Recurrence properties of a class of nonautonomous discrete systems

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Abstract

We study recurrence properties for the nonautonomous discrete system given by a sequence $(f_n)_{n=1}^{\infty}$ of continuous selfmaps on a compact metric space. In particular, our attention is paid to the case when the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to a map f or forms an equicontinuous family. In the first case we investigate the structure and behavior of an ω-limit set of (f_n) by a dynamical property of the limit map f. We also present some examples of (f_n) and f on the closed interval: (a) $\omega(x,(f_n)) \setminus \Omega(f) \neq \emptyset$ for some point x; or (b) the set of periodic points of f is closed and for some point x, $\omega(x,(f_n))$ is infinite. In the second case we create a perturbation of (f_n) whose nonwandering set has small measure.

1 Introduction

In this article, we consider difference equations that can be written in the form

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{N}$$

having an initial point $x_1 = x$ in a compact metric space X and where f_n are continuous maps of X to itself. These equations define a *nonautonomous discrete system* $(X,(f_n))$ ([17]). The sequence

$$x, f_1(x), f_2 \circ f_1(x), \dots, f_n \circ \cdots \circ f_1(x), \dots$$

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beginning at the initial point $x \in X$ is called the *trajectory of* x under (f_n). If $f_n = f$ for any $n \in \mathbb{N}$, the pair $(X, (f))$ is an autonomous discrete system. In this case, we may regard the pair as a classical discrete system.

A major objective in the study of dynamical systems is to describe the eventual demeanor of a map; we are here mostly interested in understanding the structure and behavior of limit points and nonwandering points of a nonautonomous discrete system (f_n). The set of limit points of the trajectory of x under (f_n) is called the *ω-limit set* of x, denoted by $\omega(x,(f_n))$; more precisely, $z \in \omega(x,(f_n))$ if there exists a strictly increasing sequence of positive numbers $n_i, i \in \mathbb{N}$, such that $z = \lim_{i\to\infty} f_{n_i} \circ \cdots \circ f_1(x)$. This limit set is obviously nonempty and closed. A point $x \in X$ is called *nonwandering* for (f_n) provided for every open neighborhood U of x and every $N \in \mathbb{N}$, there exists k $(N \leq k)$ such that $f_1^k(U) \cap U \neq \emptyset$, where $f_1^k = f_k \circ \cdots \circ f_1$ ([7]). The set of all nonwandering points for (f_n) is called the *nonwandering set* and is denoted by $\Omega((f_n))$. The nonwandering set is also nonempty and closed. The set $\Omega((f_n))$ contains all ω-limit sets $\omega(x,(f_n))$.

We restrict our attention to two particular cases of nonautonomous discrete systems; that is, we assume that the sequence (f_n) converges uniformly to a map f, or the sequence forms an equicontinuous family. Our motivation for studying the relation between (f_n) and f in this particular context is provided by the following results. The first one (for systems on intervals) is by R. Kempf:

Theorem 1.1 ([16, Theorem 3])**.** *Let (f_n) be a nonautonomous discrete system on a compact interval I such that the sequence (f_n) converges uniformly to a map $f : I \to I$ which has fixed points but no other periodic points. Then any ω-limit set of (f_n) is a closed interval, possibly degenerated, consisting of fixed points of f only.*

Remark 1. Theorem 1.1 still holds when "interval" is replaced by "tree" [24].

The second is the result (for systems on compact metric spaces) by J.S. Cánovas, which considers the case when the limit map f has the shadowing property.

Theorem 1.2 ([6, Theorem 1.1])**.** *Let (f_n) be a nonautonomous discrete system on a compact metric space X such that the sequence (f_n) converges uniformly to a map $f : X \to X$. If f has the shadowing property, then any limit point of any trajectory of (f_n) is a non-wandering point of f.*

In the interval case, Cánovas also established a stronger result.

Theorem 1.3 ([6, Theorem 1.2])**.** *Let (f_n) be a nonautonomous discrete system on a compact interval I such that the sequence (f_n) converges uniformly to a map $f : I \to I$. If f has the shadowing property, then any ω-limit set of (f_n) is an ω-limit set of f.*

Remark 2. This theorem is still true for systems on "graphs" following the argument in [6] together with Theorem 3.1 in [19] (see [7, Conjecture]).

It is hence important to study the structure and behavior of an ω-limit set of (f_n) and the dynamical relationship between (f_n) and f.

We state a reformulation (Theorem 3.6) of the result of ω-limit sets by Šarkovs'kiĭ [23], in terms of a nonautonomous discrete system; the formulation may help us to understand the structure of an ω-limit set. Cánovas' theorem 1.2 is expanded to the case of all limit maps, however, by using the notion of chain recurrence (Theorem 3.7). This with the structural theorem above includes a part of Kempf's theorem 1.1 (and also Sun's one [24]) (Corollary 3.8).

Kolyada and Snoha [17, Theorem H] generalized for the case of nonautonomous discrete systems the classical result ([5]) stating the topological entropy is concentrated on the nonwandering set (their definition of nonwandering set is slightly different from ours). We next explore a property of the nonwandering points of those systems. In the case when the sequence (f_n) forms an equicontinuous family, we create a perturbation of (f_n) whose nonwandering set has small measure (Theorem 4.1). This is a generalization in spirit of the fact that in many autonomous discrete systems the chain recurrent set turns to be small [13, 25].

