mu\succ$, the two translates $T_\mu(V_1), T_{2\mu}(V_1)$ of V_1 in V_2 yield two other troikas $t' = A_\mu t, t'' = A_{2\mu} t$, giving rise to the claimed unique decomposition $\mathcal{F} = t \cup t' \cup t''$.

(ii) Since $\mathrm{wt}(\lambda) = 3$, precisely one of the coordinates of λ in the basis $\mathcal{B}_\varepsilon$ is zero. First suppose λ satisfies $\xi_4 = 0$. Then for $\mathbb{P}V_2 = \mathbb{P}(\prec\lambda, \mu\succ)$ to be of kind Λ_3 the element μ must also satisfy $\xi_4 = 0$. So a point $p \in \mathcal{F}$ must be of the form U_{ijk0}. Hence that troika $\{p, A_\lambda p, A_{2\lambda} p\} \subset \mathcal{F}$ which contains p has centre

$$c = (I + A_\lambda + (A_\lambda)^2) U_{ijk0} = U_{\emptyset\emptyset\emptyset 0}. \tag{35}$$

So the same point $c = U_{\emptyset\emptyset\emptyset 0}$, which lies on the line $L_d \in \mathcal{L}_4$, is the centre of each of the three troikas in $\mathcal{F}$. Of course, if instead λ satisfies $\xi_i = 0$ for $i = 1,2,3$ then the analogous reasoning shows that the common centre of the three troikas in $\mathcal{F}$ is $U_{0\emptyset\emptyset\emptyset} \in L_a$, $U_{\emptyset 0\emptyset\emptyset} \in L_b$ or $U_{\emptyset\emptyset 0\emptyset} \in L_c$, according as $i = 1, 2$ or 3.

(iii) We are dealing with $\mathcal{S} = \theta_u(V_3)$ where V_3 is of the form $V_3 = V_2 \oplus \prec\nu\succ$, $V_2 = \prec\lambda, \mu\succ$, and where we may choose ν to have $\mathrm{wt}_\varepsilon = 4$. The fan $\mathcal{F} = \theta_u(V_2)$ determines a triplet $\mathcal{T} = \{\mathcal{F}, \mathcal{F}', \mathcal{F}''\}$ of fans, where $\mathcal{F}' = A_\nu(\mathcal{F})$ and $\mathcal{F}'' = A_{2\nu}(\mathcal{F})$, such that $\mathcal{F} \cup \mathcal{F}' \cup \mathcal{F}'' = \mathcal{S}$. Moreover if $c_{\mathcal{F}} = U_{\emptyset\emptyset\emptyset 0}$ as in (35) then $c_{\mathcal{F}'} = A_\nu c_{\mathcal{F}}$, $c_{\mathcal{F}''} = A_{2\nu} c_{\mathcal{F}}$ will be the other two points $U_{\emptyset\emptyset\emptyset 1}, U_{\emptyset\emptyset\emptyset 2}$ of the line L_d. Similarly for the other three cases considered in (ii) above, where $\{c_{\mathcal{F}}, c_{\mathcal{F}'}, c_{\mathcal{F}''}\}$ is one of the other lines of the tetrad $\mathcal{L}_4$. ∎

In the paper [11] the fact that a Segre variety $\mathcal{S} = \mathcal{S}_3(2)$ in $\mathrm{PG}(7,2)$ determines a distinguished tetrad $\mathcal{L}_4$ of lines which span $\mathrm{PG}(7,2)$ only emerged rather late, see [11, Section 4.1]. One of the motivations for the present paper was to come to a clearer understanding of the relationship between $\mathcal{S}$ and $\mathcal{L}_4$. This can now be achieved: see the next theorem, where it is shown how to obtain the same tetrad $\mathcal{L}_4$ from any of the 24 copies of a Segre variety $S_3(2)$ in the 81-set ω_4.

Theorem 26. *A Segre variety $\mathcal{S}$ in ω_4 determines precisely four triplets $\mathcal{T}_i$, $i \in (1,2,3,4)$ of fans; further the resulting four lines $L_i := L(\mathcal{T}_i)$ are the four lines of the $\mathcal{H}_7$-tetrad $\mathcal{L}_4$.*

Proof. A Segre variety in ω_4 is of the form $\mathcal{S} = \theta_u(V_3)$ where the projective plane $P = \mathbb{P}V_3 \subset \mathrm{PG}(3,3)$ is of kind $\mathcal{P}_0$. Now, see (31), there are precisely 8 elements of V_3 of weight 3, and these form 4 pairs, say $\{\pm\lambda_1\}$, $\{\pm\lambda_2\}$, $\{\pm\lambda_3\}$, $\{\pm\lambda_3\}$. Consequently the element $0000 \in V_3$ has precisely 4 extensions, namely

$\{0000, \lambda_i, -\lambda_i\}$, $i = 1,2,3,4$, to a 3-set of elements of V_3 which are Hamming distance 3 apart. Hence the point $u = \theta_u(0000)$ lies in precisely 4 troikas and, by Theorem 25(i), in precisely 4 fans $\mathcal{F}_i$, $i = 1,2,3,4$. By Theorem 25(iii) each of the resulting 4 triplets $\mathcal{T}_i = \{\mathcal{F}_i, \mathcal{F}'_i, \mathcal{F}''_i\}$ of fans determines a line $L(\mathcal{T}_i)$ of the tetrad $\mathcal{L}_4$. Further, from the proof of Theorem 25(iii), we see that these lines are distinct. ∎

4.4 Future research

We are of the opinion that some further investigation of the denizens of ω_4, and of their interactions, should prove worthwhile. Moreover such an investigation should not be confined to the 27-set denizens arising from Theorem 9, since at least some of the 9-set denizens of ω_4 arising from Theorem 10 deserve attention. In particular the 9-sets in ω_4 which arise from those lines in the table (23) which have weight pattern $(0,0,4,0)$ are certainly noteworthy. For suppose that $V(2,3) \subset V(4,3)$ is such that $L = \mathbb{P}V(2,3)$ is of kind Λ_7, and so $\mathrm{wt}_\varepsilon(\rho) = 3$ for every nonzero element $\rho \in V(2,3)$. It follows that any pair of distinct elements ρ, σ of $V(2,3)$ are Hamming distance 3 apart, and hence, by Lemma 22, the points p_ρ, p_σ are non-perpendicular: $p_\rho \cdot p_\sigma = 1$. *In this easy manner we have constructed a 9-cap* $\mathcal{N} = \{p_\rho\}_{\rho \in V(2,3)}$ *on the quadric* $\mathcal{H}_7$. Moreover under the action on $\mathcal{N}$ of $\mathcal{G}_{81}$ we will obtain a partition of the 81-set ω_4 into an ennead of quadric 9-caps.

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