We also present some examples in the last section. In particular, we construct examples of (f_n) and f on the closed interval $[0,1]$: (a) $\omega(x,(f_n)) \setminus \Omega(f) \neq \emptyset$ for some point x (Example 1); or (b) the set of periodic points of f is closed and for some point x, $\omega(x,(f_n))$ is the union of finitely many disjoint nondegenerate closed intervals consisting of periodic points of f (Examples 2 to 4). In fact, the (f_n) of Example 2 possess $\omega(x,(f_n)) = [0,1]$ for every $x \in [0,1]$. The first example (a) shows that Theorems 1.2 and 1.3 do not hold in general, if the limit map f does not have the shadowing property. The second one (b) states that an ω-limit set $\omega(x,(f_n))$ may be infinite, although the set of periodic points $P(f)$ is closed. When considering the case of an autonomous discrete system on a closed interval, there exist no minimal map, and if $P(f)$ is closed then every ω-limit set $\omega(x,f)$ must be finite (i.e., it is a periodic orbit of f). The diversity of nonautonomous discrete systems is indicated.

We finally remark that details on their motivations are also given by Kolyada and Snoha [17, Introduction] from the viewpoint of topological entropy, and the several results concerning the topological entropy or chaos can be found in [17, 20, 8].

2 Preliminaries

We now give the terminology and notation needed in what follows. A *map on X* is a continuous function $f : X \to X$ from a space X to itself; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. By a *graph*, we mean a compact connected one-dimensional branched manifold. A *tree* is a graph without cycles. For a map f on a space X, a point $x \in X$ is called a periodic point of f with period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < n$. If $n = 1$, the point is said to be fixed. We denote by $P(f)$ and $F(f)$ the set of periodic points and of fixed points of f, respectively.

We let $f : X \to X$ be a map on a compact metric space (X,d). Let $x, y \in X$. An *ε-chain from x to y* is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of X such that $x_0 = x$, $x_n = y$ and $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$. In the case when a sequence of points $\{x_i\}_{i=0}^{\infty}$ has the same metric property (i.e., $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, 2, \ldots$), the sequence is said to be an *ε-pseudo orbit* of f. The map f has the *shadowing property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit is ε-shadowed by an orbit of f (that is, $d(x_i, f^i(x)) < \varepsilon, i = 0, 1, 2 \ldots$ for some $x \in X$). We say *x can be chained to y* if for every $\varepsilon > 0$ there exists an ε-chain from x to y, and we say x is *chain recurrent* if it can be chained to itself. The set of all chain recurrent points is called the *chain recurrent set* of f and denoted by $\mathrm{CR}(f)$ ([9]). Note that the chain recurrent set contains all nonwandering points in that including the "genuine"recurrent points x (i.e., such that x belongs to the closure of its forward orbit), minimal subsets and periodic orbits. We need the following lemma by Block and Franke which gives a characteristic property of chain recurrent points.

Lemma 2.1 ([3, Theorem A])**.** *Let $f : X \to X$ be a map on a compact metric space X and $x \in X$. Then $x \notin \mathrm{CR}(f)$ if and only if there exists an open set U of X such that $x \notin \mathrm{Cl}\, U$, $f(x) \in U$ and $f(\mathrm{Cl}\, U) \subseteq U$.*

We also use the next lemma which characterizes a non-chaotic map.

Lemma 2.2 ([2, Theorem], [18, Theorem A])**.** *Let $f : T \to T$ be a map on a tree T. If the set of periodic points of f is closed, then every chain recurrent point is periodic.*

We finally state fundamental facts from geometric topology and measure theory. The dimension $\dim X$ of a space X means the covering dimension (see [11] and [22]). A space X is said to be *locally $(n-1)$-connected* if for every $x \in X$ and every neighborhood U of x in X, there exists a neighborhood $V \subseteq U$ of x in X such that every map $f : S^k \to V$ extends to a map $\tilde{f} : B^{k+1} \to U$ for every $0 \le k \le n-1$, where S^k and B^{k+1} stand for the unit k-dimensional sphere and the unit $(k+1)$-dimensional ball of the $(k+1)$-dimensional Euclidean space, respectively. We recall a characteristic property of locally $(n-1)$-connected spaces; it is slightly rephrased here.

Lemma 2.3 ([4, p. 80], [14, p. 150])**.** *Let X be a compact metric space and $k \in \mathbb{N}$. Then X is locally $(k-1)$-connected if and only if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every map $\varphi : A \to X$ from a closed set A of a compact metric space Z with $\dim Z \smallsetminus A \le n$ and $\mathrm{diam}[\varphi(A)] < \delta$, there exists an extension $\tilde{\varphi} : Z \to X$ of φ satisfying $\mathrm{diam}[\tilde{\varphi}(Z)] < \varepsilon$.*

We also need the following elementary facts later.

Proposition 2.4. *Let μ be a finite Borel measure on a compact metric space X. Then*

(1) there exists a basis of the topology consisting of sets whose boundaries have μ-measure zero;

(2) every nonatomic point $x \in X$ has a neighborhood basis $\{U_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} \mu(U_k) = 0$.

3 Structural properties of the omega-limit set

The first basic property of an ω-limit set of a nonautonomous discrete system was given by Kempf:

Lemma 3.1 ([16, Theorem 2], [6, Proposition 2.1]). *Let (f_n) be a nonautonomous discrete system on a compact metric space X such that the sequence (f_n) converges uniformly to a map $f : X \to X$. For any $x \in X$, the ω-limit set $\omega(x, (f_n))$ is strongly invariant under f, that is, $f(\omega(x, (f_n))) = \omega(x, (f_n))$.*

The next lemma is a variation of the result due to Šarkovs'kiĭ [23], in a nonautonomous discrete system. Since the statement is the key to the proof of Theorem 3.6, we include here a proof for completeness.

Lemma 3.2. *Let (f_n) be a nonautonomous discrete system on a compact metric space (X, d) such that the sequence (f_n) converges uniformly to a map $f : X \to X$. If K is a nonempty proper closed set of an ω-limit set $\omega(x, (f_n))$, then $K \cap \mathrm{Cl}[f(\omega(x, (f_n)) \setminus K)] \neq \emptyset$.*

Before proving Lemma 3.2, we need a notation and proposition which was stated in the proof of Theorem 1.1 in [6]. Let $(X, (f_n))$ be a nonautonomous discrete system. For $m \geq n$, we denote by f_n^m the composite $f_m \circ \cdots \circ f_n$. In particular, we let f_1^0 be the identity map on X.

Proposition 3.3. *Let (f_n) be a nonautonomous discrete system on a compact metric space (X, d) such that the sequence f_n converges uniformly to a map $f : X \to X$. For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $x \in X$, the sequence $\{f_1^n(x)\}_{n \geq n_0}$ is an ε-pseudo orbit of f.*

Proof. Given $\varepsilon > 0$, we let $n_0 \in \mathbb{N}$ be such that the distance between f_n and f is less than ε for all $n \geq n_0$. Then for $x \in X$ and $n \geq n_0$,

$$d(f(f_1^n(x)), f_1^{n+1}(x)) = d(f(f_1^n(x)), f_{n+1}(f_1^n(x))) < \varepsilon.$$

This implies that $\{f_1^n(x)\}_{n \geq n_0}$ is an ε-pseudo orbit of f. ∎

Proof of Lemma 3.2. The strategy of our proof comes from Lemma IV3 in [1]. Assume on the contrary that $K \cap \mathrm{Cl}[f(\omega(x, (f_n)) \setminus K)] = \emptyset$.

Let G_1 and G_2 be open sets of X such that $\omega(x, (f_n)) \setminus K \subseteq G_1$, $K \subseteq G_2$, and $f(\mathrm{Cl}\, G_1) \cap \mathrm{Cl}\, G_2 = \emptyset$. We put $\varepsilon = d(f(\mathrm{Cl}\, G_1), \mathrm{Cl}\, G_2)$. By $\omega(x, (f_n)) \subseteq G_1 \cup G_2$ and Proposition 3.3, we can take a sequence $n_1 < n_2 < \cdots$ such that $f_1^{n_k}(x) \in G_1, f_1^{n_k+1}(x) \in G_2$ for $k = 1, 2, \ldots$, and $\{f_1^n(x)\}_{n \geq n_1}$ is an $\varepsilon/3$-pseudo orbit of f.

Now, we let $y \in \mathrm{Cl}\, G_1$ be such that $y = \lim_{j \to \infty} f_1^{n_{k_j}}(x)$. For a sufficiently large j_0 satisfying $d(f(y), f(f_1^{n_{k_{j_0}}}(x))) < \varepsilon/3$, we then deduce

$$\begin{aligned} d(f(y), f_1^{n_{k_{j_0}}+1}(x)) &\leq d(f(y), f(f_1^{n_{k_{j_0}}}(x))) + d(f(f_1^{n_{k_{j_0}}}(x)), f_1^{n_{k_{j_0}}+1}(x)) \\ &< \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3; \end{aligned}$$

however, it is a contradiction. ∎

We also need two elementary facts.

Proposition 3.4. *Let (f_n) be a nonautonomous discrete system on a compact metric space X. For $k \in \mathbb{N}$ and $x \in X$, we have*

$$\omega(x,(f_n)) = \bigcup_{i=0}^{k-1} \omega(f_1^i(x),(f_{(\ell-1)k+i+1}^{\ell k+i})_\ell),$$

where for $0 \le i < k$, $(f_{(\ell-1)k+i+1}^{\ell k+i})_\ell$ means the sequence of maps

$$f_{i+1}^{k+i}, f_{k+i+1}^{2k+i}, f_{2k+i+1}^{3k+i}, \dots, f_{(\ell-1)k+i+1}^{\ell k+i}, \dots$$

Proof. Let $z \in \omega(x,(f_n))$ and let $n_1 < n_2 < \dots$ be a sequence such that $f_1^{n_\ell}(x) \to z$ (as $\ell \to \infty$). Take $0 \le p < k$, $\ell_1 < \ell_2 < \dots$, and $m_1 < m_2 < \dots$ satisfying $n_{\ell_j} = m_j k + p$ for $j \in \mathbb{N}$. Then we find that $f_{p+1}^{m_j k+p}(f_1^p(x)) = f_1^{n_{\ell_j}}(x) \to z$; hence $z \in \omega(f_1^p(x),(f_{(\ell-1)k+p+1}^{\ell k+p})_\ell)$.

The converse inclusion is obvious. ∎

Proposition 3.5. *Let (f_n) be a sequence of maps on a compact metric space (X,d) which converges uniformly to a map $f : X \to X$, and $k \in \mathbb{N}$. Then for $0 \le i < k$, the sequence $(f_{(\ell-1)k+i+1}^{\ell k+i})_\ell$ is uniformly converging to f^k.*

Proof. Since the proof is straightforward, the details are left to the reader. ∎

We establish a reformulation of the result due to Šarkovs'kiĭ [23], in terms of a nonautonomous discrete system, by employing a strategy similar to that underlying the proof of Lemma IV4 in [1]. The formulation help us to understand the structure of an ω-limit set.

Theorem 3.6. *Let (f_n) be a nonautonomous discrete system on a compact metric space (X,d) such that the sequence f_n converges uniformly to a map $f : X \to X$. Then we have the following:*

(1) an ω-limit set $\omega(x,(f_n))$ is finite if and only if it is a periodic orbit of f;

(2) if an ω-limit set $\omega(x,(f_n))$ is infinite, then no isolated point of $\omega(x,(f_n))$ is periodic with respect to f.

Proof. (1): Assume that $W = \omega(x,(f_n))$ is finite. Since $f(W) = W$ by Lemma 3.1, it follows that W contains a periodic orbit P of f. Note that $(W \setminus P) \cap \mathrm{Cl}\, f(P) = (W \setminus P) \cap P = \emptyset$. If $W \setminus P$ was nonempty, it would follow that W is infinite, by the nonclosedness of $W \setminus P$ from Lemma 3.2. This is impossible, therefore $W = P$.

The converse is obvious.

(2): Assume that $W = \omega(x,(f_n))$ contains a periodic orbit P of f with period k, and P has an isolated point y in W. We shall show $W = P$, that is, W is finite.

Using Proposition 3.4, let $j \in \{0,\dots,k\}$ be such that

$$y \in W_j = \omega(f_1^j(x),(f_{(\ell-1)k+j+1}^{\ell k+j})_\ell).$$

Then y is isolated in W_j and the sequence $(f_{(\ell-1)k+j+1}^{\ell k+j})_\ell$ converges uniformly to f^k as $\ell \to \infty$ by Proposition 3.5. As we know $(W_j \setminus \{y\}) \cap \mathrm{Cl}\, f^k(\{y\}) = (W_j \setminus \{y\}) \cap \{y\} = \emptyset$, it follows from Lemma 3.2 and the closedness of $W_j \setminus \{y\}$ that $W_j = \{y\}$. This equality implies

$$f_1^{\ell k+j}(x) = f_{(\ell-1)k+j+1}^{\ell k+j} \circ \cdots \circ f_{j+1}^{k+j}(f_1^j(x)) \to y \text{ as } \ell \to \infty.$$

For $1 \leq i < k$, using (uniform) continuity of f, Proposition 3.3 and the following inequality (for all sufficiently large ℓ)

$$\begin{aligned} &d(f^i(y), f_1^{\ell k+j+i}(x)) \\ &\leq d(f^i(y), f^i(f_1^{\ell k+j}(x))) + d(f^i(f_1^{\ell k+j}(x)), f^{i-1}(f_1^{\ell k+j+1}(x))) + \cdots \\ &+ d(f^2(f_1^{\ell k+j+i-2}(x)), f(f_1^{\ell k+j+i-1}(x))) + d(f(f_1^{\ell k+j+i-1}(x)), f_1^{\ell k+j+i}(x))), \end{aligned}$$

we can easily see that, for $0 \leq i < k$,

$$f_1^{\ell k+j+i}(x) \to f^i(y) \text{ as } \ell \to \infty.$$

This shows that, for $0 \leq i < k$,

$$\omega(f_1^i(x), (f_{(\ell-1)k+i+1}^{\ell k+i})_\ell) = \{f^{k-j+i}(y)\},$$

(note $f^k(y) = y$), namely $W = P$. Therefore W is finite. ∎

We next generalize Theorem 1.2 ([6, Theorem 1.1]) to all limit maps by using the notion of chain recurrence. We note that if f has the shadowing property, then the nonwandering set $\Omega(f)$ coincides the chain recurrent set $\mathrm{CR}(f)$.

Theorem 3.7. *Let (f_n) be a nonautonomous discrete system on a compact metric space (X, d) such that the sequence f_n converges uniformly to a map $f : X \to X$. Then any limit point of any trajectory of (f_n) is in the chain recurrent set of f.*

Proof. Assume on the contrary that $x \in \omega(x_0, (f_n)) \setminus \mathrm{CR}(f)$. By Lemma 2.1, we take an open set U in X such that $x \notin \mathrm{Cl}\, U$, $f(x) \in U$, and $f(\mathrm{Cl}\, U) \subseteq U$. Put

$$\varepsilon = \min\{d(x, \mathrm{Cl}\, U),\ d(f(\mathrm{Cl}\, U), X \setminus U)\}.$$

By uniform convergence of f_n to f, we let $n_0 \in \mathbb{N}$ be such that the distance between f_n and f is less than $\varepsilon/2$ for all $n \geq n_0$. We note here that, for $z \in U$ and $n \geq n_0$,

$$f_n(z) \in \mathbb{B}(f(\mathrm{Cl}\, U); \varepsilon/2) \subseteq U,$$

where $\mathbb{B}(f(\mathrm{Cl}\, U); \varepsilon/2)$ is the $\varepsilon/2$-neighborhood of $f(\mathrm{Cl}\, U)$.

Since $f(x)$ is also an element of $f(\omega(x_0, (f_n))) = \omega(x_0, (f_n))$, we can choose a natural number $n_1 > n_0$ such that $f_{n_1} \circ \cdots \circ f_1(x_0) \in U$. Then we find that, for all $n \geq n_1 (\geq n_0)$,

$$f_n \circ \cdots \circ f_1(x_0) \in U.$$

This implies $x \in \mathrm{Cl}\, U$; however it is a contradiction. ∎

We obtain the following which is a generalizing of Theorem 1.1.

Corollary 3.8. *Let (f_n) be a nonautonomous discrete system on a tree T such that the sequence f_n converges uniformly to a map $f : T \to T$. If the set of periodic points $P(f)$ is closed, then any limit point of any trajectory of (f_n) is in $P(f)$, and moreover, an ω-limit set $\omega(x, (f_n))$ is either a periodic orbit of f, or dense in itself (i.e., no isolated point in the set).*

Proof. The first statement is from Lemma 2.2 and Theorem 3.7, and the second one is by Theorem 3.6. ∎

Remark 3. We note (on intervals) that if $f_n = f$ for $n \in \mathbb{N}$ and $P(f)$ is closed, then $\omega(x, (f_n))$ must be finite for any $x \in I$. However, in general, $\omega(x, (f_n))$ may be infinite, although $P(f)$ is closed (see Examples 2, 3, and 4).

4 The nonwandering set and the size for a given measure

The definition of nonwandering set for nonautonomous discrete systems was early introduced by Kolyada and Snoha [17] for computing topological entropy on the nonwandering set for equicontinuous sequences. However, as Cánovas points out in [7, 8], this set (for their definition) may be quite big while the dynamics of the system can be quite simple. Consider the sequence $(g, f, f, \dots)$, where $g(x) = 0$ for $0 \le x \le 1$ and f is the tent map. Therefore, we use an alternative definition of nonwandering set (by Cánovas [7]) in order to avoid the existence of those critical examples (see Introduction for definition).

The distance between nonautonomous discrete systems (f_n) and (g_n) on a compact metric space is defined by the supremum of the uniform metrics f_n and g_n, $n \in \mathbb{N}$.

Theorem 4.1. *Let (X, d) be a k-dimensional locally $(k-1)$-connected compact metric space, where $k \ge 0$ (for $k = 0$ we simply skip the local connectedness assumption), and μ be a finite Borel measure on X without atoms at the isolated points of X. Let (f_n) be a nonautonomous discrete system on X such that the sequence (f_n) is equicontinuous. Then for every $\varepsilon > 0$ and $\alpha > 0$, there exist a nonautonomous discrete system (g_n) with $d((f_n), (g_n)) < \varepsilon$ and $\gamma > 0$ such that, for every nonautonomous discrete system (h_n) with $d((h_n), (g_n)) < \gamma$, we have $\mu(\Omega((h_n))) < \alpha$.*

Proof. We shall prove this by a process similar to that used for Theorem 3.1 in [25]. For the convenience of the reader we repeat the relevant material (concerning construction of geometric and measurable tools), thus making our exposition self-contained.

Choose $\varepsilon > 0$ and $\alpha > 0$ arbitrarily. Corresponding to the number $\varepsilon/2$, let $\delta = \delta(\varepsilon/2) < \varepsilon/2$ be as in Lemma 2.3 (or $\delta = \varepsilon/2$ in the case $k = 0$). The equicontinuity of (f_n) yields a positive number $\xi < \delta/2$ such that if $A \subseteq X$ with $\operatorname{diam}[A] \le \xi$, then $\operatorname{diam}[f_n(A)] < \delta/2$ for any $n \in \mathbb{N}$. Let $\{z_1, \dots, z_p\}$ be a $\xi/2$-dense set avoiding any atoms on μ. By Proposition 2.4 (1) and a standard argument, we can easily choose pairwise disjoint open neighborhood V_i of z_i, each of diameter at most ξ, whose union has full measure. Next, using regularity of μ, we can find open sets $W_i \ni z_i$ such that $\mathrm{Cl}\, W_i \subseteq V_i$ and $\mu(X \setminus \bigcup_{i=1}^{p} W_i) < \alpha$. In case $\dim X = 0$, we choose sets V_i which are both open and closed, and we simply let $W_i = V_i$.

Let $n \in \mathbb{N}$. For every $i \in \{1, \dots, p\}$, let $m_n(i) \in \{1, \dots, p\}$ be such that $d(f_n(z_i), z_{m_n(i)}) < \xi/2$. Then we define g_n by $g_n(x) = z_{m_n(i)}$ on $\mathrm{Cl}\, W_i$, and we let $g_n = f_n$ on the complement of the union of the V_i's. We note that the diameter of the image $g_n(\mathrm{Cl}\, V_i)$ (where defined) is at most $\mathrm{diam}[\{z_{m_n(i)}\} \cup f_n(\mathrm{Cl}\, V_i)] < \delta$. Hence, it follows from Lemma 2.3 (separately on each $\mathrm{Cl}\, V_i$) that we can extend the map to a map g_n on the whole of $\mathrm{Cl}\, V_i$ such that $\mathrm{diam}[g_n(\mathrm{Cl}\, V_i)] < \varepsilon/2$ (In case $\dim X = 0$ there exists nothing to do in this step). The map $g_n : X \to X$ is well-defined and continuous, and the distance between f_n and g_n is less than ε.

Using nonatomic property of $\{z_i\}$ together with Proposition 2.4 (2), let $\gamma > 0$ be such that $\mathbb{B}(z_i; \gamma) \subseteq W_i$ for $i \in \{1, \dots, p\}$ and

$$\mu(X \setminus \bigcup_{i=1}^{p} W_i) + \mu(\bigcup_{i=1}^{p} \mathrm{Cl}\, \mathbb{B}(z_i; \gamma)) < \alpha.$$

Put $U = \bigcup_{i=1}^{p} \mathbb{B}(z_i; \gamma)$. Then we note that, for each $i \in \{1, \dots, p\}$, $g_n(\mathrm{Cl}\, W_i) = z_{m_n(i)} \in \mathbb{B}(z_{m_n(i)}; \gamma) \subseteq U$.

For any nonautonomous discrete system $(h_n) : X \to X$ with $d((g_n), (h_n)) < \gamma$, we see that

$$h_n(\mathrm{Cl}\, U) \subseteq h_n(\bigcup_{i=1}^{p} \mathrm{Cl}\, W_i) \subseteq U \text{ for all } n;$$

hence we find that

$$\left((\bigcup_{i=1}^{p} W_i) \setminus \mathrm{Cl}\, U \right) \cap \Omega((h_n)) = \emptyset.$$

We therefore establish that the set $\Omega((h_n))$ has measure less than α. ∎

5 Examples

Example 1. We construct a nonautonomous discrete system (f_n) on the unit interval $I = [0, 1]$ which converges uniformly to a map $f : I \to I$ such that $\omega(z, (f_n)) \setminus \Omega(f) \neq \emptyset$ for some point $z \in I$, where $\Omega(f)$ means the nonwandering set of f.

[Construction]: The map $f : I \to I$ is defined by

$$f(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/6, \\ 1/3 & \text{for } 1/6 \le x \le 1/3, \\ 2x - 1/3 & \text{for } 1/3 \le x \le 2/3, \\ -3x + 3 & \text{for } 2/3 \le x \le 1. \end{cases}$$

We note that $(0, 1/3) \cap \Omega(f) = \emptyset$ and $[0, 1/3] \subseteq \mathrm{CR}(f)$.

Let $z_0 = 17/18$ and $z_1 = 37/54$ ($f(z_1) = z_0$ and $f(z_0) = 1/6$). Take

$$\cdots < z_{n+1} < z_n < \cdots < z_2 < z_1 < z_0$$

in a recursive way such that $z_n \in (1/3, 2/3)$ for $n \geq 2$ and $f(z_{n+1}) = z_n$ for $n \in \mathbb{N}$. Then note $z_n \to 1/3$. For $n \in \mathbb{N}$, the map $f_n : I \to I$ is defined by

$$f_n(x) = \begin{cases} 6z_n x & \text{for } 0 \leq x \leq 1/6, \\ z_n & \text{for } 1/6 \leq x \leq z_{n+1}, \\ f(x) & \text{for } z_{n+1} \leq x \leq 1, \end{cases}$$

see Figure 1. We can easily see that $f_n \to f$ uniformly, and for all $n \in \mathbb{N}$

$$f_1^{3(2^n-1)}(1/6) = 1/6$$

by an inductive argument. This implies $1/6 \in \omega(1/6, (f_n)) \setminus \Omega(f)$.

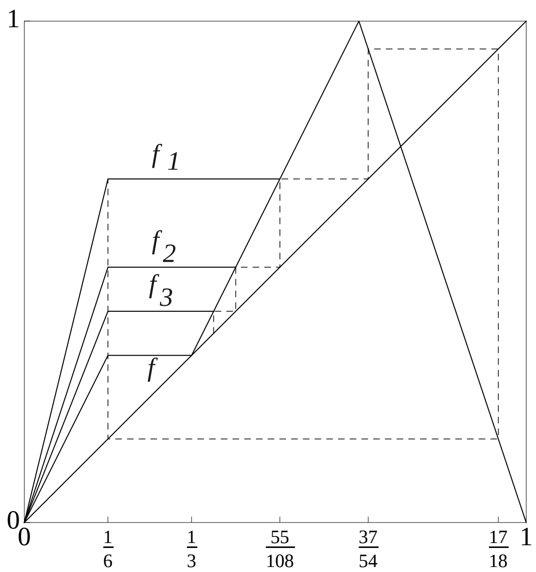

Figure 1:

Remark 4. Example 1 shows that Theorems 1.2 and 1.3 do not hold in general, if f does not have the shadowing property. We also note that $\omega(1/6, (f_n)) = \{1/6, 1/3\} \cup \{z_n \mid n \geq 0\}$ and $1/6 \notin \omega(x, f)$ for any $x \in [0, 1]$.

As stated in Remark 3, we construct three examples of nonautonomous discrete systems on the unit interval such that for some point x, the limit set $\omega(x,(f_n))$ is the union of closed intervals, although the limit map is strongly non-chaotic (i.e., every point is asymptotically periodic).

Example 2. We construct a nonautonomous discrete system (f_n) on the unit interval $I = [0,1]$ which converges uniformly to a map $f : I \to I$ such that $P(f) = F(f)$ and $\omega(0,(f_n)) = I$ consisting of fixed points of f.

[Construction]: [1] Let $f : I \to I$ be the identity map. For $k \in \mathbb{N}$, the map $g_k : I \to I$ is defined by

$$g_k(x) = \begin{cases} \min\{x + 1/2^k, 1\} & \text{if } k \text{ is odd,} \\ \max\{x - 1/2^k, 0\} & \text{if } k \text{ is even,} \end{cases}$$

(see Figure 2). We define the sequence (f_n) as

$$\underbrace{g_1, g_1,}_{2^1 \text{ times}} \underbrace{g_2, \dots, g_2,}_{2^2 \text{ times}} \underbrace{g_3, \dots, g_3,}_{2^3 \text{ times}} \cdots$$

Then $P(f) = F(f)$, $f_n \to f$ uniformly, and

$$\{f_1^n(0) \mid n = 0,1,\dots\} = \{\ell/2^k \mid \ell = 0,1,\dots,2^k, k = 1,2,\dots\};$$

hence we see $\omega(0,(f_n)) = I$. The construction also shows that $\omega(x,(f_n)) = I$ for every $x \in I$.

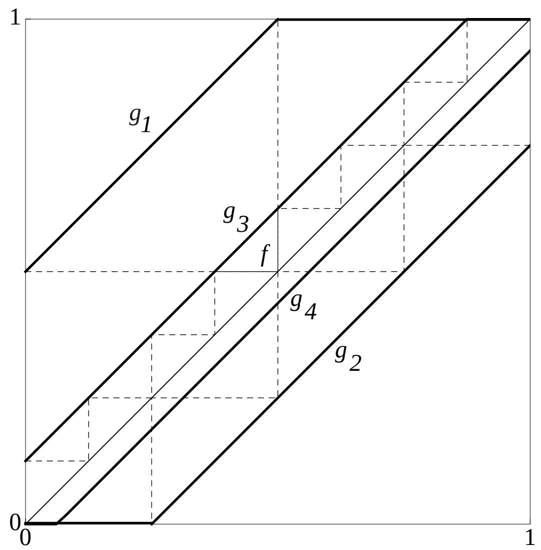

Figure 2:

[1]The referee kindly pointed out that an example similar to Example 2 is found in the paper [12].

Example 3. Let $p \in \mathbb{N}$. We construct a nonautonomous discrete system (f_n) on the unit interval $I = [0,1]$ which converges uniformly to a map $f : I \to I$ such that $P(f) = F(f^{2^{p-1}})$ and $\omega(0,(f_n)) = \bigoplus_{i=1}^{2^{p-1}} I_i$, where each I_i is a nondegenerate closed interval consisting of periodic points of f.

[Construction]: We shall construct maps by induction on p.

Let $p = 1$. Let g_k, f_n and f be as in Example 2, and put $g_{1k} = g_k$ and $f^{(1)} = f$.

Let $p = 2$. The map $f^{(2)} : I \to I$ is defined by

$$f^{(2)}(x) = \begin{cases} x + 2/3 \ (= f^{(1)}(3x)/3 + 2/3) & \text{for } 0 \le x \le 1/3, \\ x - 2/3 & \text{for } 2/3 \le x \le 1, \end{cases}$$

and by linearity elsewhere. For $k \in \mathbb{N}$, the map $g_{2k} : I \to I$ is defined by

$$g_{2k}(x) = \begin{cases} g_{1k}(3x)/3 + 2/3 & \text{for } 0 \le x \le 1/3, \\ x - 2/3 & \text{for } 2/3 \le x \le 1, \end{cases}$$

and by linearity elsewhere (see Figure 3). We define the sequence (f_n) as

$$\underbrace{g_{21}, \ldots, g_{21}}_{2^2 \text{ times}}, \underbrace{g_{22}, \ldots, g_{22}}_{2^3 \text{ times}}, \underbrace{g_{23}, \ldots, g_{23}}_{2^4 \text{ times}}, \cdots$$

Then $P(f^{(2)}) = F(f^{(2)2^1})$, $f_n \to f^{(2)}$ uniformly, and

$$\{f_1^n(0) \mid n = 0,1,\ldots\} = \{\frac{1}{3} \cdot \frac{\ell}{2^k}, \frac{2}{3} + \frac{1}{3} \cdot \frac{\ell}{2^k} \mid \ell = 0,1,\ldots,2^k, k = 1,2,\ldots\};$$

hence we see $\omega(0,(f_n)) = [0,1/3] \cup [2/3,1]$.

Suppose that we had constructed maps g_{qk} and $f^{(q)}$ $(q = 1,\ldots,p-1, k \in \mathbb{N})$. Then the map $f^{(p)} : I \to I$ is defined by

$$f^{(p)}(x) = \begin{cases} f^{(p-1)}(3x)/3 + 2/3 & \text{for } 0 \le x \le 1/3, \\ x - 2/3 & \text{for } 2/3 \le x \le 1, \end{cases}$$

and by linearity elsewhere. For $k \in \mathbb{N}$, the map $g_{pk} : I \to I$ is defined by

$$g_{pk}(x) = \begin{cases} g_{p-1\,k}(3x)/3 + 2/3 & \text{for } 0 \le x \le 1/3, \\ x - 2/3 & \text{for } 2/3 \le x \le 1, \end{cases}$$

and by linearity elsewhere (see Figure 4, the case $p = 3$). We define the sequence (f_n) as

$$\underbrace{g_{p1}, \ldots, g_{p1}}_{2^p \text{ times}}, \underbrace{g_{p2}, \ldots, g_{p2}}_{2^{p+1} \text{ times}}, \underbrace{g_{p3}, \ldots, g_{p3}}_{2^{p+2} \text{ times}}, \cdots$$

Then $P(f^{(p)}) = F(f^{(p)2^{p-1}})$, $f_n \to f^{(p)}$ uniformly, and

$$\begin{aligned} &\{f_1^n(0) \mid n = 0,1,\ldots\} \\ &= \{\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{p-1}}{3^{p-1}} + \frac{1}{3^{p-1}}\frac{\ell}{2^k} \mid a_i = 0,2, i = 1,\ldots,p-1, \\ &\qquad \ell = 0,\ldots,2^k, k = 1,2,\ldots\}; \end{aligned}$$

hence we see

$$\omega(0,(f_n)) = \bigoplus_{\substack{a_i=0,2 \\ i=1,\dots,p-1}} [\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{p-1}}{3^{p-1}}, \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_{p-1}}{3^{p-1}} + \frac{1}{3^{p-1}}].$$

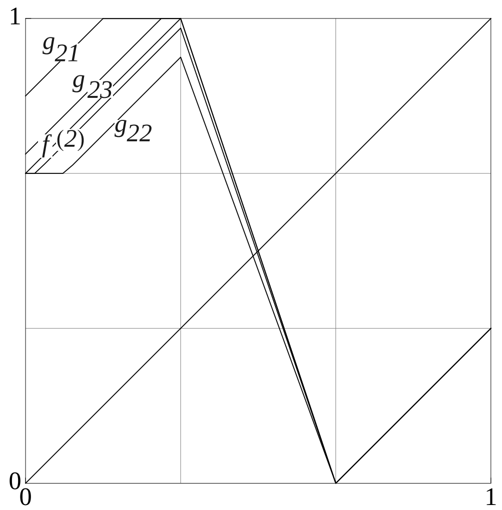

Figure 3: the case $p = 2$

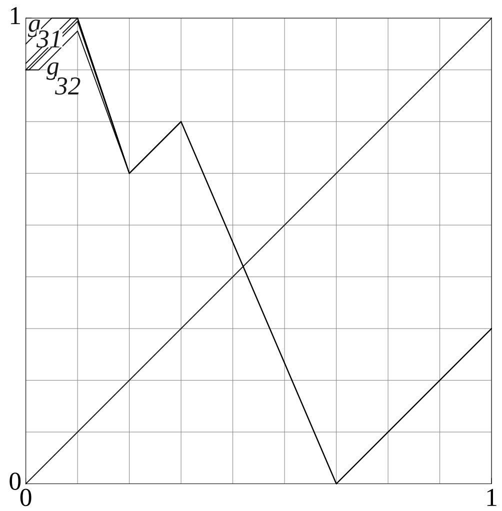

Figure 4: the case $p = 3$

Example 4. We construct a nonautonomous discrete system (f_n) on the unit interval $I = [0,1]$ which converges uniformly to a map $f : I \to I$ such that $P(f)$ is closed, and for $p \in \mathbb{N}$, there exists $x_p \in I$ satisfying $\omega(x_p, (f_n)) = \bigoplus_{i=1}^{2^{p-1}} I_i$, where each I_i is a nondegenerate closed interval consisting of periodic points of f.

[Construction]: For $p \in \mathbb{N}$, let $I_p = [1 - 1/3^{p-1}, 1 - 2/3^p]$. We use the maps $g_{pk}, f^{(p)}$ which were constructed in Example 3. The map $f : I \to I$ is defined by

$$f(x) = \begin{cases} f^{(p)}(3^p(x - 1 + 1/3^{p-1}))/3^p + 1 - 1/3^{p-1} & \text{for } x \in I_p \ (p \in \mathbb{N}), \\ 1 & \text{for } x = 1, \end{cases}$$

and by linearity elsewhere (see Figure 5). For $k \in \mathbb{N}$, the map $g_k : I \to I$ is defined by

$$g_k(x) = \begin{cases} g_{p\ k-p+1}(3^p(x - 1 + 1/3^{p-1}))/3^p + 1 - 1/3^{p-1} \\ \qquad\qquad\qquad\qquad \text{for } x \in I_p \ (p = 1, \dots, k), \\ x \qquad\qquad\qquad\qquad \text{for } 1 - 1/3^k \le x \le 1, \end{cases}$$

and by linearity elsewhere. We note that $f(I_p) = I_p$ and $g_k(I_p) = I_p$ for $p, k \in \mathbb{N}$. We define the sequence (f_n) as

$$\underbrace{g_1, g_1}_{2^1 \text{ times}}, \underbrace{g_2, \dots, g_2}_{2^2 \text{ times}}, \underbrace{g_3, \dots, g_3}_{2^3 \text{ times}}, \cdots$$

Then $P(f)$ is closed, $f_n \to f$ uniformly, and we see $\omega(1 - 1/3^{p-1}, (f_n)) = \bigoplus_{i=1}^{2^{p-1}} I_i$, where $I_i \cong I$, $i = 1, \dots, 2^{p-1}$.

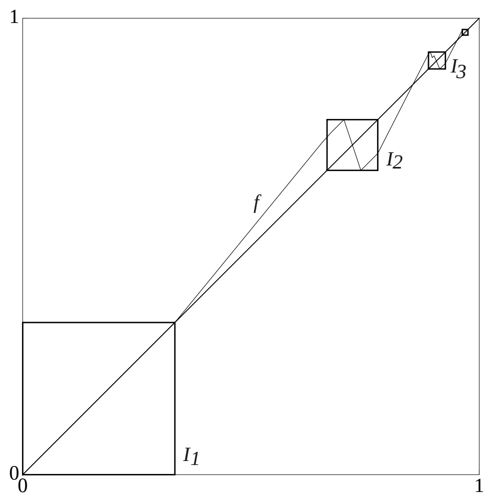

Figure 5:

We note by the following examples that the first conclusion of Corollary 3.8 is false in general for maps on two-dimensional spaces and dendrites, even if the case of an autonomous discrete system.

Example 5. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The map $f : D \to D$ is defined by $f(z) = \sqrt{|z|}z \cdot e^{i2\pi\alpha}$, where α is a fixed irrational number. Let $f_n = f$ for $n \in \mathbb{N}$. Then $F(f) = P(f) = \{(0,0)\}$ and $\omega(z,(f_n))$ is the boundary of D for $z \neq (0,0)$.

Example 6. Let C denote the Cantor Middle-Third set which is constructed on $[0,1] \times \{0\} \subseteq \mathbb{R}^2$, and let G denote the dendrite [21, 10.39 Example] which has C as the base space and $* = (1/2, 1/2)$ as the top point; as drawn below in Figure 6. Let $h : C \to C$ be the adding machine (c.f. [10, p.137]); we only need the property that every orbit of h is dense in C, that is, any ω-limit set coincides C. In [15], Kato constructed a map $H : G \to G$ such that $H|_C = h$, $H(*) = *$, and $d(z,*) > d(H(z),*)$ for $z \in G \setminus (C \cup \{*\})$. Let $H_n = H$ for $n \in \mathbb{N}$. Then $F(H) = P(H) = *$ and $\omega(z,(H_n)) = C$ for $z \in C$.

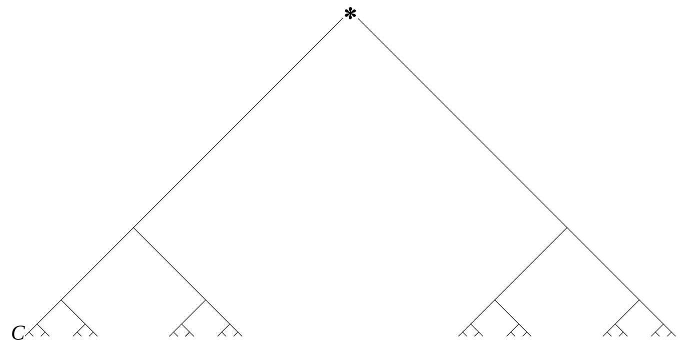

Figure 6:

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